

IMPULSIVE DISTURBANCE IN A ROTATING STRATIFIED FLUID

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ABSTRACT

A theoretical investigation is made of the response of a rotating stratified fluid to the sudden initiation of a line sink. The fluid is assumed to be inviscid and non-diffusive and occupies all space. The velocity components are derived in Laplace space for a sink with whose strength has arbitrary time dependence. The Laplace inversion is performed in detail for the case of impulsive sink flow and the significance of the resulting solution is discussed.

INTRODUCTION

This study aims to shed light on the nature of the time evolution and eventual steady state flow field caused by a sink in a rotating stratified fluid. It is motivated by the proposition that some sink flows in geophysical and even limnological situations may be influenced by rotation as well as the usual density stratification (Monismith & Maxworthy, 1989; Whitehead, 1980). Examples of such sink flows include the withdrawal from a water supply reservoir and the entrainment of ambient fluid by hydrothermal plumes (Speer, 1989).

For the case of a non-rotating but stratified fluid the action of buoyancy forces is to cause fluid entering the sink to come from a narrow layer located at the level of the sink. This is called selective withdrawal and is a well known and much studied problem. For a review of this subject see Imberger & Patterson (1989). By analysing the initial value problem of suddenly initiating the sink flow, Pao & Kao (1974) were able to show that the withdrawal layer is established by internal waves of zero frequency but finite group velocity travelling outward from the sink. These waves are called columnar disturbances or shear waves. Given the analogy between the dynamics of rotating homogeneous fluids and stratified non-rotating fluids (Veronis, 1967) similar waves are responsible for the vertical withdrawal layer above a sink in a rotating homogeneous fluid (see, for example, Pao & Shih, 1973). It should be pointed out that similar waves are responsible for the growth of Taylor columns in rotating fluids and blocking in stratified fluids (see Bretherton, 1967).

However, for a fluid which is both rotating and stratified, internal wave frequencies are bounded above and below by the inertial and buoyancy frequencies (Gill, 1982 p.258). Hence there exist no freely propagating waves which carry energy at vanishing frequencies. It follows that, within the framework of linear, inviscid theory at least, selective withdrawal does not occur in a rotating stratified fluid and some other withdrawal flow pattern oc-

curs. It is the purpose of this study to investigate such flows.

In the light of the preceeding discussion, to gain an understanding of the physics of the establishment and possible steady state flow due to a sink in a rotating stratified fluid, an initial value problem for a line sink is formulated in an unbounded fluid. The strength of the sink may be a function of time. It is formulated in two dimensions for ease of analysis and although this configuration is difficult to achieve in the laboratory, the physics of the flow will be essentially the same as would be observed for, say, the axisymmetric case. The method of analysis is similar to that of Hendershott (1969) who studied axisymmetric oscillations in a rotating stratified fluid caused by a finite spherical source. For the case when the sink flow is impulsive, i.e. the time dependence can be represented by a Dirac delta function, the velocity field is derived in detail. This solution is then used to discuss the withdrawal flow for a maintained sink of constant strength in a rotating stratified fluid.

FORMULATION AND SOLUTION

Consider an inviscid, non-diffusive fluid which is stably stratified with constant buoyancy frequency N occupying all space. Let (x, z) be a coordinate system such that z is antiparallel to the gravity vector and the x axis is perpendicular to the z axis. The properties of the fluid motion are assumed to be independent of the transverse (y) coordinate and therefore may be considered two dimensional. Further, the fluid is rotating about the z axis with uniform angular velocity $f/2$. Initially the fluid is at rest in this coordinate system. At time $t = 0$ a line sink parallel to the y axis located at the origin of the coordinate system with strength $Q(t) = Q_0 \delta(t)$, $t > 0$, is suddenly switched on. The dimensions of Q are $m^2 s^{-1}$. The buoyancy frequency N is defined by $N^2 = -\frac{1}{\rho_0} (d\rho_0/dz)$, where $\rho_0(z)$ is the undisturbed density.

At sufficient distance (to be quantified later) from the sink, the induced velocities will be small and the resulting motion is governed by the following set of linearised nondimensional equations

$$u_x + w_z = -q(t)\delta(x)\delta(z), \quad (1)$$

$$\rho_0(u_t - Bv) = -P_x, \quad (2)$$

$$v_t + Bu = 0, \quad (3)$$

$$\rho_0 w_t = -P_z - \rho, \quad (4)$$

$$\rho_t - w\rho_z = 0, \quad (5)$$

where $B = f/N$, is the ratio of the Coriolis parameter to the buoyancy frequency and is a measure of the relative strengths of rotation and stratification. Here u is the

velocity component in the x direction, w is the velocity component in the z direction, v the swirl velocity along the axis of the sink, P' is the pressure perturbation from hydrostatic pressure and $\rho(x, z, t)$ is the variation of the density from $\rho_0(z)$.

To recover the dimensional quantities the following transformations are used

$$t \rightarrow t/N, \quad (x, z) \rightarrow L(x, z), \quad (6), (7)$$

$$(u, v, w) \rightarrow \frac{Q}{L}(u, v, w), \quad \rho_0 \rightarrow \rho' \rho_0, \quad (8), (9)$$

$$\rho \rightarrow \frac{QN}{gL} \rho', \quad P \rightarrow \rho' QNP, \quad (10), (11)$$

where L is a length scale and ρ' is a reference density, here, the density at $z = 0$. Formally, for the linearisation to be valid the length scale must be chosen so that the Froude number, $Fr = Q/NL^2$ be small i.e. $Fr \ll 1$. Further, for the Coriolis terms to be important relative to the nonlinear advective terms it is also required that the Rossby number, $Ro = Q/fL^2$, be small i.e. $Ro \ll 1$. Thus for a given Q , f and N equations (1)–(5) are good approximations to the fully nonlinear Euler equations at sufficiently large distance from the sink.

Initially, just after the sink flow has been turned on, stratification has no effect since its influence is manifested through forces proportional to the displacement of fluid particles. Given the analogy between rotating and stratified flows in two dimensions (Veronis, 1967) a similar remark can be said about the initial effect of rotation. Hence, since the fluid is incompressible, the appropriate initial condition to use is that of potential flow, in particular the potential flow caused by a line sink. Therefore at $t = 0$, $u = \phi_x$, $w = \phi_z$, $v = 0$, $\rho = 0$, where $\phi = q(0)/2\pi \log(\sqrt{x^2 + z^2})$.

Following Hendershott (1969), the Laplace transform in the time variable of (1)–(5) is taken where the Laplace transform has the usual definition

$$\bar{g}(s) = \int_0^\infty g(t)e^{-st} dt.$$

This yields, using the appropriate initial conditions

$$\bar{u}_x + \bar{w}_z = -\bar{q}(s)\delta(x)\delta(z), \quad (12)$$

$$\rho_0(s\bar{u} - B\bar{v}) = -\bar{p}_x, \quad (13)$$

$$s\bar{v} + B\bar{u} = 0, \quad (14)$$

$$\rho_0 s\bar{w} = -\bar{p}_z - \bar{\rho}, \quad (15)$$

$$s\bar{\rho} - \bar{w}\rho_0 z = 0, \quad (16)$$

where $\bar{P} = \rho_0 \phi + \bar{p}$ and $\delta(x)$ is the delta function. Eliminating \bar{v} from (13) and (14) and $\bar{\rho}$ from (15) and (16) and then substituting the resulting expressions for \bar{u} and \bar{w} in (12) gives, after some rearrangement, the following equation for $\bar{p}(x, z, s)$

$$\begin{aligned} \bar{p}_{xx} + \frac{s^2 + B^2}{s^2 + 1} \bar{p}_{zz} + \beta \frac{s^2 + B^2}{s^2 + 1} \bar{p}_z \\ = -\rho_0 \bar{q}(s) \frac{s^2 + B^2}{s} \delta(x)\delta(z). \end{aligned} \quad (17)$$

Where the density, by virtue of the constancy of N , has been written in the form $\rho_0 = \exp(-\beta z)$, $\beta = N^2 L/g$.

The details of the solution of (17) are not presented for sake of brevity but the solution obtained through use of Fourier transforms and requiring that the velocity components vanish at infinity is given by

$$\bar{p}(x, z, s) = -\rho_0 \frac{\bar{q}(s)\sqrt{s^2 + B^2}\sqrt{s^2 + 1}}{2\pi s} K_0(\xi), \quad (18)$$

where K_0 is zeroth order modified Bessel function of the second kind and

$$\xi = \frac{\beta}{2} \left(\frac{s^2 + B^2}{s^2 + 1} x^2 + z^2 \right)^{\frac{1}{2}}.$$

The velocity components are then calculated using (18). For example the horizontal velocity is

$$\bar{u} = \frac{\bar{q}(s)}{2\pi} \sqrt{\frac{s^2 + B^2}{s^2 + 1}} \frac{x}{\frac{s^2 + B^2}{s^2 + 1} x^2 + z^2} \xi K'_0(\xi),$$

where the prime indicates differentiation with respect to ξ . The expressions for the other velocity components are similar.

The limit $\beta \rightarrow 0$ corresponds to the Boussinesq approximation, since the terms involving vertical variations of density in computing rates of change of momentum in the inertial terms are ignored when deriving (19). This can be stated mathematically that in the Boussinesq approximation vertical distances L from the sink are such that $N^2 L/g \ll 1$. The Boussinesq approximation is valid to a good approximation in most physical applications and it is employed from here on. For $\beta \rightarrow 0$ then $\xi \rightarrow 0$ and in this limit it can be deduced, using results from Abramowitz & Stegun (1972) p.375, that

$$\lim_{\xi \rightarrow 0} \xi K'_0(\xi) = -1.$$

The velocity components are then

$$\bar{u} = \frac{-1}{2\pi} \bar{q}(s) \sqrt{\frac{s^2 + 1}{s^2 + B^2}} \frac{x}{x^2 + \frac{s^2 + 1}{s^2 + B^2} z^2}. \quad (19)$$

$$\bar{w} = \frac{-1}{2\pi} \bar{q}(s) \sqrt{\frac{s^2 + 1}{s^2 + B^2}} \frac{z}{x^2 + \frac{s^2 + 1}{s^2 + B^2} z^2}. \quad (20)$$

Further, using (14), the transformed swirl velocity is determined to be

$$\bar{v} = \frac{B}{2\pi s} \bar{q}(s) \sqrt{\frac{s^2 + 1}{s^2 + B^2}} \frac{x}{x^2 + \frac{s^2 + 1}{s^2 + B^2} z^2}. \quad (21)$$

It now remains for a given $q(t)$ to invert the above Laplace transforms to obtain the velocity field. Note that as $s \rightarrow \infty$ (i.e. $t \rightarrow 0$) the velocity field approaches that of potential flow caused by a line sink as would be expected.

It is worth noting that essentially what has been done in this section is to derive a Greens function for the linearised time dependent flow of a rotating stratified fluid due to a line singularity. It is then possible by the method of images and through Greens theorem to generate solutions for sink flows in various flow geometries, and also to study time dependent flows over various topographies. Such investigations are presently being carried out by the author.

IMPULSIVE SINK FLOW

The response of a rotating stratified fluid is examined in detail when $q(t) = \delta(t)$. Physically this represents turning the sink on and off in a very small time. Rearranging (19) and noting that the Laplace transform of $\delta(t)$ is unity, it is necessary to evaluate the Laplace inversion contour integral given by

$$I = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \frac{\sqrt{s^2 + B^2}\sqrt{s^2 + 1}}{s^2 + \omega^2} ds, \quad (22)$$

where $\omega^2 = (B^2 x^2 + z^2)/(x^2 + z^2)$, in order to find $u(x, z, t)$.

The integral can be evaluated by closing the contour at infinity and evaluating the contributions to the integral from the enclosed singularities. However since the transform part of the integrand in (22) tends to a constant as $s \rightarrow \infty$ there is a contribution to the integrand by the contour at infinity. This is of the delta function type and is not important for large times, which is the case being considered here, and is therefore ignored. Alternatively, the problem can be avoided by letting $q(t) = [H(t) - H(t-a)]/a$, evaluating the inverse transform, and then taking the limit $a \rightarrow 0$.

The integrand has simple poles at $s = \pm i\omega$ and branch points at $s = \pm iB$ and $s = \pm i$. The case where the buoyancy frequency is greater than or equal to the inertial frequency will be considered here i.e. $B \leq 1$. If $B > 1$ then the results are identical but with the x and z axes interchanged. Note also that, from the definition of ω , it follows $B < \omega < 1$ for all x and z .

It can be shown that the parts of the contour integral involving the branch cuts at $\pm i$ and $\pm iB$ represent decaying oscillations (at the buoyancy and inertial frequencies) which behave like $t^{-3/2}$ for large t . Therefore at times large compared to f^{-1} and N^{-1} the main contribution to the integral comes from the simple poles at $s = \pm i\omega$. To find this contribution the residues at $s = \omega e^{\pm i\pi/2}$ are calculated, where the poles have been written in exponential form to avoid the ambiguity that occurs when branch points exist. The details of the residue calculation are not presented but the sum of the residues gives the following horizontal velocity

$$u = \frac{-x}{2\pi(x^2 + z^2)} \frac{\sqrt{1-\omega^2}\sqrt{\omega^2-B^2}}{\omega} \cos(\omega t). \quad (23)$$

Similarly the w and v velocities can be calculated. The results are given;

$$w = \frac{-z}{2\pi(x^2 + z^2)} \frac{\sqrt{1-\omega^2}\sqrt{\omega^2-B^2}}{\omega} \cos(\omega t), \quad (24)$$

$$v = \frac{Bx}{2\pi(x^2 + z^2)} \left(\frac{B}{\omega^2} + \frac{\sqrt{1-\omega^2}\sqrt{\omega^2-B^2}}{\omega^2} \sin(\omega t) \right). \quad (25)$$

The oscillations are a result of internal waves of frequency ω being radiated by the sink which cause fluid particles to oscillate in the radial direction. The velocity field in the (x, z) plane, thus represents persistent oscillations whose frequency and amplitude are functions of position. The behaviour of the constant phase lines of the waves can be deduced from the following argument: the phase of the waves is given by ωt and so as time increases, ω must decrease accordingly and so the lines of constant phase, which are straight lines through the origin, rotate toward the horizontal axis for $B < 1$. The rotation of fluid properties toward the horizontal axis is illustrated in figure 1 which shows the time evolution of contours of equal velocity in the first quadrant of the (x, z) plane for $B = 0.5$. The contours are seen to rotate toward the x axis. The number of these sets of contours and the number of lines of constant phase increases with the time. Similar behaviour also occurs in the impulsive displacement of a cylinder in a rotating homogenous fluid (see Bretherton, 1967). Since the group velocity is perpendicular to the phase velocity energy is radiated in a radial direction.

DISCUSSION

The discussion and significance of the above results relies on some elementary properties of internal-inertial waves.

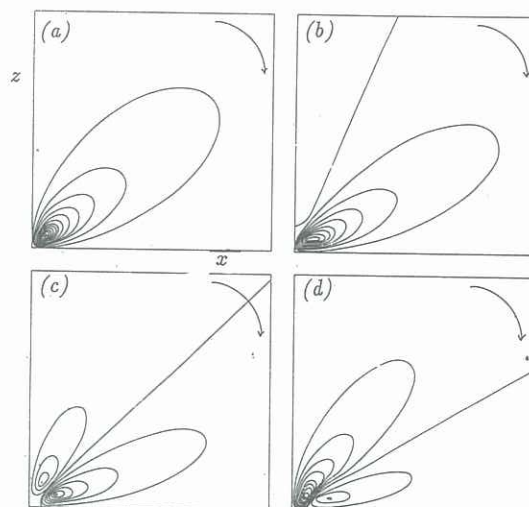


Figure 1: Time evolution of velocity field for $B=0.5$. (a) $t=4$, (b) $t=5$, (c) $t=6$ and (d) $t=7$.

For internal waves with wavenumber k and frequency ω the magnitude of the group velocity is (Gill (1982))

$$|C_g| = \frac{(1-B^2) \cos \theta \sin \theta}{\omega |k|}, \quad (26)$$

where θ is the angle made with the horizontal axis. From (26), the maximum group velocity or energy radiated, in general, occurs along the line $z = Bx$ and is carried by waves of frequency $\omega = \sqrt{2}B/\sqrt{B^2+1}$. This also corresponds, not unexpectedly, to the maximum amplitude of oscillation of the fluid particles. The parameter B is then of critical importance in determining the direction at which the maximum amount of energy is radiated. Thus as $B \rightarrow 0$ more and more of the energy radiated by the sink is concentrated along the horizontal axis and is carried by waves of decreasing frequencies. Only when the parameter B is only equal to zero (i.e. when the fluid is not rotating) is the (maximum) energy carried at zero frequencies.

For non-zero B waves of zero frequency do not exist, and the maximum amount of energy is carried by waves of finite frequency to the horizontal. This suggests that phenomena that occur in stratified non-rotating fluids such as selective withdrawal and blocking do not occur in a fluid that is both rotating and stratified.

To illustrate the effects of the direction of maximum energy propagation and the frequency of the waves that carry this energy, consider the velocity field caused by a sink whose strength is constant in time. The velocity field can be formally obtained by integrating (23) – (25) from 0 to t in addition to the contribution to the pole at $s = 0$. Physically this is equivalent to “summing up” the contributions of the arrival of each successive wave at a particular point. The results are given (in dimensional quantities)

$$u = \frac{-x}{2\pi(x^2 + z^2)} \left(\frac{B}{\omega^2} + \frac{\sqrt{1-\omega^2}\sqrt{\omega^2-B^2}}{\omega^2} \sin(\omega t) \right), \quad (27)$$

$$w = \frac{-z}{2\pi(x^2 + z^2)} \left(\frac{B}{\omega^2} + \frac{\sqrt{1-\omega^2}\sqrt{\omega^2-B^2}}{\omega^2} \sin(\omega t) \right), \quad (28)$$

$$v = \frac{Bx}{2\pi(x^2 + z^2)} \left(\frac{Bt}{\omega^2} - \frac{\sqrt{1-\omega^2}\sqrt{\omega^2-B^2}}{\omega^3} \cos(\omega t) \right) \quad (29)$$

Firstly consider the case of a non-rotating fluid, so $B = 0$ and it follows $v = 0$ and u , from (27), reduces to

$$u(x, z, t) = \frac{-x}{2\pi(x^2 + z^2)^{3/2}} \sqrt{1 - \omega^2} \frac{\sin \omega t}{\omega}. \quad (30)$$

There is a similar expression for the vertical velocity w . In the limit $t \rightarrow \infty$, $\sin \omega t / \omega \rightarrow \pi \delta(\omega)$ in equation (30) and it then follows that $u \rightarrow -\frac{1}{2} \delta(z)$ and $w \rightarrow 0$ for large times. The sink flow is selective in the sense that the withdrawn fluid comes from the level of the sink as represented by the delta function. Essentially this illustrates the result of integrating (23) with respect to time and then taking the large time limit i.e.

$$\lim_{t \rightarrow \infty} \int_0^t \cos \omega \tau d\tau = \delta(\omega),$$

that is, the effect of summing waves arriving at a particular location due to successive sink pulses gives no nett contribution to the flow field *except* those of zero frequency along the horizontal axis. Since a stratified non-rotating fluid can support such waves of zero frequency the flow due to a line sink eventually collapses to a singular jet along the x -axis and selective withdrawal occurs.

For the case $B > 0$, then $\omega > 0$ and so the delta function distribution gives zero contribution to the velocity field. That is, since energy is carried by waves of finite frequency the effect of summing all such waves at a particular location is to produce no nett response. The steady state velocity is then given by the pole at $s = 0$ and represents a balance between the baroclinic production of vorticity and the rotational production of vorticity by Coriolis forces. In a coordinate frame in which the vertical axis is stretched by a factor of B the streamlines are identical to that caused by a sink in potential flow. Note that superimposed on the steady state velocity field is a swirl velocity which increases linearly with time. This is a consequence of the conservation of angular momentum, since a fluid particle moving toward the sink with a horizontal component of velocity must increase its swirl velocity in order to conserve its angular momentum. Figure two shows the horizontal velocity

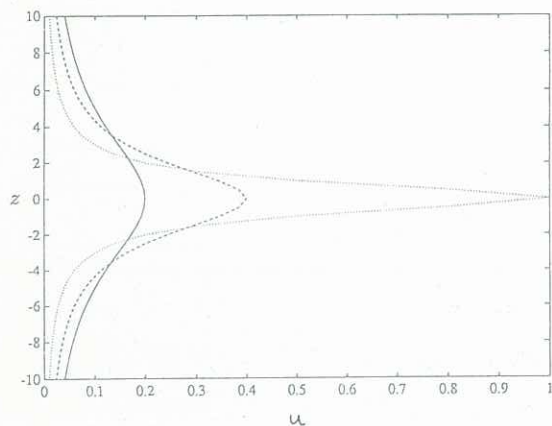


Figure 2: Normalised horizontal velocity profile at a particular value of x for $B = 1.0$ (—), $B = 0.5$ (---), $B = 0.2$ (.....).

profiles given by (27) at large times for various values of B . There is an increasing tendency for a well defined narrow jet to form as B decreases i.e. as stratification dominates over rotation. In the limit $B \rightarrow 0$ the horizontal velocity collapses to a line jet as expected.

It should be noted that the results of this study have been derived through the use of linearised inviscid theory. Real fluids are viscous and also the Froude number is finite meaning that eventually either the effects of viscosity and/or nonlinearity become important. In the case of a stratified fluid this prevents the flow collapsing to a line

jet. The importance of rotation on the withdrawal problem may then be judged on the relative sizes of the withdrawal layer thickness produced by a balance between stratification and either viscous, nonlinear or rotational effects

CONCLUSIONS

The velocity components have been derived in Laplace space for the initial value problem of suddenly initiating a two dimensional line sink flow in a rotating stratified fluid. The Laplace transform inversion was performed in detail for the case when the sink flow is impulsive. The solution exhibits non-propagating transient oscillations at the inertial and buoyancy frequencies as well as persistent oscillations caused by the radiation of internal waves whose phase lines rotate toward the horizontal axis when $N > f$. The waves responsible for selective withdrawal in a stratified fluid do not exist in the presence of rotation and for a maintained sink flow of constant strength the flow resembles potential flow in a stretched coordinate frame with a superimposed swirl velocity which increases linearly with time.

I would like to thank Jörg Imberger for his guidance of this work. Thanks also go to Simon Clarke, Graeme Hocking and John Taylor for helpful comments regarding this article and to Belindah Murray for checking the spelling.

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