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A THEORETICAL STUDY OF UNSTEADY SHOCK WAVE/BOUNDARY LAYER INTERACTION IN TWO DIMENSIONS

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ABSTRACT

A theoretical study has been made of weak, unsteady shock wave/boundary interaction in two dimensions. A simple linear model is first used to demonstrate the modelling technique. It is then extended by taking an approach based on Lighthill's steady 'triple-deck' model. The resulting two-point boundary value problem is solved numerically to obtain natural solutions which indicate that the extent of upstream influence decreases with increasing frequency of perturbation.

INTRODUCTION

A shock wave/boundary layer interaction is characterised by distortion of a boundary layer flow through pressure gradients associated with an impinging shock wave. The effects of this distortion propagate both upstream and downstream through the boundary layer. The flow outside the boundary layer is affected in turn by the expansion and compression waves emitted from the disturbed boundary layer. This phenomenon can lead to flow separation in supersonic and hypersonic intakes and at wing-fuselage junctions.

Shock wave interactions with turbulent boundary layers are inherently unsteady. Unsteady interactions can also occur in supersonic panel flutter and in transonic compressors where shock waves from rotor blades can impinge momentarily and periodically on stator blades. This paper reports theoretical studies of unsteady, two-dimensional shock wave boundary layer interactions. Small perturbations arising from weak shock waves are considered in turbulent boundary layers. The analysis is based first on a simple model and then on an extension of the Lighthill (1953) 'triple-deck' model.

LIGHTHILL'S TRIPLE-DECK MODEL.

Tsien and Finsten (1949) investigated the steady, two-dimensional problem of a weak shock wave interacting with a boundary layer. They simplified the boundary layer to a uniform subsonic flow, with the obvious weakness that the Mach number at the wall was not zero. Furthermore, their theory gives no indication of the value of the Mach number in the boundary layer. Lighthill (1950) attempted to improve this model by replacing the uniform layer with a shear layer next to the wall, with Mach number falling continuously from the free stream Mach number to zero at the wall. This model failed to predict the measured extent of upstream influence because it was everywhere inviscid. Lighthill (1953) resolved this problem with his 'triple-deck' model.

In the 'triple-deck' model the external supersonic stream is the outer deck, with the boundary layer represented by a rotational, compressible, inviscid flow as the middle deck and a viscous, incompressible flow as the inner deck. The middle deck is both supersonic and subsonic. It occupies most of the boundary layer and incorporates the physical mechanisms that primarily determine the interaction. Lighthill showed that the thin, viscous sublayer of thickness L^* determines a distance, $0.78L^*$, from the wall at which an effective wall can be located and above which viscosity can be ignored. Hereafter, we take $y^* = 0$ at the effective wall to be the bottom of the inner deck. (Note that starred quantities are dimensional).

Lighthill defines the length of upstream influence in terms of an inverse logarithmic decrement K_1^{-1} (i.e. the distance in which the disturbance due to the shock wave is reduced by the factor of e^{-1}). His second-order approximation to K_1^{-1} is given by

$$K_1^{-1} = \frac{M_1^2}{\beta}\left[\int_0^{\delta^*}\left(M^{-2}(y^*)-1\right)dy^* + \frac{\beta^2}{M_1^4}\int_0^{\delta^*}M^2(y^*)dy^*\right]. \quad (1)$$

$M(y^*)$ is the Mach number profile across the middle deck of thickness δ^*, M_1 is the free stream Mach number and $\beta^2 = M_1^2 - 1$. At $y^* = 0$, $M = M_2 \neq 0$. Lighthill showed that K_1^{-1} given by (1) is independent of the precise value chosen for δ^*. He also showed that the Mach number M_2 is given by (2).

$$M_2 = 0.78\, M'(0)\, L^* . \quad (2)$$

Lastly, Lighthill derived (3) for the thickness L^* of the viscous sublayer in terms of the kinematic viscosity ν_w^* at the wall and the velocity gradient $U^{*\prime}(0)$.

$$L^* = \left[\frac{\nu_w^*}{K_1\, U^{*\prime}(0)}\right]^{1/3} . \quad (3)$$

Clearly, an iterative procedure and assumed $U(y^*)$ and $M(y^*)$ are needed to find the inverse logarithmic decrement K_1^{-1} from the implicit set of equations (1) to (3).

A SIMPLE UNSTEADY MODEL

The essential features of an unsteady, two-dimensional shock wave/boundary layer interaction can be determined from an extension of the steady model of Tsien and Finsten (1949). With a judicious choice of the arbitrary Mach number at the wall, the simple model does provide some understanding over the full range of frequency of unsteadiness, whereas the authors' extension of the triple-deck model at present is limited to low frequencies. The simple model also provides the framework for the more complicated analysis of the triple-deck model.

The simple model is described in detail by Mahkri and Simmons (1987). It comprises a main supersonic stream at Mach number M_1 and a uniform subsonic layer next to the wall at arbitrary Mach number M_2. The subsonic layer is treated as a perfect gas with constant specific heats and zero viscosity and thermal conductivity. The unsteady, two-dimensional equations of continuity and momentum, with x^* in the main stream direction and y^* normal to the wall, are

$$\frac{\partial}{\partial x^*}\left(\rho^* u^*\right) + \frac{\partial}{\partial y^*}\left(\rho^* v^*\right) + \frac{\partial \rho^*}{\partial t^*} = 0 , \qquad (4)$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + \frac{\partial u^*}{\partial t^*} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*} , \qquad (5)$$

$$u^* \frac{\partial v^*}{\partial x^*} + u^* \frac{\partial v^*}{\partial y^*} + \frac{\partial v^*}{\partial t^*} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial y^*} . \qquad (6)$$

The fact that entropy remains constant along the streamlines is expressed by

$$u^*\frac{\partial p^*}{\partial x^*} + v^*\frac{\partial p^*}{\partial y^*} + \frac{\partial p^*}{\partial t^*} = a^{2*}\left[u^*\frac{\partial \rho^*}{\partial x^*} + v^*\frac{\partial \rho^*}{\partial y^*} + \frac{\partial \rho^*}{\partial t^*}\right] . \qquad (7)$$

The basic flow is perturbed by time-varying and spatially-varying pressure in the main stream. Within the interaction zone the velocities in the x^*- and y^*- directions are $U^* + u^*$ and v^*, the density is $R^* + \rho^*$, the pressure is $P^* + p^*$ and the local speed of sound is $A^* + a^*$. By neglecting second-order terms, the governing linear equations for the perturbation quantities are obtained. They are made non-dimensional by introducing $x = x^*/\delta^*$, $y = y^*/\delta^*$, $u = u^*/U^*$, $v = v^*/U^*$, $t = t^*U^*/\delta^*$, $\rho = \rho^*/R^*$ and $p = (p^*-P^*)/(\gamma P^*)$, where δ^* is the thickness of the layer.

The y-momentum equation becomes trivial in this simple model. The non-dimensional, linearised equations for continuity, x-momentum and constant entropy are (8), (9) and (10).

$$\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \rho}{\partial t} = 0 , \qquad (8)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = -\frac{1}{M_2^2}\frac{\partial p}{\partial x} , \qquad (9)$$

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} - \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial t} = 0 . \qquad (10)$$

Equations (8) and (10) can be combined to eliminate density. Then only (11) and (12) are needed

$$\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial p}{\partial t} = 0 , \qquad (11)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = -\frac{1}{M_2^2}\frac{\partial p}{\partial x} \qquad (12)$$

To reduce the number of unknowns in (11) and (12) to two, some approximations are needed. First, consider the term $\partial v/\partial y$. To a first approximation,

$$\frac{\partial v^*}{\partial y^*} = \frac{v^*}{\delta^*} \qquad (13)$$

Now follow the motion of a fluid particle in the boundary layer but at the interface with the main stream. Then

$$v^* = U^*\frac{\partial y_1^*}{\partial x^*} + \frac{\partial y_1^*}{\partial t^*} . \qquad (14)$$

where y_1^* is the value of y^* at the interface. It follows that, in non-dimensional form,

$$\frac{\partial v}{\partial y} = \frac{\partial y_1}{\partial x} + \frac{\partial y_1}{\partial t} . \qquad (15)$$

Next consider the spatial and temporal pressure perturbations, p, at the interface between the boundary layer and the main stream. It is assumed that the spatial and temporal perturbations combine linearly. The spatial component of perturbations due to change in the x-direction is given by the steady Prandtl-Meyer result, which is, in non-dimensional form,

$$p = \frac{M_1^2}{\sqrt{M_1^2 - 1}}\frac{\partial y_1}{\partial x} \qquad (16)$$

The temporal component of perturbation can be predicted by linearising the expression (17) for the pressure on the surface of a piston moving into a duct of constant cross-section, where the gas was initially at rest (Liepmann and Roshko, 1967).

$$\frac{p^*}{P^*} = \left[1 + \frac{\gamma-1}{2}\frac{\partial y^*/\partial t^*}{A^*}\right]^{2\gamma/(\gamma-1)} . \qquad (17)$$

The linearised non-dimensional result is

$$p = M_1\frac{\partial y_1}{\partial t} . \qquad (18)$$

Hence, combining spatial and temporal perturbations, and noting that pressure is independent of y in the boundary layer for this simple model, it follows that

$$p = \frac{M_1^2}{\sqrt{M_1^2 - 1}}\frac{\partial y_1}{\partial x} + M_1\frac{\partial y_1}{\partial t} \qquad (19)$$

Substituting (15) and (19) into (11) and (12) yields the governing equations (20) and (21) in two dependent variables, u and y_1. In practice, these perturbations are generated by applied perturbations in pressure.

$$\frac{\partial u}{\partial x} + \frac{\partial y_1}{\partial x} + \frac{\partial y_1}{\partial t} + D\frac{\partial^2 y_1}{\partial x^2} + E\frac{\partial^2 y_1}{\partial x \partial t} + F\frac{\partial^2 y_1}{\partial t^2} = 0 , \qquad (20)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + B\frac{\partial^2 y_1}{\partial x^2} + C\frac{\partial^2 y_1}{\partial x \partial t} = 0 , \qquad (21)$$

where $B = \dfrac{M_1^2}{M_2^2 \cdot \sqrt{M_1^2 - 1}}$, $C = \dfrac{M_1}{M_2^2}$,

$D = \dfrac{M_1^2}{\sqrt{M_1^2 - 1}}$, $E = M_1 + \dfrac{M_1^2}{\sqrt{M_1^2 - 1}}$,

$F = M_1$.

If spatial and temporal perturbations are assumed to be periodic, (20) and (21) define an eigenvalue problem with solution

$$u = \bar{u}\, e^{\sigma x} e^{\omega t} \qquad (22)$$

and

$$y_1 = \bar{y}_1\, e^{\sigma x} e^{\omega t} . \qquad (23)$$

Here $\bar{u}$ and $\bar{y}_1$ are complex, defining both relative magnitude and phase relations,

$$\sigma \quad = \quad \kappa + i2\pi n \tag{24}$$

and

$$\omega \quad = \quad \Omega \; + \; i\beta \tag{25}$$

κ is the logarithmic decrement of upstream influence, n the wave number or reciprocal of the wavelength, and β the dimensionless frequency. The measure of temporal decay is Ω.

Substitution of (22) and (23) into (20) and (21) yields, in matrix form,

$$\begin{bmatrix} \sigma & (\sigma + \omega + D\sigma^2 + E\sigma\omega + F\omega^2) \\ (\sigma + \omega) & (B\sigma^2 + C\sigma\omega) \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{y}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{26}$$

It follows that, for non-trivial solutions,

$$(B - D)\sigma^3 + [(C - D - E)\omega - 1]\sigma^2 - [2\omega + (E + F)\omega^2]\sigma - \omega^2 - F\omega^3 = 0 . \tag{27}$$

For a given value of the complex decay rate σ, the corresponding complex frequency ω can be determined from (27).

Results.

Solutions of (26) which are stationary in time are sought by setting $\Omega = 0$ in (25). Then for given x, u and y_1 become periodic in time, with non-dimensional circular frequency β. Equation (27) then becomes

$$(B - D)\sigma^3 + [-1 + i(C - D - E)\beta]\sigma^2 + [(E + F)\beta^2 - i2\beta]\sigma + [\beta^2 + iF\beta^3] = 0. \tag{28}$$

For a given β, (28) is a cubic equation in complex σ with complex coefficients that are functions of free stream Mach number M_1 and subsonic boundary layer Mach number M_2. It has been solved numerically over a range of β, M_1 and M_2. For each β there is only one complex solution for σ with a positive real part. The other two are discarded because they are physically unrealistic. Of particular interest is the real part κ of σ because its inverse is the distance upstream of a station x at which the perturbations have fallen to 1/e of their magnitudes at x. Thus κ^{-1} is a measure analogous to the inverse logarithmic decrement K_1^{-1} in Lighthill's steady triple-deck model. However, κ^{-1} is dependent on frequency β.

Typical variations of upstream influence κ^{-1} with frequency β are shown in Figure 1. The arbitrary values of M_2 have been chosen to give values of upstream influence at $\beta = 0$ that are in reasonable agreement with those predicted for steady turbulent boundary layers by the rigorous triple-deck model. By doing this, some physical significance can be assigned to the results. In particular κ^{-1} falls from a limit at low β to a smaller limit at high β. It must be noted that the value of this simple model is that it is not restricted by a low frequency assumption. The following unsteady extension of the triple-deck model is restricted to low frequencies.

AN UNSTEADY TRIPLE-DECK MODEL.

In this section an unsteady two-dimensional model is based on an extension of Lighthill's steady triple-deck model. Governing linearised equations are found for the middle deck. Boundary conditions for this deck are set at the top (y=1) by the main stream and at the bottom (y=0) by the Lighthill steady theory. Unlike the simple model, flow properties now vary with distance y through the layer. By following a procedure similar to that used for the simple model, the linearised non-dimensional equations (29), (30) and (31) result for combined continuity and entropy and for x- and y-momentum for the rotational, compressible and inviscid middle deck.

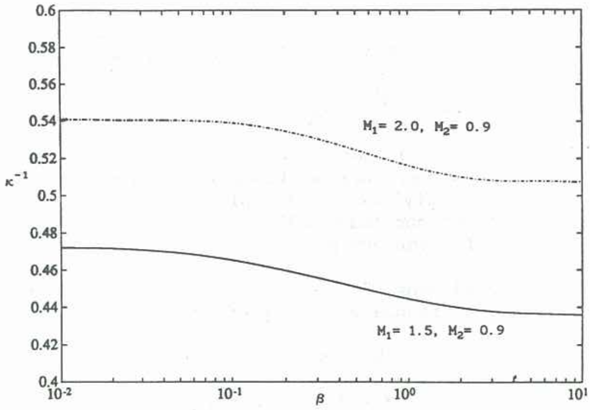

Figure 1 Simple model predictions of dependence of upstream influence κ^{-1} on frequency β.

$$U(y)\,\frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{29}$$

$$\frac{\partial u}{\partial x} + \frac{v}{U(y)}\frac{\partial U(y)}{\partial y} + \frac{1}{U(y)}\frac{\partial u}{\partial t} = -\frac{1}{M^2(y)}\,U(y)\,\frac{\partial p}{\partial x} , \tag{30}$$

$$U(y)\,\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = -\frac{1}{M^2(y)}\,U^2(y)\,\frac{\partial p}{\partial y} . \tag{31}$$

Velocities are normalised by the mean x-velocity, U_1, at the top of the middle deck and U(y) is the non-dimensional mean velocity profile.

It is convenient to replace v with the local flow direction η such that

$$v = \eta\, U(y) . \tag{32}$$

Equations (29), (30) and (31) then yield (33), (34) and (35) after some manipulation.

$$\frac{\partial \eta}{\partial y} = \left[M^{-2}(y) - 1\right]\frac{\partial p}{\partial x} - \frac{1}{U(y)}\frac{\partial p}{\partial t} + \frac{1}{U^2(y)}\frac{\partial u}{\partial t} , \tag{33}$$

$$\frac{\partial u}{\partial x} + \frac{1}{U(y)}\frac{\partial u}{\partial t} + \eta\,\frac{\partial U(y)}{\partial y} = -\,U(y)\,M^{-2}(y)\,\frac{\partial p}{\partial x} , \tag{34}$$

$$\frac{\partial \eta}{\partial x} + \frac{1}{U(y)}\frac{\partial \eta}{\partial t} = -\,M^{-2}(y)\,\frac{\partial p}{\partial y} . \tag{35}$$

If spatial and temporal perturbations are assumed to be periodic, (33) to (35) define an eigenfunction problem with solution

$$p = \bar{p}(y)\, e^{\sigma x}\, e^{\omega t} , \tag{36}$$

$$u = \bar{u}(y)\, e^{\sigma x}\, e^{\omega t} , \tag{37}$$

$$\eta = \bar{\eta}(y)\, e^{\sigma x}\, e^{\omega t} . \tag{38}$$

The complex velocity $\bar{u}$ can be eliminated, thereby reducing the equations to a set of two, written in matrix form as (39).

$$\begin{bmatrix} \frac{\partial \eta}{\partial y} \\ \frac{\partial p}{\partial y} \end{bmatrix} = \begin{bmatrix} G & H \\ J & 0 \end{bmatrix} \begin{bmatrix} \bar{\eta} \\ \bar{p} \end{bmatrix} \tag{39}$$

where $G = \dfrac{-\,\omega}{U(y)[U(y)\sigma + \omega]}\,\dfrac{\partial U(y)}{\partial y}$

$$H = (M^2(y) - 1)\sigma - \frac{\omega}{U(y)} - \frac{\sigma\omega M^2(y)}{U(y)\sigma+\omega} ,$$

$$J = -\frac{M^2(y)[U(y)\sigma + \omega]}{U(y)} .$$

Again, σ and ω are complex but we seek solutions that are stationary in time ($\omega = i\beta$). We choose U(y) and M(y) empirically, set boundary conditions and solve (39) for $\sigma = \kappa + i2\pi n$ over a range of frequency β.

Following Stalker (1984) we assume simple turbulent boundary layer profiles,

$$U = (y + 0.78L)^{1/7} \qquad (40)$$

and

$$M = M_1(y + 0.78L)^{0.25} , \qquad (41)$$

where $L = L^*/\delta^*$ is given by (3) and is small so that $y \cong 1$ at the top of the middle deck.

In non-dimensional form, (3) becomes

$$L = 0.78 \left[\frac{\nu_w}{K_1\, U'(0)}\right]^{1/3} , \qquad (42)$$

where K_1 is obtained from Lighthill's steady theory and non-dimensional ν_w becomes the Reynolds number based on δ^*, free stream velocity U_1^* and kinematic viscosity ν_w^*. (Mahkri and Simmons, 1987).

Following Lighthill, we assume the boundary condition at the effective wall at the bottom of the middle deck to be

$$\bar{\eta} = 0 + i0 \quad \text{at} \quad y = 0 . \qquad (43)$$

At the top of the middle deck (y = 1) we assume a small perturbation

$$\bar{p} = \hat{p} + i0 \qquad (44)$$

and the steady Prandtl-Meyer relation,

$$\bar{\eta} = \bar{p}\,\frac{\sqrt{M_1^2 - 1}}{M_1^2} . \qquad (45)$$

This amounts to imposing a low frequency restriction on the model. The main stream at y = 1 must traverse a disturbance wavelength $1/n$ in a time that is short compared with the disturbance period, $2\pi/\beta$. These two times are equal when $\beta \cong 2\pi$.

With (39) being four real equations and (43), (44) and (45) being six real boundary conditions, the problem is overdetermined and, for a given β, there are only solutions for special values of σ (eigenvalues). It is a two-point boundary value problem that has been solved numerically by a shooting method, shooting from y = 1 to y = 0.

Results.

Typical variations of upstream influence κ^{-1} with frequency β are shown in Figure 2 for two Mach numbers and $\hat{p} = 0.05$. Values of κ^{-1} at $\beta = 0$ (less than one boundary layer thickness) are consistent with those obtained by Lighthill (1953) and many experiments. The decrease of upstream influence with increasing β is apparent, although the model is restricted to low β (less than about 2 or 3). The chosen Reynolds number ν_w based on δ^* is 1×10^4, ensuring that the boundary layer is turbulent. In all cases, M_2 was low, in the range 0.17 to 0.27, and the thickness L of the viscous sublayer was typically one percent of the boundary layer thickness.

The fact that the upstream influence cannot

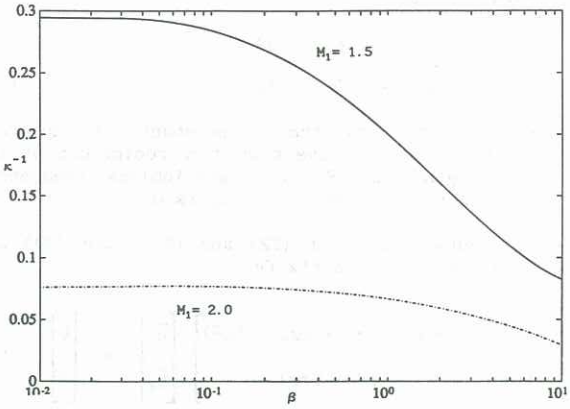

Figure 2 Unsteady predictions of dependence of upstream influence κ^{-1} on frequency β.

be predicted for Mach numbers greater than about 2 is not surprising. The upstream influence through the subsonic part of the boundary layer is then annulled by the downstream transmission of perturbations in the supersonic part of the free stream along characteristics in the boundary layer, a phenomenon noted by Lighthill. It is possible, however, that more refined assumptions for velocity and Mach number profiles will increase the free stream Mach number to which upstream influence can be predicted.

CONCLUSIONS

Both the simple model and the triple-deck model predict a degree of upstream influence which is consistent with steady experiments (Lighthill, 1953) and which decreases with increasing frequency. It must be stressed that only the natural response of the boundary layer, expressed in terms of eigenvalues and eigenfunctions, has been found. However, a similar extent of upstream influence can be expected in situations involving small, forced perturbations arising, for example, from an oscillating, weak shock wave.

The unsteady triple-deck model contains empiricism in the formed of assumed mean velocity and Mach number profiles. This empiricism provides flexibility; more refined turbulent and laminar boundary layer profiles can easily be included.

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