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APPLICATION OF A PSEUDOSPECTRAL (COLLOCATION) METHOD TO UNSTEADY FLOW PROBLEMS

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ABSTRACT

A pseudospectral numerical model is developed for solving time-dependent linear and non-linear partial differential equations. This method approximates a solution to a differential equation at certain selected points with a series expansion of orthogonal polynomials. Chebyshev polynomials are used in the numerical model and the selected points at which the numerical approximation is done are known as the collocation points. These collocation points are the extrema of the Chebyshev polynomial. The usage of the Chebyshev polynomial as the interpolating function enables one to evaluate the spectral representation (in terms of the Chebyshev coefficients) of the actual function using the Fast Fourier Transform (FFT) technique. The numerical approximations of the spatial derivatives of the function in question are derived from the global consideration of the Chebyshev coefficients of the function. Numerical solutions to three problems are obtained and are discussed in this paper. Agreement with analytical solutions is good.

INTRODUCTION

Many useful versions of spectral methods have been developed and some are described by Fletcher (1984). Gottlieb & Orszag (1986) also provide a comprehensive discussion of the theory and the application of some of these spectral methods. For instance, the spectral technique is often used for predicting and modelling the transition from laminar to turbulent flow. This numerical method has also been applied to other areas of computation fluid dynamics. Global weather modelling, heat transfer, reacting flows and magnetohydrodynamics are just a few of the numerous applications of the spectral methods to the wider area of fluid dynamics.

The pseudospectral technique, using the Chebyshev polynomial as the basis interpolation function, is especially popular with the study of boundary layer flows. This is because of the distribution of the collocation points in the calculation region. As these collocation points are the extrema of the Chebyshev polynomial, they concentrate at the regions near the boundaries and are sparsely distributed in the region between them. Since the boundary layer is a thin layer adjacent to the surface on which it develops, such an allocation of computational grid points is very useful.

This paper discusses the pseudospectral or collocation method which is popular in the field of incompressible fluid dynamics modelling. The ability to apply the FFT to transform the function from its physical representation to its spectral representation (in terms of Chebyshev coefficients) or vice versa, makes it very attractive especially when non-linearities are encountered in a partial differential equation. This feature makes the evaluation of the non-linear terms economical on computational times compared to, for example, the Galerkin spectral method. The model described in this paper is being further developed to aid in the study of the receptivity of a laminar boundary layer to a disturbance generated by a vibrating leading edge. This paper, however, describes some of the techniques used in the model and only one-dimensional cases are considered. The reliability of the model (built for eventual use in the study of the receptivity problem) as well as its versatility to solve various types of unsteady fluid dynamic problems are shown.

The numerical solutions to three unsteady fluid dynamic problems are discussed. Two types of time-differencing schemes are used for these time-dependent problems. These three problems are chosen to demonstrate the effectiveness of this method in relation to solving the linear and non-linear partial differential equations in unsteady flows. More examples will be presented at the conference.

EQUIPMENT

The software developed for solving partial differential equations (linear and non-linear) with the pseudospectral (collocation) method were written in the C programming language. The development and actual execution of the numerical simulations were performed on a Macintosh II personal computer. This computer has an MC68020 Motorola 32-bit architecture, 15.67 MHz clock frequency processor. It also has an in-built Motorola 68881 floating-point coprocessor.

PSEUDOSPECTRAL APPROXIMATIONS OF SPATIAL DERIVATIVES

The codes to evaluate the first and second spatial derivatives were tested with the following functions:

$$f(x) = 1 + \sin(2*\pi*x + \pi/4) \quad (1)$$

The derivatives of the above functions are calculated in the range $(0 \le x \le 1)$. The numerical results are shown in Figure (1).This

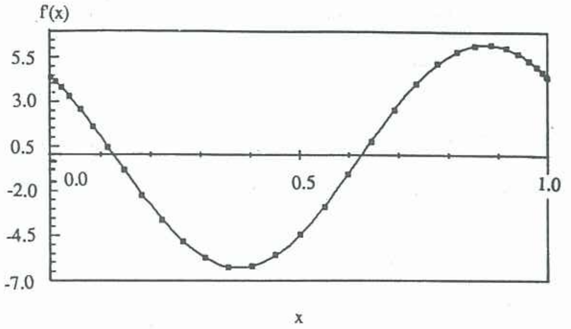

Figure 1. Comparison of the numerically determined first derivative of test function and that obtained analytically in the range $(0 \le x \le 1)$.
(o - numerical solution, —— - analytical solution)

range is used in the numerical solutions of the unsteady flow problems presented in this paper. These numerically determined values of the derivatives are compared with the solution obtained by differentiating the functions directly. The solid lines in the figures represent the solution obtained by direct differentiation of the test function concerned whereas the open squares are the numerically calculated ones. The percentage error in these cases was less than 0.01%. There is also no evident phase shift of the numerical solutions relative to the analytical results. The ability of this numerical technique to approximate the spatial derivatives a function is crucial as numerical solutions to partial differential equatins involves manipulation of such derivatives.

This method of determining the spatial derivatives of a function numerically is known to be more accurate than the other methods of numerical approximations such as the finite-difference or the finite-element method. This is because the latter methods use only the local information to estimate the derivatives whereas the method presented here utilizes the global information of the function over the entire domain (Hussaini & Zang (1987)). This is clearly advantageous. However, pseudospectral numerical simulation involving compressible fluid dynamics had to be carefully dealt with. Any discontinuities in the values of the function in question will lead to two point oscillations of wavelength $\lambda = 2*\Delta x_j$ over the entire mesh. As the mesh is not uniform, the spectrum of these oscillation is large (Cornille (1982)). These oscillations can, however, be removed by filtering. The subject of such filtering will not be discussed here since the flow problems considered are incompressible. This subject is dealt with by Cornille (1982).

TIME DIFFERENCING SCHEMES

In the numerical solutions to the time-dependent partial differential equations described, the following time differencing schemes used are:

(1) Crank-Nicholson (when solving linear problems with non-zero viscosity),
(2) Adams-Basforth-Crank-Nicholson (when solving non-linear problems with non-zero viscosity).

These are implicit time differencing schemes which will give rise to a set of linear ordinary differential equations. The boundary conditions are absorbed into the right-hand side of this set of ordinary differential equation. The differential operator is written in matrix form using the finite difference approximation. This system of equations and hence the time incremented value of the variable of interest, can be solved by pre-multiplying the set of equation by the inverse of the matrix representation of the differential operator. Since the set of matrix equations is tri-diagonal, it can also be solved by standard Guassian elimination without pivoting. This method of solution is more economical on computer storage. These are the recommended time-differencing schemes when the viscosity is non-zero. The numerical solutions obtained with these time-differencing schemes were averaged periodically (about every 20 time steps) to avoid the appearance of oscillations due to any numerical instability. This method is similar to that described in Roger & Taylor (1982).

NUMERICAL SOLUTIONS

1 One rigid boundary moved suddenly and one held stationary

This problem is described schematically in Figure (2). The fluid between the infinite plates is initially at rest and the lower plate is brought suddenly to a constant velocity U in its own plane with the upper plate remaining stationary. The fluid between these plates will be brought into motion by the viscous stress at the plate with the velocity distribution determined by the following normalized equation:

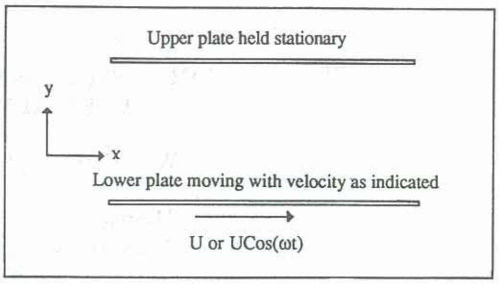

Figure 2. Schematic representation of the unsteady flow problem

$U_t=U_{yy}/Re$, (Re=Reynolds' No) (2)
with boundary conditions,

$U(0,t) = U = 1.0$, and $t > 0$ (3)
$U(1,t) = 0.0$ $t > 0$ (4)

The initial condition used is;

$U(y,0) = 0.0$ $0 \leq y \leq H$ (5)

The velocity distribution is calculated numerically using the pseudospectral technique with Re=100 and a time step of 0.001 sec. Thirty two collocation points were used. The result is shown in Figure (3). The analytical solution to this problem is obtained from Batchelor (1987). The percentage error in this particular example is less than 0.01%. This model clearly shows the transient motion of the fluid within the plates when the bottom plate is given a constant velocity when t > 0. The agreement between the analytical and the numerical solution is good.

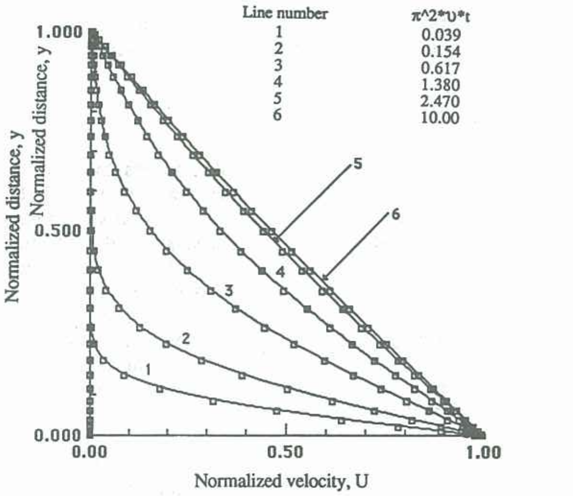

Figure 3. Numerical and analytical solution to problem (1).
(Reynolds' number = 10)
(o - numerical solution, ——— - analytical solution)

2 Flow due to an oscillating plane boundary (zero mean flow)

The lower plate in problem (1) is now subjected to an oscillatory velocity, Ucos(ωt) (where ω is the angular frequency of oscillation). This situation is also depicted in Figure (2) with an oscillatory U. It is expected that fluid motion between the plates will go through a transient and then finally achieve a steady state. Batchelor (1987) showed that the velocity around the lower plate will gradually become a harmonic function of time, t, with the same frequency as the velocity of the oscillating boundary. The steady state velocity profile, which is a function of time t, is also given.

The governing equation for this problem is the same as that given in eqn (2). The oscillating plate is introduced as a boundary condition,

$$U(0,t) = U \cos(\omega t). \quad (6)$$

Numerical calculations of the velocity distributions were performed with the pseudospectral technique and an example of the solution is shown in Figure (4). The value of Re used is 10 and a time step size of 0.001 sec is employed. The solid lines represent the analytical solution whereas the numerically calculated ones are denoted by the open squares. These figures show that the fluid velocity oscillations are confined to a layer a small distance from the oscillating plate which is commonly referred to as the Stoke's layer. To accommodate a better comparison with the analytical solution, the velocity calculations are performed only in a layer close to the plate as shown in Figure (5). Very good agreement between the numerical and the analytical solution is obtained.

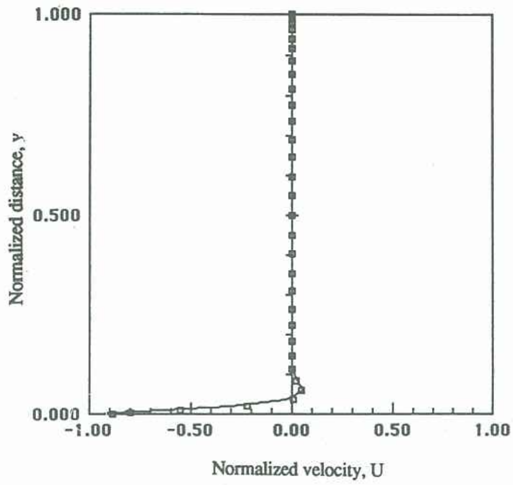

Figure 4. Numerical and analytical solution to problem (2) at a certain time step. (Reynolds' number = 10) (o - numerical solution, ——— - analytical solution)

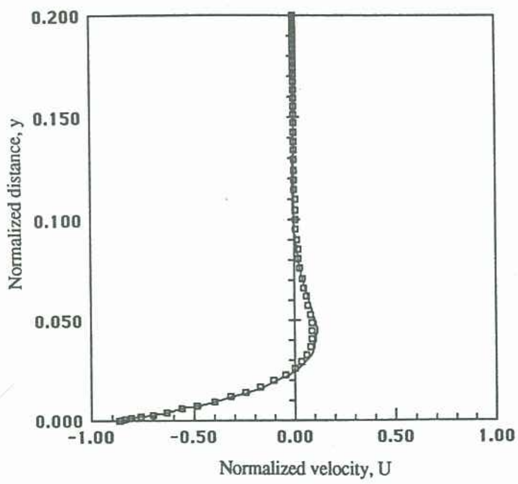

Figure 5. Numerical and analytical solution to problem (2) at a certain time step. (Reynolds' number = 10) (o - numerical solution, ——— - analytical solution)

3 Burger's Equation

The normalized Burger's equation in one-dimension is

$$U_t + U(U_x) - U_{xx}/Re = 0.0 \quad (7)$$

This model is a good test of the codes for eventual use for solving the Navier-Stoke's equation since it represents the simplest case where there exists both the viscous and the non-linear convective terms (Wadia & Payne (1979)). This equation is solved numerically with the initial condition:

$$U(x,0) = \sin(\pi * x), \qquad 0 \le x \le 1 \quad (8)$$

and the boundary condition

$$U(0,t) = U(1,t) = 0.0 \qquad t \ge 0 \quad (9)$$

Numerical solutions obtained in this paper correspond to Re = 100 and a time step size of 0.0001 sec. These results are shown in Figure (6) and Figure (7). The results obtained show good agreement with the numerical results obtained by Wadia and Payne (1979). The results show that the initial sine wave shows a tendency to develop into a steep front near the normalized value of x = 1 as time progresses. After a while, this steep front broadens because of viscous dissipation and eventually only a sine wave remains. The eventual sine wave obtained is of a much smaller magnitude than that of the initial one due the viscosity of the fluid. This observation also corresponds very well with the qualitative description given by Cole (1951).

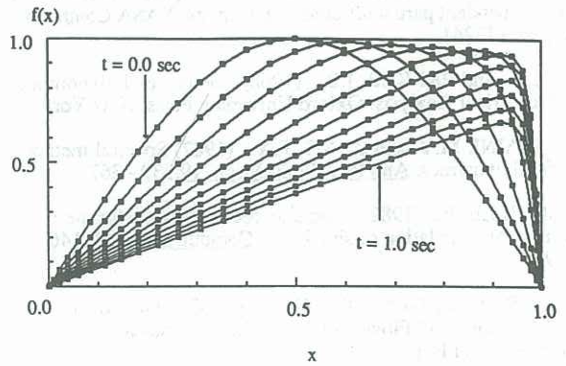

Figure 6. Numerical solution to Burger's equation (Reynold's number for this problem is 100 and the time step size used is 0.001 sec, each line represents a time interval of 0.1 sec)

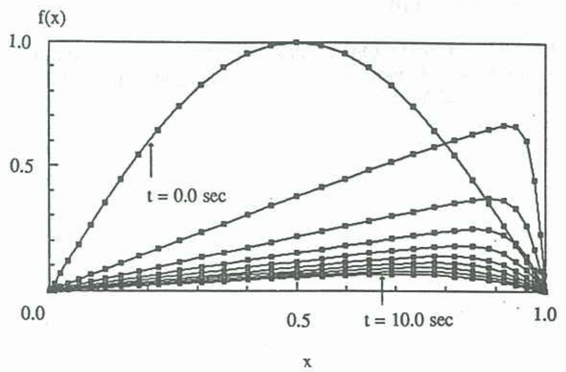

Figure 7. Numerical solution to Burger's equation (Reynold's number for this problem is 100 and the time step size used is 0.001 sec, each line represents a time interval of 1.0 sec)

CONCLUSION

The results presented in this paper show the effectiveness of the pseudospectral method to unsteady flow problems. The main conclusions are:

(1) The choice of the interpolating functions are very important in solving non-linear differential equations. In the cases described in this paper, the use of the Chebyshev polynomial facilitates the use of the FFT in the transformation from physical to spectral space and vice versa. This not only simplifies the codes but is also very time-efficient.

(2) An unevenly spaced mesh is obtained when one uses Chebyshev polynomial as the interpolating functions. The meshes are closely spaced at the end points and sparse in the middle. This is desirable especially when modelling unsteady boundary layer flow where the activities are located very close to the plate. This is clearly shown in the numerical solution to problem (2).

(3) The use of global data for the calculation of the derivatives gives a very good estimate of the spatial derivatives. Care, however, must be taken when handling discontinuities.

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