VISCOUS FLOW IN PROPAGATING CRACKS: THE TRANSPORT OF MOLTEN ROCK THROUGH THE EARTH'S LITHOSPHERE

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ABSTRACT

Buoyancy-driven flows resulting from the introduction of fluid of one density into a crack embedded in an elastic solid of different density are analysed. Scaling arguments are used to investigate the various regimes of crack propagation. Nonlinear equations governing the shape and rate of spread of the propagating crack are solved for the cases of vertical propagation of buoyant fluid released into a solid of greater density and of lateral propagation of fluid released at an interface between an upper layer of lesser density and a lower layer of greater density. The theoretical solutions provide simple models for the vertical transport of molten rock, or magma, through the Earth's lithosphere and in the lateral intrusion of magma at a neutral-buoyancy level close to the Earth's surface.

1. INTRODUCTION

The transport of magma by fissures, or dykes, opened by fluid-induced fracture of the Earth's lithosphere is an important and intriguing phenomenon. Magma-fracture is responsible for the transport through the lithosphere of nearly all the melt produced in the underlying mantle. However, the impossibility of making direct observations of the formation of dykes has limited our understanding of the controlling parameters and physical balances. It is our purpose to analyse the governing balance of stresses for a propagating, fluid-filled fracture and to present solutions of problems of crack propagation which are directly relevant to the transport of magma.

Magmas produced in the upper regions of the Earth's mantle are less dense than the surrounding rock and rise to collect at the base of the overlying cold and brittle lithosphere. Subsequent transport of the magma towards the surface takes place in dykes which crack through the lithosphere at speeds of order a few metres per second, driven by the buoyancy of the magma. During this stage of magma-transport, we are concerned with the vertical propagation of a buoyant fluid-filled crack which is fed from a source at its base. Most melts are less dense than the uppermost few kilometres of the Earth's lithosphere. In such cases, the propagation of vertical dykes ceases near the neutral-buoyancy level of the melt (Ryan 1987; Walker 1989). The dyke system may subsequently cut the Earth's surface and cause fissure eruptions but, more commonly, it is observed that the dykes propagate laterally rather than vertically (Rubin & Pollard 1987) and that the majority of the magma fails to reach the surface. During this stage of magma-transport, we are interested in the lateral propagation of a crack in a stratified solid, where the crack is fed with fluid at its neutral-buoyancy level.

Many previous studies of dykes have examined the exposed remains of solidified intrusions and related them to theoretical solutions for the shape of a stationary fluid-filled crack. These solutions are sometimes extended to give a quasi-static description of a propagating crack in which the criterion for propagation is defined by the static stress field and dynamical effects, such as the viscous pressure drop in the fluid, are ignored. A solution which does incorporate the dynamical interaction between the fluid-mechanical and elastic forces was derived for two-dimensional cracks in which buoyancy forces are negligible (Spence & Sharp 1985). Here we present dynamical solutions which incorporate the geophysically important effects of buoyancy. Specifically, we consider solutions appropriate to the geophysical regime for the cases of vertical and lateral propagation described above (see figure 1). Throughout, we consider only laminar flow; it is, however, possible to solve for turbulent flows (Lister 1989b).

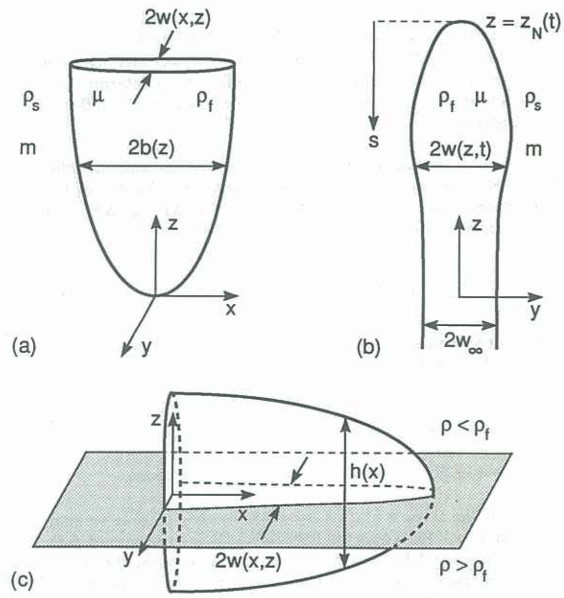

FIG. 1 (a) A buoyant fluid of density ρ_f and viscosity μ rises from a point source through a crack in a solid of density ρ_s and elastic modulus $m = G/(1-\nu)$; (b) A two-dimensional crack rises in the x,z-plane from a linear source; (c) A fluid-filled crack propagating at its neutral-buoyancy level, $z = 0$.

2. PRELIMINARY ANALYSIS OF FLUID-FRACTURE

Before deriving specific analytic solutions, we first discuss the magnitudes of the various stresses which play a role in magma-fracture. Secondly, we derive the equations that give the thickness and flow rate in a fluid-filled crack in terms of the pressure distribution in the fluid.

2.1 Analysis of pressure scales and flow regimes

Consider a fluid-filled crack embedded in an infinite elastic solid. Suppose that the solid has shear modulus G, Poisson's ratio ν, density ρ_s and stress-intensity factor (defined below) K and that the fluid is incompressible and has dynamic viscosity μ, density ρ_f and prescribed volume $V(t)$. Let $\Delta\rho = \rho_s - \rho_f$ and $m = G/(1-\nu)$. For simplicity, suppose that the crack lies in a vertical plane and define h to be the vertical extent, b the horizontal extent and w the width of the crack. It can easily be shown that $w \ll b, h$. Let u be a typical velocity scale for the fluid flow and let l denote the extent of the crack (either b or h depending on the context). Where appropriate, we use the parameter values $m = 2 \times 10^{10}$ Pa, $K = 10^6$ Pa m$^{1/2}$, $\mu = 10^2$ Pa s and $\Delta\rho = 300$ kg m^{-3} to ascertain whether a particular expression is relevant in a geophysical setting. Injection rates dV/dt vary greatly from $O(1\,\mathrm{m^3s^{-1}})$ to $O(10^6\,\mathrm{m^3s^{-1}})$.

The relative magnitudes of four pressure scales control the regime of crack propagation. These are (i) the pressure required to open the crack against elastic forces

$$\Delta P_e \sim \frac{mw}{l}, \tag{1}$$

(ii) the hydrostatic pressure due to the density difference

$$\Delta P_h \sim g\Delta\rho h, \quad (2)$$

(iii) the viscous pressure drop caused by flow in the crack

$$\Delta P_v \sim \frac{\mu u l}{w^2} \quad (3)$$

and (iv) a crack-extension pressure defined by

$$\Delta P_c \sim \frac{K}{l^{1/2}}. \quad (4)$$

This last pressure is that required for the stress intensity at the crack-tip to have a material-dependent critical value K. If the stress intensity were smaller than this value then the crack would not propagate. Conversely, if it were maintained at a larger value then the crack would propagate at about 40% of the speed of sound in the solid, which is inconsistent with fracture driven by viscous flow into the crack-tip. Conservation of fluid volume leads to the relations $u \sim l/t$, $hbw \sim V$ (3-D flows) and $lw \sim V$ (2-D flows), where t is the time since the initiation of the crack. These relations, together with the estimates (1)–(4) of the pressure scales, are sufficient to determine the dimensions and rate of spread of a fluid-filled crack in the different parameter regimes.

Suppose, first, that ΔP_h is negligible in comparison with the other pressure scales. The width of the crack is then given by one of two possible balances: $\Delta P_e \sim \Delta P_c$ or $\Delta P_e \sim \Delta P_v$. If $\Delta P_c \ll \Delta P_v$ then the resultant balance between (1) and (3) leads to

$$l \sim \left(\frac{V^3 m t}{\mu}\right)^{1/(3n+6)}, \quad (5)$$

where $n = 0$ for a two-dimensional crack and $n = 1$ for an axisymmetric crack. The neglect of ΔP_c is valid provided that

$$\frac{V}{t l^n} \gg \frac{K^4}{\mu m^3} = 10^{-9}\,\mathrm{m^2 s^{-1}}. \quad (6)$$

This condition is easily satisfied in geophysical applications.

Secondly, if the crack is stationary or slowly moving then we may neglect ΔP_v and look for a balance between ΔP_e and the sum of ΔP_h and an excess pressure ΔP_0 in the fluid. These pressures cannot exceed ΔP_c since rapid crack propagation is inconsistent with the neglect of ΔP_v. From (1), (2) and (4) it follows that such a crack has a maximum height and width given by

$$h \sim \left(\frac{K}{g\Delta\rho}\right)^{2/3} = 50\,\mathrm{m}, \qquad w \sim \left(\frac{K^4}{g\Delta\rho m^3}\right)^{1/3} = 0.3\,\mathrm{mm}. \quad (7a, b)$$

It is clear that such narrow cracks would be incapable of transporting significant volumes of magma through the lithosphere.

From these arguments we see that ΔP_c is negligible and ΔP_v provides the dominant resistance for the propagation of both vertical and horizontal cracks in the lithosphere. It remains to determine the dominant driving pressure. From (1) and (2) we find that $\Delta P_h \sim \Delta P_e$ when

$$\frac{h^2}{w} \sim \frac{m}{g\Delta\rho} = 7 \times 10^6\,\mathrm{m}. \quad (8)$$

The vertical extent of feeder dykes through the lithosphere is such that h^2/w is likely to be greater than this value (*e.g.* $h > 3$ km, $w < 1$ m), in which case we may neglect ΔP_e and the dominant pressure balance is between ΔP_h and ΔP_v. This balance is equivalent to that governing the flow of a fluid down an inclined plane and, consequently, the thickness of the crack is governed by the kinematic-wave equation (Huppert 1982). If the rate of injection Q varies with time then the thickness near the source varies according to $w \sim (Q\mu/g\Delta\rho)^{1/3}$ and these variations in thickness propagate away from the source at a velocity $g\Delta\rho w^2/\mu$; elastic effects are only significant near the crack-tip where they play a role in the resolution of the leading kinematic shock-wave. The solution for a two-dimensional crack fed by a constant flux from a linear source is derived in §3.2.

If a crack rises from a localized source, it will tend to spread laterally due to the variations in w and ΔP_e in any horizontal cross-section (*cf.* the downstream spreading of a gravity current on an inclined plane due to cross-stream variations in its thickness). Two pressure balances are possible for this spread: $\Delta P_e \sim \Delta P_c$ or $\Delta P_e \sim \Delta P_v$. Solutions for each of these balances are derived in §3.1 and it is shown that, for geophysical parameters, the dominant horizontal balance of pressures is between ΔP_e and ΔP_v.

An important consequence of the dominant vertical balance between ΔP_h and ΔP_v is that lithospheric cracks have little tendency to propagate through a level at which the density of the solid decreases below that of the melt. If such a crack reaches the neutral-buoyancy level of the melt then the crack will subsequently propagate along this level. Solutions for this lateral flow are given in §4.

In conclusion, the resistance to fracture K is unimportant in geophysical applications; thin fractures could propagate much faster than magma would be able to intrude behind them. If the crack has a sufficiently large vertical extent, as defined by (8), then the dominant balance for the vertical motion is between ΔP_h and ΔP_v. If the crack rises to a density interface at which the density difference between the solid and fluid is reversed then further propagation will be lateral and along the interface.

2.2 Theoretical results for thin cracks

We consider a crack of width $2w(x,z)$ lying in the plane $y = 0$ and derive the equations that govern the elastic and fluid-mechanical responses to the fluid pressure p in the crack.

Let the crack be sufficiently narrow and the fluid sufficiently viscous that $\rho_f w^3|\nabla w.\nabla p|/\mu^2 \ll 1$. It follows that the flow satisfies the conditions of lubrication theory and that the variations in the width of the crack are given by the averaged equation of continuity for Poiseuille flow

$$\frac{\partial w}{\partial t} = \frac{1}{3\mu}\nabla.(w^3\nabla p). \quad (9)$$

Now suppose that the crack is two-dimensional and has width $2w(s)$, where s may be either x or z. The assumption of two-dimensionality is appropriate for a crack rising from a long, linear source (see §3.2) or as a local approximation to the shape of a crack with $h \gg b$ (see §3.1) or $b \gg h$ (see §4). The elastic pressure in the plane $y = 0$ is given by

$$p = -m\mathcal{H}(\mathrm{d}w/\mathrm{d}s), \quad (10)$$

where $\mathcal{H}$ denotes a Hilbert transform. Equation (10) may be inverted to give

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \frac{1}{m\pi}\int_0^\infty p(\sigma)\sqrt{\frac{\sigma}{s}}\frac{\mathrm{d}\sigma}{\sigma - s} + \frac{c_1}{\sqrt{s}} \qquad (w \neq 0 \text{ for } s > 0)$$

$$= \frac{1}{m\pi}\int_{-s_*}^{s_*} p(\sigma)\sqrt{\frac{s_*^2-\sigma^2}{s_*^2-s^2}}\frac{\mathrm{d}\sigma}{\sigma - s} + \frac{c_2}{\sqrt{s_*^2 - s^2}} \qquad (w \neq 0 \text{ for } |s| < s_*) \quad (11a, b)$$

depending on whether the crack is semi-infinite or finite in extent. The constants c_1 and c_2 are found from the boundary conditions that w is finite as $s \to \infty$ and $w = 0$ at the edges of the crack.

3. VERTICAL PROPAGATION OF A BUOYANT CRACK

3.1 A Point Source

Consider the release of an incompressible fluid of density ρ_f into a crack in an infinite elastic solid of greater density ρ_s. Suppose that the rate of release is a constant Q. The fluid will rise, driven by its buoyancy, thus causing a planar crack to propagate upwards. We define the origin to be the point of release and take the z-direction to be vertically upwards (figure 1a). Let the crack occupy $|y| < w(x,z,t)$ and let the edges of the crack be at $x = \pm b(z,t)$. We assume that the crack has propagated a sufficient distance that the height h of the crack satisfies $h \gg b$. As we noted earlier, $b \gg w$.

The fluid pressure is given by the sum of the buoyancy force and the elastic pressure exerted by the solid. Since $h \gg b$, the crack may be treated as being locally two-dimensional and the elastic pressure is given by (10), where the Hilbert transform is taken with respect to x. Thus the total pressure is given by

$$p_T = -g\Delta\rho z - m\mathcal{H}\frac{\partial w}{\partial x}. \quad (12)$$

After the initial crack-propagation front has passed, the crack and the flow will approach a steady state in which $\partial w/\partial t = 0$ and the flux through any cross-section is given by Q. Accordingly, we substitute into (9) to obtain

$$g\Delta\rho\frac{\partial w^3}{\partial z} + m\frac{\partial}{\partial x}\left(w^3\frac{\partial^2}{\partial x^2}\mathcal{H}w\right) = 0 \quad (13a)$$

$$\frac{2g\Delta\rho}{3\mu}\int_{-b(z)}^{b(z)} w^3\,\mathrm{d}x = Q. \quad (13b)$$

We define similarity variables ξ and W, where

$$x = b_N \left(\frac{3Q\mu m^3 z^3}{2(g\Delta\rho)^4}\right)^{1/10} \xi \tag{14a}$$

$$w(x,z) = b_N^3 \left(\frac{27Q^3\mu^3}{8m(g\Delta\rho)^2 z}\right)^{1/10} W(\xi) \tag{14b}$$

and b_N is chosen so that $W(\pm 1) = 0$. Equations (13) become

$$(W^3(\mathcal{H}W)'')' = \tfrac{3}{10}(\xi W^3)', \qquad b_N = \left(\int_{-1}^{1} W^3 \, d\xi\right)^{-1/10}. \tag{15a, b}$$

We integrate (15a) twice and use $W(\pm 1) = 0$ to obtain

$$\mathcal{H}W' = \tfrac{3}{20}\xi^2 + c_4, \tag{16}$$

where c_4 is a constant of integration. The Hilbert transform in (16) may be inverted from tables of standard transforms or by substitution into (11b). The resultant solution has a dimensionless stress intensity of $-c_4 - \frac{3}{40}$ at $\xi = \pm 1$. As shown in §2, K is negligible in geophysical problems. Therefore, $c_4 = -\frac{3}{40}$ and

$$W = \tfrac{1}{20}(1-\xi^2)^{3/2}, \qquad b_N = \left(\frac{2048000}{63\pi}\right)^{1/10}. \tag{17a, b}$$

The smooth closure of the crack at $\xi = \pm 1$ follows from the neglect of the resistance of the medium to fracture. At very large values of z, however, the cross-stream elastic pressures decrease sufficiently that the resistance to fracture can no longer be neglected. For the regime in which this resistance is dominant, we note that a vertical crack with an elliptic cross-section satisfies the fluid and elastic equations exactly: such a crack with cross-section $x^2/b^2 + y^2/w^2 = 1$ is held open by a constant internal pressure $p_0 = mw/l_0$, where $l_0 = b + \frac{1-2\nu}{2(1-\nu)}w$, and contains a velocity distribution

$$u = \frac{\Delta\rho g b^2 w^2}{2\mu(b^2+w^2)}\left(1 - \frac{x^2}{b^2} - \frac{y^2}{w^2}\right).$$

We equate the volume flux to Q and the stress intensity $mw/l_0^{1/2}$ to K, make the approximation $w \ll b$ and deduce that

$$b = \left(\frac{4m^3\mu Q}{\pi K^3 \Delta\rho g}\right)^{2/5}, \qquad w = \left(\frac{4K^2\mu Q}{\pi m^2 \Delta\rho g}\right)^{1/5}. \tag{18a, b}$$

If we assume that the transition from negligible to dominant fracture resistance occurs where the widths given by (14a) and (18a) are equal then we conclude that equations (14) and (17) hold when

$$z < \frac{1}{5\pi}\left(\frac{21}{32}\right)^{1/3} \frac{m^3 Q\mu}{K^4}. \tag{19}$$

With geophysical parameters the transition height is $4 \times 10^7 Q$ m, which, for reasonable values of Q, is much greater than the thickness of the lithosphere; hence, the lateral spread of dykes is given by (14a).

3.2 A Linear Source

Now suppose that the source is long in comparison to the scale of the crack and that the rate of release per unit length is a constant q. Consider, therefore, a two-dimensional crack propagating into a uniform elastic solid (figure 1b). Let $z_N(t)$ be the location of the crack-tip and $w(z,t)$ the half-width of the crack. The flow in the crack is driven by a total effective pressure

$$p_T = -g\Delta\rho z - m\mathcal{H}\frac{\partial w}{\partial z} \tag{20}$$

and the variation of w is given by (9). Since the flux into the crack is constant, we seek travelling-wave solutions which propagate at some fixed speed c. Far from the crack-tip w tends to a constant value, w_∞, and the flux tends to q. We define $s = ct - z$, substitute into (9) and integrate once to obtain

$$w_\infty = \left(\frac{3\mu q}{2g\Delta\rho}\right)^{1/3}, \qquad c = \frac{q}{2w_\infty}, \qquad p_T' = \frac{3\mu c}{w^2}. \tag{21a-c}$$

We non-dimensionalize w, s, p and K with respect to the scales $\hat{w} = w_\infty$, $\hat{s} = (mw_\infty/g\Delta\rho)^{1/2}$, $\hat{p} = m\hat{w}/\hat{s}$ and $\hat{K} = \hat{p}\hat{s}^{1/2}$. In dimensionless variables we find that

$$p' = w^{-2} - 1, \tag{22}$$

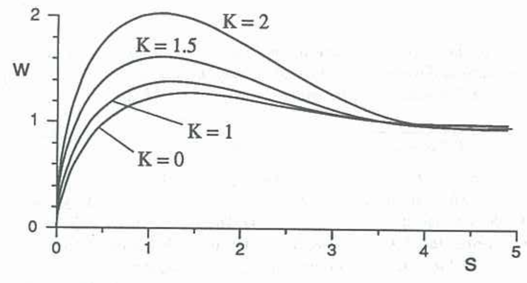

FIG. 2 The half-width w of a two-dimensional crack.

where the elastic pressure p is given by $p' = p_T' - 1$. We integrate (11a) by parts to obtain

$$w(s) = \frac{1}{\pi}\int_0^\infty p'(\sigma)\left((s-\sigma)\ln\left|\frac{\sqrt{s}+\sqrt{\sigma}}{\sqrt{s}-\sqrt{\sigma}}\right| - 2\sqrt{s\sigma}\right) d\sigma. \tag{23}$$

From (10) and (23) it may be shown that the requirement that the stress intensity at $s = 0$ be equal to the critical value K leads to

$$\frac{\sqrt{8}}{\pi}\int_0^\infty \sigma^{1/2} p'(\sigma)\, d\sigma + K = 0. \tag{24}$$

Numerical solutions of (22)–(24) are found for all values of K and some of the calculated crack widths are shown in figure 2. The solution relevant to geophysical applications is for $K = 0$.

At the crack-tip $w \to 0$ and from (22) $p \to -\infty$. This behaviour will be common to all solutions representing the extension of a fluid-filled crack and is due to the large pressure gradients required to drive a viscous liquid into a narrow gap. Vapour will be exsolved from the fluid under such low pressures and, consequently, small volumes of relatively inviscid volatiles will be present in the tips of extending cracks. Solutions with exsolved volatiles are derived by Lister (1989a).

4. LATERAL PROPAGATION OF A NEUTRALLY BUOYANT CRACK

In §3 we considered the vertical propagation of a fluid-filled crack through a solid of greater density than the fluid. We now analyse the lateral propagation of the crack for the case in which the solid is horizontally stratified in density and the crack has risen to the neutral-buoyancy level of the fluid. The distribution of stresses in the walls of a fluid-filled crack is such that we expect the lateral propagation of the crack along this level to continue in the vertical plane defined by the rising feeder crack; this is in agreement with geological observations (Rubin & Pollard 1987).

After a sufficient length of time the horizontal extent of the laterally propagating crack will be much greater than the width of the feeder crack. Thus we may consider a point source of fluid at the origin of coordinates in which the plane of neutral buoyancy is at $z = 0$ (figure 1c). Let the lateral crack occupy $|y| < w(x,z,t)$ for $h_l(x,t) < z < h_u(x,t)$ and $-x_N(t) < x < x_N(t)$ (with symmetry about $x = 0$). We have $x_N \gg h_l, h_u$ and, as usual, $h_l, h_u \gg w$. Therefore, the pressure in the fluid is given by the sum of the hydrostatic value and a constant excess pressure; the width of the crack is given by (11b), where the Hilbert transform is taken with respect to z. We suppose that the crack is fed at such a rate that the total volume of fluid is given by $2Qt^\alpha$. Let the solid have density ρ_u in $z > 0$ and density ρ_l in $z < 0$ and, for simplicity, equal shear modulus and Poisson's ratio in the two regions. Stratifications in which the density varies away from the neutral-buoyancy level according to a power law (*e.g.* a linear gradient) may be also be analysed (Lister 1989b).

Motivated by the conclusions of §2, we suppose that the resistance to fracture of the solid is very much less than the available hydrostatic stresses. Therefore, h_l, h_u and the excess pressure in the fluid will be related in such a way that the stress intensities at the upper and lower edges of the crack are both zero. If the excess pressure is too large then the stress intensities will be positive, the vertical extent of the crack will increase and the excess pressure will decrease. If the average level of the crack is too low relative to the neutral-buoyancy level then the stress intensity will be greater at the upper edge of the crack than the lower and the crack will rise.

Let $\theta = (\rho_l - \rho_f)/(\rho_l - \rho_u)$ and $\overline{\theta} = (\rho_f - \rho_u)/(\rho_l - \rho_u) = 1 - \theta$, where ρ_f is the density of the fluid. Thus the difference between the hydrostatic pressure in the fluid and in the solid is given by

$$p = p_0(x) - \overline{\theta}(\rho_l - \rho_u)gz \qquad (0 < z < h_u) \tag{25a}$$

$$p = p_0(x) + \theta(\rho_l - \rho_u)gz \qquad (h_l < z < 0) \tag{25b}$$

where p_0 is the excess pressure in the fluid. The crack is held open by this pressure difference and its width may be calculated from (11b). We define a dimensionless pressure P_0 and width W by

$$P_0 = \frac{p_0}{\theta\overline{\theta}(\rho_l - \rho_u)gh}, \qquad W = \frac{mw}{\theta\overline{\theta}(\rho_l - \rho_u)gh^2}. \qquad (26a, b)$$

Let $h(x,t) = h_u - h_l$ be the total height of the crack, $\zeta = z/h$, $\zeta_l = h_l/h$ and $\zeta_u = h_u/h$. The unknowns P_0, ζ_l, ζ_u and W are found by solving the problem consisting of equation (10), the equation $\zeta_u - \zeta_l = 1$ and the requirements of zero stress intensity at both edges of the crack. Since this problem is independent of x, the solutions depend only on the parameter θ, that is P_0, ζ_l and ζ_u are constants and W is a function of ζ. The calculation of these constants and of $W(\zeta;\theta)$ is described by Lister (1989b); solutions for W are shown in figure 3. We take W to be known and calculate the variation of h with x.

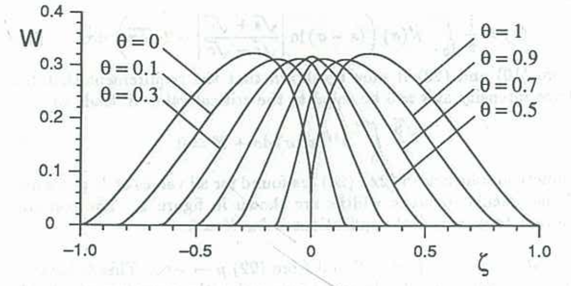

FIG. 3 The half-width W of a fluid-filled crack at the density step $\zeta = 0$ in a stratified solid, where $\theta = (\rho_l - \rho_f)/(\rho_l - \rho_u)$.

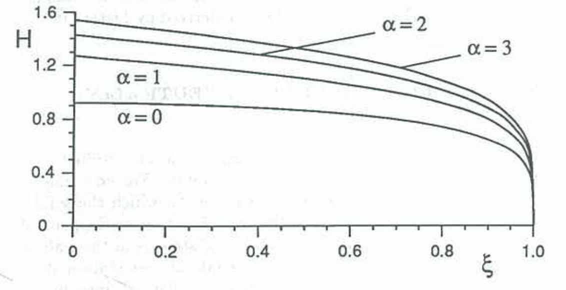

FIG. 4 The height H of a crack of volume Qt^α propagating at a density step as a function of lateral position ξ.

The lateral variations in the pressure given by (25) drive a flow in the crack. We integrate (9) with respect to z and substitute from (25) and (26) to obtain

$$I_1 \frac{\partial h^3}{\partial t} = \frac{I_3 P_0 \left(\theta\overline{\theta}(\rho_l - \rho_u)g\right)^3}{3\mu m^2} \frac{\partial}{\partial x}\left(h^7 \frac{\partial h}{\partial x}\right), \qquad (27a)$$

where $I_j(\theta) = 2\int_{\zeta_l}^{\zeta_u} W^j(\zeta)\,\mathrm{d}\zeta$. The problem is completed by the equation of global conservation of volume

$$\frac{I_1 \theta\overline{\theta}(\rho_l - \rho_u)g}{m} \int_0^{x_N} h^3 \,\mathrm{d}x = Qt^\alpha. \qquad (27b)$$

We observe that equations (27) have a similarity solution and that any solution with sufficiently smooth initial conditions will tend to this similarity form. We define similarity variables ξ and H by

$$x = \xi_N \left(\frac{I_3^3 P_0^3 \left(\theta\overline{\theta}(\rho_l - \rho_u)g\right)^4 Q^5 t^{5\alpha+3}}{27 I_1^8 \mu^3 m}\right)^{1/11} \xi \qquad (28a)$$

$$h(x,t) = \xi_N^{2/5} \left(\frac{3\mu m^4 Q^2 t^{2\alpha-1}}{I_1 I_3 P_0 \left(\theta\overline{\theta}(\rho_l - \rho_u)g\right)^5}\right)^{1/11} H(\xi), \qquad (28b)$$

where x_N is chosen so that $H(1) = 0$. Equations (27) become

$$\alpha H^3 - \frac{5\alpha+3}{11}(\xi H^3)' = (H^7 H')', \qquad \xi_N = \left(\int_0^1 H^3 \,\mathrm{d}\xi\right)^{-5/11}. \qquad (29a, b)$$

Numerical solutions of (29a) are shown in figure 4. In the interesting case of a fixed-volume release ($\alpha = 0$) we can integrate (29) analytically to obtain the exact solution

$$H = \left(\tfrac{15}{22}(1 - \xi^2)\right)^{1/5}, \qquad \xi_N = \left(\frac{11}{30}\right)^{3/11} \left(\frac{\Gamma(16/5)}{\Gamma^2(8/5)}\right)^{5/11}. \qquad (30a, b)$$

6. DISCUSSION

We have provided a physical understanding of the forces that govern the rise of magma through the Earth's lithosphere and its subsequent emplacement near the surface at the neutral-buoyancy level of the magma. Analytic solutions have been derived to model each of these vertical and lateral stages of magma-transport. These solutions have been evaluated for typical geophysical parameters by Lister (1989a, b). The predicted values of w, b, h and x_N are in broad agreement with geological observations. The sizes of the leading elastic shock and of the volatile-filled tip are found to be small in comparison to the scale of the dyke and have no dynamical effect. Detailed application of our results to the emplacement of dykes, however, may require our solutions for the fluid motion and elastic deformation to be coupled to the thermal problem of heat transfer from the magma to the colder country rock (Bruce & Huppert 1989).

Interesting parallels exist between the solutions of §§3 and 4 and those for viscous gravity currents down a sloping plane (Smith 1973; Huppert 1982) and at an interface (Lister & Kerr 1989). The dominant downstream balance between buoyancy forces and the viscous pressure drop in a thin layer is common to all these problems. Analogies may be drawn between the elastic shock at the tip of a propagating crack and the surface-tension dominated region at the front of a gravity current, between the critical stress-intensity at which a crack-tip will propagate and the critical contact angle at which a contact-line will move and between cross-stream spreading due to elastic and to hydrostatic pressures caused by variations in the thickness of the flow. Such analogies are, of course, qualitative rather than quantitative but they are useful aids when considering the behaviour of propagating cracks.

To sum up, we have analysed the balance of forces in a propagating fluid-filled crack and derived solutions to the governing equations in three model geometries. These solutions increase our understanding of the dynamics of dyke emplacement and, hence, of the origins of the igneous intrusions that contain many of the world's valuable ore deposits.

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