

AN ANALYTIC SOLUTION TO THE SEDIMENT DIFFUSION EQUATION

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ABSTRACT : The diffusion equation for the transported phase with small concentrations is rederived and some of the approximations inherent in the standard form of the equation are noted. An analytic solution for the diffusion equation in two dimensions is developed under the hypothesis of constant diffusion coefficients ϵ_x, ϵ_y . This solution is applied to a variety of injection modes in steady flow. The results are then compared with the classical experiments of Jobson and Sayre (1970).

Introduction: The diffusion equation for a non-reacting, transported phase can be solved by a number of analytical and numerical methods. Methods of the latter type have predominated lately, but it is felt that in many cases the analytic solutions remain useful for their more direct application and the possibility of checking the asymptotic behaviour of the numerical computation. The solution herein developed applies to the case of steady source of pollutant, near the surface of a two dimensional stream.

ANALYSIS :

Following the standard analysis of Daily and Harleman (1966), the mass flux vector may be written as:

$$\vec{q} = \rho_o (u_s c - \vec{\epsilon} \nabla c) \quad (1)$$

where ρ_o = dispersed phase density
 \vec{u}_s = velocity of dispersed phase (sediment)
 c = volumetric concentration.
 $\vec{\epsilon}$ = diffusivity tensor.

If the sediment has a fall velocity $\vec{\omega}$ then the velocity vector \vec{u}_s may be related to the fluid velocity \vec{u}_f by :

$$\vec{u}_s = \vec{u}_f - \vec{\omega} \quad (2)$$

This assumption presupposes small inertia for the sediment particles. Combining these relations into the dispersed phase continuity equation :

$$\frac{\partial c}{\partial t} + \frac{1}{\rho_o} \nabla \cdot \vec{q} = 0 \quad (3)$$

there results:

$$\frac{\partial c}{\partial t} + \nabla \cdot [(\vec{u}_f - \vec{u}_s)c - \vec{\epsilon} \nabla c] \quad (4)$$

In turn, the equation of conservation of mass for the fluid is :

$$\frac{\partial}{\partial t} (1-c) + \nabla \cdot (\vec{u}_f (1-c)) = 0 \quad (5)$$

From (4) and (5) the divergence of the fluid phase is:

$$\nabla \cdot \vec{u}_f = \nabla \cdot ((\vec{u}_f - \vec{u}_s)c + \vec{\epsilon} \nabla c) \quad (6)$$

The usual simplifying assumption that $\nabla \cdot \vec{u}_f = 0$ is then true only for small concentrations c when the RHS of (6) would be negligible, or when equilibrium conditions in the sediment phase have been reached. With this simplification, equation (4) reads:

$$\frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \frac{\partial}{\partial x_j} (\omega_j c + \epsilon_j \frac{\partial c}{\partial x_j}) \quad (7)$$

In turbulent flow, the turbulent diffusion coefficient defined by the Fickian analogy:

$$-u_j c' = \epsilon_j \frac{\partial c}{\partial x_j}$$

is much larger than the molecular coefficient; so that the final form of the sediment mass diffusion equation is :

$$\frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \frac{\partial}{\partial x_j} (\epsilon_j \frac{\partial c}{\partial x_j} + \omega_j c) \quad (8)$$

Equation (8) has been the starting point for most studies of sediment and pollutant diffusion, notably those of Kalinske (1940), Dobbins (1943), and many others including the paper by Jobson and Sayre cited before.

Consider now a flow where :

$$u_j = (U, 0, 0) \quad \omega_j = (0, \omega, 0) \quad (9)$$

represent the mean velocities of the transported phase.

$$\frac{\partial c}{\partial t} + U \frac{\partial c}{\partial x} = \frac{\partial}{\partial x} (\epsilon_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y} (\epsilon_y \frac{\partial c}{\partial y} + \omega c) \quad (10)$$

For steady, uniform flow (10) reduces to :

$$\epsilon_y \frac{\partial c}{\partial y} + \omega c = 0$$

the well known Schmidt equation, henceforth referred as the "equilibrium" condition.

By omitting the first term on the RHS of (10) the sediment diffusion equation has been solved analytically by Kalinske, Dobbins and Mei (1967), under different boundary conditions. Numerical solutions have been obtained, among others by Apmann (1969), Jobson and Sayre (1970) and O'Connor (1970). In all these cases, the term $\epsilon_x \frac{\partial c}{\partial x}$ has been assumed small.

BOUNDARY CONDITIONS:

The boundary condition chosen here is that no net transport occurs across the free surface or the bottom

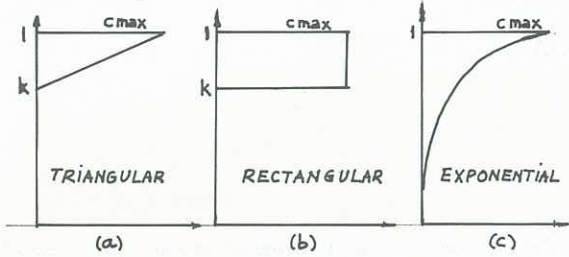
$$\epsilon_y \frac{\partial c}{\partial y} + \omega c = 0 \quad \text{for } y=0 \quad (11)$$

$$y=y_o$$

A second boundary condition specifies the initial concentration profile at $x=0$. Following Keulegan (1944), no other boundary condition need be specified. The first boundary condition is different

from that employed by Kalinske, Apmann and Mei, as they were treating a problem where the source of the material was at the channel bottom. Our boundary condition is more appropriate for the surface injection of sediment or neutrally buoyant pollutant. SURFACE INJECTION CONDITIONS :

The dispersed phase (sediment, pollutant) is assumed to be injected at the surface following the modes shown in Figure 1.



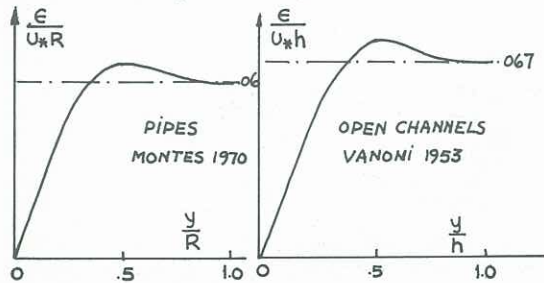
Assumptions regarding the coefficients of the differential equation (10):

- Longitudinal velocity U independent of the depth
- Diffusion coefficient ϵ_y independent of the depth
- Diffusion coefficient ϵ_x constant with x .
- Sediment fall velocity w is also assumed constant.

Although these assumptions are, to some extent, an oversimplification of the true behaviour of these variables, only the constancy of ϵ_x with x needed modification if an improved definition of c for small x is expected.

Vertical Diffusion Coefficient: As shown by Hinze (1959) and Montes (1970) the diffusion coefficient for momentum is practically constant for about 70% of the depth in the case of pipes and channels. If the distribution of ϵ_y is calculated from the experiments by Vanoni (1944, 1953), it is found that the shape of the distribution remains similar to the momentum diffusion coefficient in pure water flow, but the maximum values are only about 2/3 as large. From the similarity of the results in pipes and channels the chosen value of the vertical diffusion coefficient is :

$$\epsilon_y = 0.067 u_* h \quad (12)$$



Uniform velocity distribution: This assumption was prompted by the wish to simplify the analytic solution. It may be justified by noting that, in turbulent flows, point velocities do not differ more than 10% from the mean. In a smooth channel with logarithmic velocity distribution the difference is less than 8% of the mean.

Longitudinal dispersion coefficient : As Harleman (1970) points out, the coefficient ϵ_x in (10) should be properly called a dispersion coefficient as U is taken to be independent of y . Its magnitude is much larger than that of ϵ_y , as confirmed by Taylor (1954), Elder (1959) and Fischer (1964).

ANALYTIC SOLUTION TO THE DIFFUSION EQUATION

The following set of dimensionless variables may be defined:

$$y_* = \frac{y}{h} ; x_* = \frac{x}{h} ; \epsilon_{x,y} = \frac{\epsilon_{x,y}}{U h} ; \omega_* = \frac{\omega}{U} \quad (13)$$

The sediment diffusion equation (10) is then rewritten in terms of these variables. All terms become dimensionless.

A solution:

$$c = c_e(y) + c'(x, y) \quad (14)$$

is postulated, where $c_e(y)$ is the equilibrium solution achieved for large x , and $c'(x, y)$ is the "perturbation" of the equilibrium solution at small values of x . For the perturbation solution it will be assumed that it is of the "separation of variables" form:

$$c'(x, y) = X(x) Y(y) \quad (15)$$

Upon introduction into equation (10) we are led to:

$$\frac{X'}{X} - \epsilon_x \frac{X''}{X} = \epsilon_y \frac{Y''}{Y} + \omega \frac{Y'}{Y} = -k^2 \quad (16)$$

where k^2 is the separation constant. Equation (16) is equivalent to :

$$X'' - \frac{1}{\epsilon_x} X' - \frac{k^2}{\epsilon_x} X = 0 \quad (17)$$

$$Y'' + \frac{\omega}{\epsilon_y} Y' + \frac{k^2}{\epsilon_y} Y = 0 \quad (18)$$

The first equation has the solution :

$$X(x) = C_2 \exp\left(\frac{1}{2\epsilon_x} - \sqrt{\frac{1}{4\epsilon_x^2} + \frac{k^2}{\epsilon_x}}\right) \quad (19)$$

The second a similar one :

$$Y(y) = \exp\left(-\frac{\omega}{2\epsilon_y}\right) (C_3 \cos(\alpha y) + C_4 \sin(\alpha y)) \quad (20)$$

where

$$\alpha = \frac{\omega}{2\epsilon_y} \left(4 \frac{k^2 \epsilon_y}{\omega^2} - 1\right)^{\frac{1}{2}} \quad (21)$$

The complete solution to (10) is then:

$$c = c_e(y) + \exp(qx - py) (C_3 \cos \alpha y + C_4 \sin \alpha y) \quad (22)$$

where $q = \frac{1}{2\epsilon_x} - \sqrt{\frac{1}{4\epsilon_x^2} + \frac{k^2}{\epsilon_x}}$; $p = +\frac{\omega}{2\epsilon_y}$

The constants C_3, C_4 and α must be determined from the boundary conditions. As the net transport across the bottom and the free surface for the equilibrium solution (large x) must equal 0, it follows that for $y=0$:

$$C_4 = -\frac{\omega}{2\epsilon_y} C_3 \quad (23)$$

and for $y=h$

$$\left(\alpha + \left(\frac{\omega}{2\epsilon_y}\right)^2 \frac{1}{\alpha}\right) \tan \alpha = 0 \quad (24)$$

This last conditions requires :

$$\alpha = n\pi \quad (n=1, 2, \dots, \infty) \quad (25)$$

The solution to the equilibrium equation with a constant vertical diffusion coefficient is also straightforward:

$$c = A \exp\left(-\frac{\omega}{2\epsilon_y} y\right) \quad (A = \text{a constant}) \quad (26)$$

Hence the complete solution is, by the linear property of (22):

$$c = A \exp\left(-\frac{\omega}{\epsilon_y} y\right) + \exp(qx - py) \sum_{n=1}^{\infty} C_n \left(\cos n\pi y - \frac{p \sin n\pi y}{n\pi}\right) \quad (27)$$

The constants C_n in (27) are determined so as to satisfy the concentration distribution at $x=0$, and constant A by requiring that the mean concentration in the equilibrium region must equal the initial mean concentration, C_m

$$A = \frac{C_m 2p}{1 - e^{-2p}} \quad (p = \frac{\omega}{2\epsilon_y}) \quad (28)$$

DETERMINATION OF THE CONSTANTS C_n :
The initial concentration profile at $x=0$ is described by :

$$c = f(y)$$

where $f(y)$ is a known function of the depth. To secure the constants C_n , one equates (27) to this function and integrates from $y=0$ to $y=1$, after having multiplied both sides of the equation by:

$$\left(\cos n\pi y - \frac{p}{n\pi} \sin n\pi y \right)$$

from the orthogonality condition of integrals of the type :

$$\int_0^1 \cos m\pi y \cos n\pi y = 0 \quad (m \neq n) \quad (29)$$

it is found that C_n may be written as :

$$C_n = \frac{2}{1 + \frac{p^2}{n^2\pi^2}} \int_0^1 f(y) Y(n) dy \quad (30)$$

$$f(y) = \left(f(y) - A e^{-2py} \right) e^{py}; \quad Y(n) = \cos n\pi y - \frac{p}{n\pi} \sin n\pi y$$

TRIANGULAR INJECTION OF SEDIMENT :

This type of injection is defined in figure 1-a. The concentration is normalised by setting

$$\int_0^1 c dy = 1$$

so that the constant A in (28) is defined uniquely in terms of p . For the triangular injection:

$$\int_0^1 c dy = (1 - k) c_{max} / 2 \quad (31)$$

From (30):

$$C_n = \frac{2}{1 + \frac{p^2}{n^2\pi^2}} \left\{ \int_0^1 e^{2py} f(y) Y(n) - A \int_0^1 e^{-py} Y(n) dy \right\}$$

the second integral can be shown to be equal to zero. For this type of injection $f(y)$ is defined as follows :

$$f(y) = 0 \quad 0 < y < k$$

$$f(y) = \frac{2}{(1-k)^2} (y-k) \quad k < y < 1$$

Hence C_n equals:

$$C_n = \frac{4}{1 + \frac{p^2}{n^2\pi^2}} \int_k^1 e^{\frac{py}{(1-k)^2}} Y_n dy$$

This integral can be solved in closed form, obtaining finally:

$$C_n = \frac{8pe^p}{(1-k)^2 (n^2\pi^2 + p^2) \left(1 + \frac{p^2}{n^2\pi^2}\right)} \left\{ \left((1-k) - \frac{(3p^2 - n^2\pi^2)}{2p(n^2\pi^2 + p^2)} \right) (-1)^n + \frac{e^{-p(1-k)}}{2p(n^2\pi^2 + p^2)} \left[(3p^2 - n^2\pi^2) \cos n\pi k - \frac{p}{n\pi} (p^2 - 3n^2\pi^2) \sin n\pi k \right] \right\} \quad (32)$$

RECTANGULAR INJECTION :

This case is described in figure 1-b. Here the function $f(y)$ is defined mathematically by:

$$f(y) = 0 \quad 0 < y < k$$

$$f(y) = c_{max} \quad k < y < 1$$

Following identical procedure to the case of triangular injection, one obtains for this case:

$$C_n = \frac{2}{(1-k)n^2\pi^2 \left(1 + \frac{p^2}{n^2\pi^2}\right)} \left[\left(2p(-1)^n e^p \right) - e^{pk} \left(2p \cos n\pi k + \left(n\pi - \frac{p^2}{n^2\pi^2} \right) \sin n\pi k \right) \right] \quad (33)$$

For a neutrally buoyant pollutant, equations (32) and (33) can be considerably simplified, as the parameter $p=0$. Thus :

$$C_n = \frac{4}{(1-k)^2 n^2\pi^2} \left((-1)^n - \cos n\pi k \right)$$

Triang. Injection (34)

$$C_n = -\frac{2}{n\pi(1-k)} \sin n\pi k \quad \text{Rect. Injection (35)}$$

EXPONENTIAL INJECTION:

This mode of injection is defined in figure 1-c. In this case :

$$f(y) = A e^{-2p(1-y)}$$

The coefficients for the series for c are very laborious to obtain. The final result is :

$$C_n = \frac{8pe^{-2p} (e^{3p} (-1)^n - 1)}{(n^2\pi^2 + 9p^2) (1 + p^2/\pi^2 n^2)} \quad (36)$$

when a neutrally buoyant pollutant is injected, equation (36) may be shown to reduce to :

$$C_n = \frac{4p}{4p^2 + \pi^2 n^2} \left[(-1)^n - e^{-2p} \right] \quad (37)$$

Although other forms of injection could be defined, the general technique remains the same and the constants may be calculated in most cases. The simplified form of the diffusion equation solved by other researchers admits of an almost identical solution, namely :

$$C = C_e + \sum_{n=1}^{\infty} C_n e^{-\epsilon_y n^2 x} \left(\cos n\pi y + \frac{2p}{\pi n} \sin n\pi y \right) \quad (38)$$

The series present in equations (32) to (36) are only slowly convergent and about 100 terms need be computed for the remainder to become negligible. However, this operation takes only a short time on a modern computer.

COMPARISON OF THE ANALYTIC SOLUTION WITH THE AVAILABLE EXPERIMENTAL RESULTS :

The dispersed phase used in the experiments of Jobson and Sayre consisted of

- a) Glass spheres with a fall velocity of $w=17.3$ mm/s, and
- b) Neutrally buoyant pollutant.

The results of the analytic prediction naturally depend on the value of the longitudinal dispersion coefficient which is much affected by the lateral dispersion of the stream (Fischer 1966). Many analytic and field studies have been conducted in this regard. It is now accepted that field results in rough channels give dispersion coefficients which are approximately up to 2 orders of magnitude larger than for laboratory channels. Thus Fischer found that when ϵ_x is expressed in the dimensionless form:

$$\epsilon_x = \frac{\epsilon_x}{U_* h}$$

the spread of the values is:

$\epsilon_x = 10.1$	Pipe Flow (G.S. Taylor)
6	2-D Channel Flow (Elder)
25	Various River Data (Fischer)
660	" " "

Expressed as a ratio of the vertical diffusion coefficient, this ratio seems to have an order of magnitude of 16 to 10,000 depending on the roughness and curvature of the stream. By comparison with Jobson's experiments conducted in a laboratory canal, it was found that a "best fit" was obtained when $\gamma = 1000$. Such best fit took place some distance from the point of injection. This indicates that the hypothesis of a constant ϵ_x must be revised if accurate concentrations are to be determined close to the injection point.

A close examination of the experimental results of Jobson and Sayre indicates that to obtain proper agreement between the present solution and the data, not only the longitudinal dispersion coefficient had to be scaled as discussed, but also an improved fit was obtained when the scaling ratio with respect to ϵ_y was made variable with distance. For sediment injection the best fit was obtained by:

$$\epsilon_x = (3000 e^{-0.2x}) \epsilon_y \quad (39)$$

and for neutrally buoyant particles:

$$\epsilon_x = (20000 e^{-0.2x}) \epsilon_y \quad (40)$$

For large values of x/h the importance of the longitudinal dispersion coefficient decreases, and it follows that the simplified solution of Dobbins is also satisfactory. At closer distances to the injection point the inclusion of the term:

$$\epsilon_x \frac{\partial^2 c}{\partial x^2}$$

will improve the accuracy of the solution, given a correct value of the dispersion coefficient. The following figures show a comparison of the calculated and measured concentration distribution for triangular injection near the surface, for sediment.

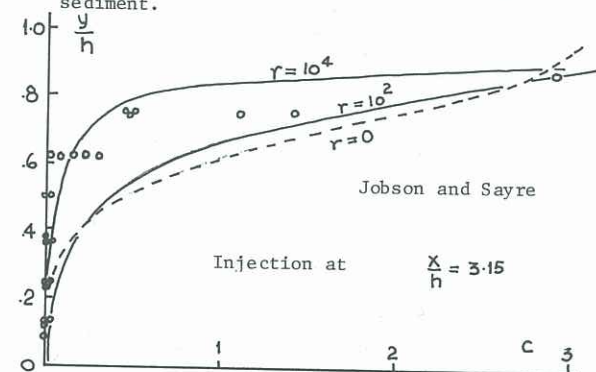


Figure 3

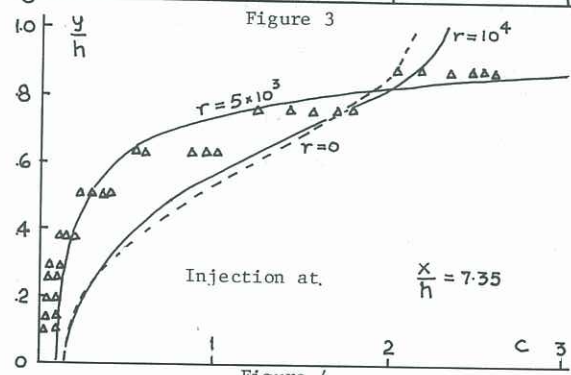


Figure 4

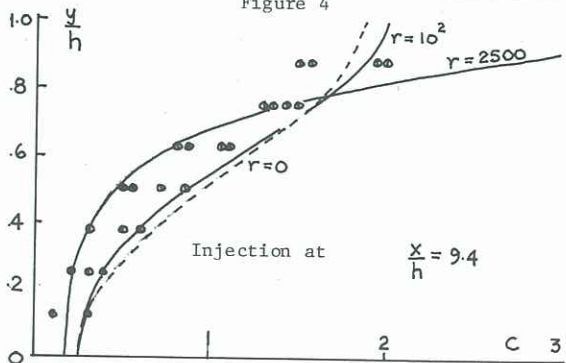


Figure 5

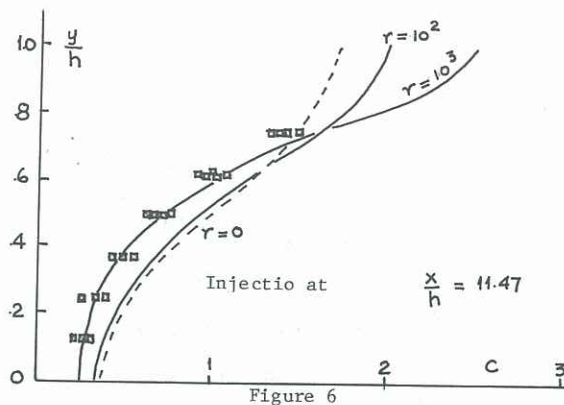


Figure 6

Conclusions: A relatively simple analytic solution to the sediment diffusion equation has been developed. The main assumptions are the constancy of the vertical and longitudinal diffusion coefficients. By comparison with the experimental results of Jobson and Sayre (1970) it is found that the analytic solution gives a fair to good account of the concentration distribution. However, for values of x/h less than about 10, this particular comparison shows the need for a more precise definition of the longitudinal dispersion coefficient. At large values of x/h the agreement is quite good.

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