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THE NONLINEAR BEHAVIOUR OF SHEAR WAVES†

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ABSTRACT

Shear waves or columnar disturbances occur in a variety of situations in stratified fluids. For withdrawal from a stratified fluid the transient behaviour following the initiation of discharge is described by an infinite series of shear waves. However when the shear waves have finite amplitude linear theories are inadequate and a nonlinear theory is required to describe the evolution of the flow. A weakly nonlinear model for long, dispersive, internal waves is used, leading to a Korteweg-de Vries (KdV) equation for stratified fluids. The linearized KdV, or Airy, equation is solved for either the initial value problem or a mixed boundary, initial value problem to yield solutions in agreement with the linear theories for shear waves. Previous scaling analysis for surface waves is extended to determine when the KdV model is valid for shear waves. When nonlinear terms are included, the shear waves are weakly nonlinear forms of internal undular bores. The solution obtained using modulation theory is used as an initial guide to the nonlinear behaviour of shear waves and is complemented by numerical solutions for particular cases.

INTRODUCTION

When a sink is initiated in a stratified reservoir, it is found for certain conditions the fluid withdrawn will only come from about the level of the withdrawal sink. In practice the withdrawal layer is often only metres thick, kilometres downstream from the sink. This effect, known as selective withdrawal, is now well understood and is used extensively for water quality control. The development of the withdrawal layer is described in detail by Imberger *et al.* (1976). Their analysis found that the behaviour was dependent on the parameter $R = Q(NL^4\nu^2)^{-\frac{2}{3}}$, where Q is the flow rate per unit width, N is the buoyancy frequency, L is the length of the reservoir and ν is the kinematic viscosity of the fluid. Three different regimes can be identified dependent on the magnitude of R. If $R \ll 1$, withdrawal will be described by a viscous–buoyancy balance, for $R \sim 1$, viscous and diffusive terms will no longer be important and when $R > 1$, inertial terms will dominate. Selective withdrawal will no longer occur in a stratified fluid once the internal Froude number, $F = Q/Nh^2$ where h is the depth of the duct or reservoir, is greater than π^{-1}. Once this occurs the flow will be described by potential theory.

The establishment of selective withdrawal in a horizontal duct was described by Pao & Kao (1974). They found that withdrawal was caused by columnar disturbances, which propogated away from the duct with the appropriate long wave speed $c_n = Nh/n\pi$, where n is the modal number for that wave (*i.e.* the n^{th} mode has a vertical velocity, $w = A \sin n\pi z$). A shear wave or columnar disturbance is able to propagate away from the sink if its speed is greater than the speed of fluid travelling towards the duct, $u_m = Q/h$. If $u_m > c_1$ or $F > \pi^{-1}$, the first mode shear wave cannot propogate into the duct and hence no withdrawal layer will form. As higher mode waves become able to propagate into the duct, the withdrawal layer thickness decreases. The structure of shear waves was independently verified by McEwan & Baines (1974), who studied shear flows and found that the flow was established by the same waves. They found that the shear wave frontal width, defined as the distance for the velocity difference to change from 5% to 95%, was given by

$$b_n = \frac{h(Nt)^{\frac{1}{3}}}{n} \tag{1}$$

Experiments performed in both of these studies, in the linear regime, showed good agreement with the theoretical results.

The motivation for the present study commences with an experimental series on unsteady selective withdrawal performed by Monismith *et al.*(1986). They studied problems similar to selective withdrawal when the sink was suddenly opened, allowed to flow at a constant rate and then shut. As would be expected from linear theory, it was found that ' positive' shear waves were produced when the sink was opened and 'negative' shear waves when the sink was closed. The negative shear waves had the effect of annulling the velocity field produced initially. Downstream from the sink the behaviour of these sets of shear waves could be well observed, since the differential wave speed of each mode allowed the waves to separate into pairs. As the flowrate was increased nonlinear effects were found to become important and manifested themselves in two ways: only the amplitudes of the lowest mode waves could now be calculated accurately, and the separation between positive and negative fronts was significicantly different from linear predictions.

In the following sections a nonlinear model for shear waves is derived, with the appropriate initial and boundary conditions. As the linearised form of this model is different to the previous models for shear waves, the linearised solution to a similar problem to selective withdrawal will be given, together with a numerical solution for the case of selective withdrawal or shear flows. In the final section, criteria for using the nonlinear model are shown. A modulation solution for the case where the nonlinear shear waves are influenced by the solid boundary is formulated and numerical solutions are shown to verify this solution and display the behaviour of a nonlinear shear wave with negative initial amplitude.

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NONLINEAR MODEL

Nonlinear models for internal waves in stratified fluids are well known. Benney (1966), proposed a model based on two perturbation parameters ϵ and μ, where $\epsilon = a/h$, $\mu = h^2/\lambda^2$, a is a typical amplitude of the wave and λ is a typical wavelength. If x, z and t are nondimensionalized by the scales $h, \mu^{-1/2}h$ and $\mu^{-1/2}(h/g)^{1/2}$, respectively, and u by the scale $\epsilon(gh)^{1/2}$, to first order in ϵ and μ the stream function can be written as $\psi = A(x,t)\phi(z)$, where ϕ satisfies the eigenvalue problem:

$$(\bar{\rho}\phi_z)_z - \frac{\bar{\rho}_z}{c^2}\phi = 0, \qquad \phi(0) = \phi(1) = 0 \tag{2}$$

and c is the appropriate long wave speed. The amplitude, A, satisfies the KdV equation:

$$A_t + cA_x + \epsilon rAA_x + \mu sA_{xxx} = 0. \tag{3}$$

The nonlinear and dispersive coefficents of equation (3) are given respectively by:

$$r = \frac{3\int_0^1 \bar{\rho}\phi_z^3 dz}{2\int_0^1 \bar{\rho}\phi_z^2 dz}, \quad s = \frac{c\int_0^1 \bar{\rho}\phi^2 dz}{2\int_0^1 \bar{\rho}\phi_z^2 dz}. \tag{4}$$

The KdV equation can be transformed to its canonical form by introducing the variables:

$$\chi = x - ct, \quad \tau = \mu st, \quad u = \frac{\epsilon r}{6\mu s}A. \tag{5}$$

The resulting form of equation (3) is:

$$u_\tau + 6uu_\chi + u_{\chi\chi\chi} = 0. \tag{6}$$

From (5), it is seen that dependent on the sign of r, the initial amplitude will be positive or negative, which is critical to the nonlinear behaviour of the shear waves.

Equation (2) will have an infinite number of solutions, giving eigenvalues c_n and eigenmodes ϕ_n. For an exponentially stratified fluid, given by $\bar{\rho} = e^{-\beta z}$, Benney (1966) showed that

$$c_n = \frac{\beta^{\frac{1}{2}}}{((n\pi)^2 + (\beta/2)^2)^{\frac{1}{2}}}, \quad \phi_n = e^{\beta z/2}\sin n\pi z. \tag{7}$$

For a linear stratification, $\bar{\rho} = 1 - \beta z$, ϕ_n can be expressed in terms of zeroth order Bessel functions and c_n is the solution of a related transcendental equation. In the limit of $\beta \to 0$, this and the solution for exponential stratification will both have the same limit. For an exponential stratification the dispersion coefficient is given by $s_n = c_n/2(n\pi)^2$. Numerical integration can be used to show that for linear stratification s_n will be similar. If equation (2) is solved numerically and Simpson's rule is used to evaluate r_n, it is found that slight variations in stratification can produce marked variations in the parameter r_n. An analytical expression for r_n can only be obtained for exponential stratification and is given by:

$$r_n = -\frac{6\beta(n\pi)^3((-1)^n e^{\beta/2} - 1)}{((\frac{\beta}{2})^2 + (n\pi)^2)((\frac{\beta}{2})^2 + 9(n\pi)^2)}. \tag{8}$$

In figure 1, this is compared with numerical values for r_1 for the cases of linear and logarithmic ($\bar{\rho} = 1 - \log(1 - \beta z)$) stratifications. In the limit of $\beta \to 0$ these stratifications are identical but as can be seen all three have a linear dependence on β with widely varying slopes. Since r varies in sign, while s stays of constant sign, the initial amplitude

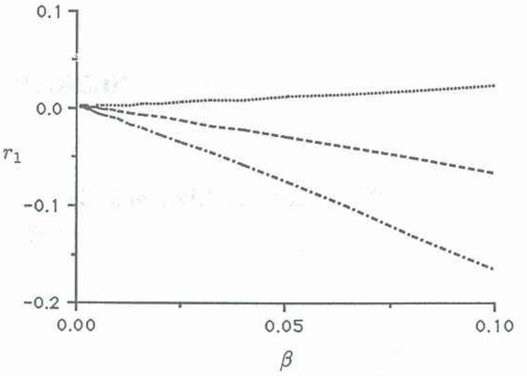

FIGURE 1: The behaviour of the first mode nonlinear coefficient for ······ exponential, — — — linear and · — · — · — logarithmic stratifications.

and hence the nonlinear behaviour of shear waves will be markedly different for each stratification.

The nonlinear model proposed by Benney (1966) applies to a single mode propogating in a duct. In studying the problems of selective withdrawal and shear flows, the streamfunctions are combinations of an infinite sum of modes given by $\psi = \sum_{n=1}^{\infty} A_n(x,t)\phi_n(z)$. If we assume that in the weakly nonlinear limit interactions between modes are negligible (see Smyth,1987), for each mode an equation of the form of (3) will apply coupled with the initial and boundary conditions:

$$A(0,t) = A_0 \quad \lim_{x\to\infty} A = \lim_{x\to\infty} A_x = 0 \qquad t > 0, \qquad A(x,0) = 0 \quad 0 \le x < \infty. \tag{9}$$

These boundary and initial conditions correspond to an induced velocity at a solid boundary, resulting in a wave propogating towards infinity. The initial amplitude for each mode, A_0, can be obtained by Fourier series from the boundary condition for ψ. For shear flows and selective withdrawal, the difference is only this initial amplitude.

A similar problem to a wave propogating away from a solid boundary, which will be used as a comparison, is an initial step profile in an infinite domain. The boundary and initial conditions for this are:

$$\lim_{x\to-\infty} A = A_0 \quad \lim_{x\to\infty} A = 0 \quad \lim_{|x|\to\infty} A_x = 0 \qquad t > 0, \qquad A(x,0) = \begin{cases} A_0 & x < 0 \\ 0 & x > 0. \end{cases} \tag{10}$$

For the canonical form of the KdV equation, the corresponding boundary and initial conditions to (9) are:

$$u\left(\chi = -\frac{c\tau}{\mu s}, \tau\right) = u_0 \quad \lim_{\chi\to\infty} u = \lim_{\chi\to\infty} u_\chi = 0 \qquad \tau > 0, \qquad u(\chi, 0) = 0 \quad 0 \le \chi < \infty \tag{11}$$

and the infinite boundary and initial conditions for an initial step profile are equivalent to (10), with appropriate substitutions.

LINEARISED SOLUTION

To compare the proposed model against previous models, we take the linear limit $\epsilon \to 0$ of equation (3). This gives the dispersive wave equation or Airy's equation. Together with the conditions (9), this system is equivalent to that solved by McEwan & Baines (1974) and Pao & Kao (1974). To show the behaviour we will firstly consider Airy's equation:

$$A_t + cA_x + \mu s A_{xxx} = 0, \tag{12}$$

with the conditions (10) and $A_0 = 1$, since the solution is independent of amplitude.

As outlined in Vleigenthart (1971), dispersive wave techniques can be used to give:

$$A(x,t) = \int_{\xi}^{\infty} Ai(\eta)\, d\eta, \tag{13}$$

$$\text{where:}\ \xi = \frac{x - ct}{(3\mu st)^{\frac{1}{3}}} \quad Ai(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(\frac{1}{3}k^3 + \eta k\right)\, dk$$

The infinite solution is shown in figure 2(a) for $t = 20$, $c = 1$ and $\mu s = 0.1$. If the solution is shown in terms of the similarity variable ξ, the frontal width extends from $\xi = 2$ to $\xi = -2$ and therefore returning to the original variables, it can be shown that the frontal width is:

$$b \approx \frac{4h}{n\pi}\left(\frac{3}{2}Nt\right)^{\frac{1}{3}}, \tag{14}$$

in agreement with equation (1). To determine the behaviour of shear waves in the presence of a solid boundary, equation (12) coupled with the semi-infinite boundary conditions (9) was solved using a linearized version of the finite-difference scheme proposed by Vleigenthart and a boundary discretisation similar to that proposed by Chu *et al.* (1983). This solution is shown in figure 2(b), for the same conditions of those of figure 2(a). Examination of the two solutions shows that for both the frontal width is almost identical. In figure 2(b) the solid boundary is positioned at $x = 0$, hence the solutions can only be compared ahead of this point. As would be expected, the restriction of $A = 1$ at the boundary significantly reduces the amplitude of the wavetrain trailing the leading wave, however as with the frontal width the wavelength is not changed in any way. Comparison of the leading wave for each case, shows that in the presence of a solid boundary the amplitude of this is also reduced. The numerical solution shown in figure 2(b) and other solutions not shown, indicate that for the presence of a solid boundary the amplitudes of the leading and trailing waves will gradually build up from zero and for large times asymptotically approach those for the full infinite solution, equation (13).

To describe the behaviour of shear waves McEwan & Baines (1974) and Pao & Kao (1974) both used approximation methods. McEwan & Baines evaluated the relevant integral using the method of stationary phase, while Pao & Kao mapped the integral to an infinite series. From experimental observations McEwan & Baines stated that the oscillatory contributions were only a small fraction of that of the front and hence they only evaluated that part of the integral giving the shear front. As has been seen this gives the same basic structure as equation (13). McEwan & Baines experimental observations were only taken a small way down the duct and in the same manner as the solutions of equation (12), the oscillatory contributions would have only been a small part of their asymptotic magnitude. The oscillatory contributions from the method of stationary phase can be evaluated, where upon the solution is almost identical to the full infinite solution, equation (13). In terms of the amplitude of the leading wave and frontal shape, the series solution of Pao & Kao shows good agreement with the solid boundary solution in figure 2(b).

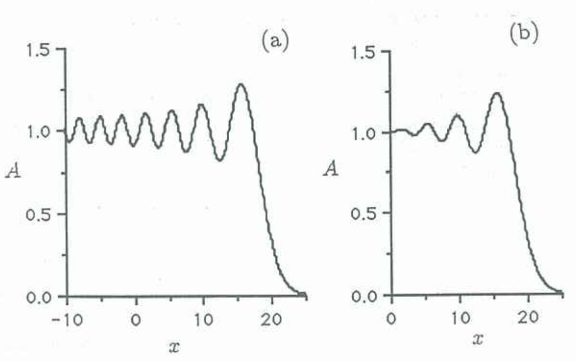

FIGURE 2: Shear wave solutions of Airy's equation, for (a) an initial step profile in an infinite domain and (b) propogation away from a solid boundary at $x = 0$.

NONLINEAR SOLUTIONS

Hammack & Segur (1976) considered modelling criteria for surface waves developing from an initial displacement $\eta(x, 0)$, which can be adapted for stratified waves. Their analysis results in criteria based upon a localized Ursell number U_0, which measures the ratio of nonlinear to dispersive effects. If $U_0 \sim 1$, then the appropriate equation is the KdV equation. If $U_0 \ll 1$, linearised dispersive theory will apply up to a timescale $\tau \sim 10^4$, after this the nonlinear KdV theory is again applicable. For scaling purposes the stop/start sink of Monismith *et al.* (1986) and paddle of McEwan & Baines (1974), can be thought of as an initial value problem in an infinite domain, where the sink or paddle has a characteristic period T_0. Hammack & Segur's localized Ursell number for shear wave propogation in stratified fluids becomes:

$$U_0 = \frac{(T_0 N h)^2 |r| u_m}{(n\pi)^2 6 s\mu (gh)^{\frac{1}{2}}} \tag{15}$$

and u_m is the maximum induced shear velocity for shear flows and the average induced velocity for selective withdrawal. For stratified flow in a duct the relevant viscous timescale is given by $\tau_\nu \sim \mu^{1/2} N h^2 / 2(n\pi)^3 \nu$. With typical laboratory situations this gives $\tau_\nu \sim 3 \times 10^2$ and so we see where linear dispersive theory is initally applicable, the nonlinear KdV theory will not apply at any time. For the upper limit of experiments performed by McEwan & Baines (1974), if we assume a value of $T_0 \sim 20\ s$ the localized Ursell parameter has a value of $U_0 \sim 0.6$ for the first mode shear wave. With the experiments of Monismith *et al.*(1986), for a sink at the base of the duct rather than the centre (resulting in odd mode waves being excited rather than just even modes) we obtain $U_0 \sim 0.2$ for the first mode.

If the properties of a nonlinear, dispersive wavetrain are averaged over a wavelength then the equations (3) and (6) can be transformed to a set of characteristic equations

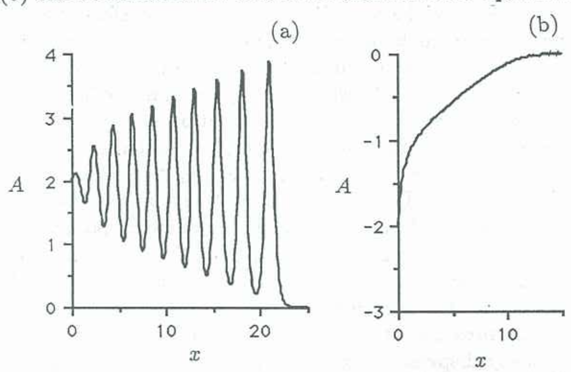

FIGURE 3: Numerical solutions of the KdV equation for an initial (a) positive and (b) negative amplitude, when the wave packets are affected by the presence of a solid boundary.

in terms of these relevant properties. Gurevich & Pitaevskii (1974) used this modulation theory to obtain a solution of the infinite system in terms of the mean height α, the amplitude of the oscillations a and wavenumber k. This solution was adapted by Smyth (1987) for the case of a stationary wall at $\chi = 0$ and can be simply extended for the moving wall problem (Smyth, personal communication). If $\alpha = u_0$ at $\chi/\tau = -c/\mu s$ instead of at $\chi = 0$, as is the case for the stationary wall problem, the following solution is obtained:

$$\alpha = B\left(m - 1 + \frac{2E(m)}{K(m)}\right), \quad a = Bm, \quad k = \frac{\pi B^{1/2}}{K(m)}, \tag{16}$$

$$B = u_0\left(\frac{2E(m_w)}{K(m_w)} + m_w - 1\right)^{-1},$$

where: $$\frac{\chi}{\tau} = 2B\left(1 + m - \frac{2m(1-m)K(m)}{E(m) - (1-m)K(m)}\right),$$

$$m_w \leq m \leq 1 \quad -c_0 \leq \frac{\chi}{\tau} \leq 4B,$$

and m_w is the solution of

$$-\frac{c}{u_0\mu s} = \frac{2\left(1 + m_w - \frac{2m_w(1-m_w)K(m_w)}{E(m_w)-(1-m_w)K(m_w)}\right)}{\frac{2E(m_w)}{K(m_w)} + m_w - 1} = f(m_w). \tag{17}$$

The functions E and K are complete elliptic integrals of the first and second kind respectively. The function $f(m_w)$ has a range extending from -6 at $m_w = 0$ to ∞ at $m_w = 1$, with the x intercept occuring at $m_w \approx 0.63$.

The two limits of this solution are firstly, if $c = 0$ the solution reduces to that for the stationary wall problem. Secondly, if $u_0 \geq c/6\mu s$ then the modulation packet is unaffected by the presence of the wall and the solution reduces to that for the infinite case, with the parameters of equation (16) becoming $B = u_0$ and $m_w = 0$. The limitation of the modulation solution is that it is only valid for $u_0 > 0$. However when $u_0 < 0$ we see from Fornberg & Whitham (1978) for the infinite case that a significantly longer front will form with the oscillatory tail being smoothed out. Eventually for $u_0 > 0$ solitons will form at the front whereas for $u_0 < 0$ no solitons will develop.

To verify the modulation solution and also examine the behaviour of the wave packet when $u_0 < 0$, two numerical solutions are shown in figure 3. These are solutions of equation (3) coupled with the boundary conditions (9), solved using an explicit method similar to Chu *et al.* (1983). The values of the parameters used are $c = 1$, $\mu s = 0.1$, $\epsilon r = 1$ and $A_0 = \pm 2$. The nonlinear shear waves are shown in both figures at time $t = 10$. The parameters used result in the modulation packet being effected by the presence of the boundary, which if we compare figure 3(a) & (b) against the infinite solutions of Fornberg & Whitham (1978) is correct. As would be expected, figure 3(a) shows the same characteristics as the solution of Chu *et al.* for the case $c = 0$. Substituting into the modulation solution, we obtain from equation (17) that $m_w \approx 0.58$ and hence $B \approx 1.066u_0$. Therefore we should have at the boundary that $a \approx 0.6$. Extrapolating the modulation packet, it is found that the numerical and modulation solution show good agreement at this point. Figure 3(b) shows that the character of a negative step will change considerably due to the presence of a boundary. No longer will we obtain a slowly dispersive front, but instead an abrupt change in amplitude with no oscillatory waves.

CONCLUSION

The KdV model for stratified fluids is applicable to describe shear waves once the Ursell parameter is such that $U_0 \sim 1$. For $U_0 \ll 1$, the times to reach nonlinear behaviour are significantly greater than the viscous time scale associated with the flow. For certain stratifications, those with $\epsilon r > 0$, modulation theory can be used to describe the nonlinear behaviour of the shear waves. The form of the modulation solution is dependent on whether the wave packet is effected by the boundary, which is governed by $u_0 \geq c/6\mu s$. When the waves are unaffected by the boundary, their form is that of an undular bore. In this case the amplitude of the leading wave will asymptotically develop to twice the initial amplitude, which is significantly greater than the amplitude predicted by linear theory. If $\epsilon r < 0$, numerical solutions show that the effect of the boundary is governed by the same constraints. The nonlinear form of the shear waves when not effected by the boundary will be a slowly dispersive front, rather than an evolution to solitons. For this case the amplitude will be less than the linear prediction. When the wave packet is affected by the boundary, modulation and numerical solutions show significant differences for both positive and negative initial amplitude. In both cases, the form of the shear waves is much more abrupt, with a further increase in amplitude for positive initial amplitude and a compressing of the dispersive front for negative initial amplitude.

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