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THE TOPOLOGY AND VORTICITY DYNAMICS OF A THREE-DIMENSIONAL PLANE COMPRESSIBLE WAKE

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ABSTRACT

The three-dimensional aspects of transition in a low Mach number plane compressible wake are studied numerically. Comparisons are made between the topology of the velocity field and the vorticity dynamics of the flow based on results from direct numerical simulations of the full compressible Navier Stokes equations. The velocity field is integrated to obtain instantaneous streamlines at different stages in the evolution. A generalized three-dimensional critical point theory is applied to classify the critical points of the velocity field.

1. INTRODUCTION.

A description of the three-dimensional topology of a wake using critical point theory provides a concise framework for interpreting the voluminous amount of numerical data that results from a three-dimensional simulation. The critical points are points in the flow where any vector field, such as velocity, vorticity, or pressure gradient is indeterminate, i. e. the norm of the vector field at the critical points is zero. Recently, Chong, Perry, & Cantwell [1988] devised a generalized approach for the classification of three-dimensional flow patterns for incompressible as well as compressible flows. Their approach is based on the determination of three matrix invariants (P, Q, and R) of the gradient tensor of the vector field used to describe the flow. These invariants identify topologically distinct regions of the flow.

Previous studies concerning the topology of wakes and jets have relied on 2D phase plane methods applied to a velocity field. Based on these methods and sectional streamline patterns, there is some evidence that the flow pattern for a three-dimensional wake may consist of "limit cycles" of the type described in Perry & Tan [1984] and Perry & Steiner [1987]. These limit cycles are bifurcation lines which have closed on themselves. Based on the presence of both foci spiraling inward and foci spiraling outward, Perry & Steiner [1987] conjectured that regions of vortex stretching and vortex compression can coexist in a three-dimensional wake. Recent results of direct simulations also show the presence of both vortex compression and stretching regions in a three-dimensional wake. In Chen, Mansour, & Cantwell [1989] localized regions of vortex compression alternating with localized regions of vortex stretching inside the spanwise rollers were identified. The strong compression regions were attributed to large negative strain rates generated by a quadrapole of streamwise vorticity located inside the spanwise rollers. The effect of the spanwise variation in strain rate is to produce "hoop-like" vorticity structures similar to those which appear in incompressible 3D mixing layers (Rogers & Moser [1989]).

In the present work, the topology of a 3D compressible wake is determined by classifying the critical points in the flow associated with the *velocity* vector field. The *vorticity* and *pressure gradient* vector fields have also been studied but are presented elsewhere due to limitations in space (Chen [1989]). The critical points of the velocity field are located and classified according to the three-dimensional method of Chong, Perry, & Cantwell [1988]. The following issues are addressed in this paper. What is the topology of a 2D compressible wake in both its early and fully-developed stages? What is the difference in topology of a 3D wake compared with a 2D wake? Do "limit cycles" exist, and if so what is their relation to other topological features in the flow? Finally, can the topology of the initial linear eigenfunctions be used to explain the nonlinear development of the vorticity and strain rate fields?

2. NUMERICAL METHOD

Only a brief description of the numerical method is given here since details of the method are presented in Chen [1989]. In our direct simulations we solve the compressible continuity, momentum, and total energy equations in full using a spectral collocation method. The wake is assumed to evolve temporally which is related to the spatially-evolving wake through the Taylor hypothesis. The perfect gas equation of state is used. The dynamic viscosity (μ) is allowed to vary with temperature through a power law: $\mu = T^{2/3}$. The Prandtl number is taken to be 1.0 and the specific heat at constant pressure is assumed to be constant. Therefore, the thermal conductivity also varies with temperature.

Since we are dealing with a time-developing flow, initial conditions need to be prescribed. Here, the initial flow is assumed to consist of Gaussian mean velocity profile given by

$$\overline{u}_1 = 1 - \Delta u_c e^{-c_1 x_2^2} \quad (1)$$

where Δu_c is the velocity deficit at the wake centerline, c_1 is a scaling factor, and x_2 is the distance from the centerline (the normal coordinate). We choose c_1 to be 0.69315 in order to make the initial wake half-width, b, equal to 1.0, and the initial velocity deficit at the wake centerline, Δu_c, equal to 0.692 corresponding to the measurements by Sato & Kuriki [1964].

From the Crocco-Busemann relation, the dependence of the mean temperature, $\overline{T}$, on $\overline{u}_1$ is obtained

$$\overline{T} = 1 + 0.5M^2(\gamma - 1)(1 - \overline{u}_1^2) \quad (2)$$

where M denotes the freestream Mach number. In this work all length scales have been normalized by the initial half-width, b, all velocities have been normalized by the freestream velocity, U_∞, temperature and density have been normalized by their freestream values, and pressure has been normalized by ρU_∞^2.

The simulations are forced with linear eigenfunctions obtained from a linear stability analysis (Chen, Cantwell, & Mansour [1989]). The least stable 2D Kelvin-Helmholtz wave, together with a pair of oblique 3D waves of equal and opposite angle, are superimposed on the mean flow at the beginning of a simulation:

$$\mathbf{b}(x_1,x_2,x_3,0) = \overline{\mathbf{b}}(x_2) + Real\Big[\epsilon_{2D}\hat{\mathbf{b}}_{2D}(x_2)e^{i(\alpha x_1+\Theta)} + \epsilon_{3D}\hat{\mathbf{b}}_{3D}(x_2)e^{i(\alpha x_1+\beta x_3)} + \epsilon_{3D}\hat{\mathbf{b}}_{3D}(x_2)e^{i(\alpha x_1-\beta x_3)}\Big] \quad (3)$$

where the vector $\mathbf{b}$ denotes the three velocity components, temperature, and density, $\overline{\mathbf{b}}$ denotes their mean parts, and $\hat{\mathbf{b}}_{2D}$ and $\hat{\mathbf{b}}_{3D}$ denote the eigenfunctions of the 2D and 3D disturbances. The 2D eigenfunctions correspond to a streamwise wave number, α, of the least stable wave predicted from linear stability theory. The 3D eigenfunctions correspond to a pair of oblique waves, with spanwise wave numbers, $\pm\beta$, and a streamwise wave number, α, corresponding to the fundamental. The angle the oblique waves make with the streamwise direction is assumed to be 30 degrees. The relative phase difference between the 2D and oblique waves, Θ, is assumed to be 0 degrees. Finally, the amplitudes of the eigenfunctions, ϵ_{2D}, and ϵ_{3D} are chosen to be 5% of their freestream values.

The temporally-evolving wake is solved with x_1 in the streamwise direction, x_2 in the transverse direction, and x_3 in the spanwise direction. Periodic boundary conditions are applied in x_1 and x_3, since the flow is statistically homogeneous in the (x_1-x_3) planes. Periodic freestream boundary conditions are applied in x_2 at $x_2 = \pm\infty$.

3. VELOCITY FIELD

In this section, the results of a Mach 1.0, Re=300 simulation are used to illustrate the topological features in a wake.

2D streamline pattern

First we examine the flow pattern for a 2D wake which serves as a baseline case with which to compare the 3D flow patterns. The critical points of the velocity field are located corresponding to the grid points where the magnitude of the velocity field is zero. The velocity field is then expanded in a Taylor series about each critical point. The gradient of the velocity vector field, evaluated at each critical point, is obtained by spectral differentiation of the velocity field. The two matrix invariants of the velocity gradient tensor, P and Q, are found and displayed on a phase diagram shown in Figure 1. The critical points in the flow consist of stable foci and saddle points. Centres ($P = 0$) can not exist since the velocity field is not solenoidal at Mach 1.0. In physical space, the relation of the foci and saddle points is shown in Figure 2a in a plot of the instantaneous streamlines obtained by integrating the velocity field. The observer is assumed to move at the phase speed of the 2D linear disturbances, $c_{ph} = 0.71$, obtained from a linear stability analysis. Initially, note that the saddle points, representing regions of high strain rates, connect foci on the same side of the wake. This is a different topology from the developed wake in which the saddle points connect spanwise rollers of opposite sign. To verify that there is no interaction between the two sides of the wake initially, we start several rake points in the streamwise direction, all located in the "alleyway" near the centerline where the fluid is moving upstream. The streamlines originating in the upper side of the alleyway all wind up in the upper side of the wake, whereas, those originating in the lower side of the alleyway, all wind up in the lower side of the wake. This confirms our conjecture that the wake behaves as two independent mixing layers initially in which fluid in the spanwise rollers is entrained from both the freestream and the slower moving fluid located in the alleyway. Eventually the fluid in the alleyway will be exhausted as it is entrained into the rollers, and at this point, the topology must bifurcate in order for mass to be conserved.

The instantaneous streamlines at $t = 40$ after the fundamental has saturated are shown in Figure 2b. Note that now the saddle points connect both sides of the wake and the fluid initially in the alleyway has collapsed to a line dividing the two sides of the wake. In the developed wake fluid in the spanwise rollers is entrained from the freestream and from the opposite side of the wake.

3D streamline pattern

The instantaneous streamline pattern for the 3D case is shown in Figure 3 at $t = 0$, and at $t = 25$ during the nonlinear roll-up stage. As in the 2D case, initially the observer is assumed to move at the phase speed of the disturbances. After the spanwise vortices have rolled-up the observer is assumed to move with the centroid of the rollers. For the case studied here, the phase speed for the 2D and 3D disturbances is the same, $c_{ph} = 0.71$. At $t = 0$, we initialize the rake points in the two spanwise planes of symmetry, $x_3 = 0$ and $x_3 = L_3/2$. Note that in both planes, there are foci and saddle points connecting foci on the same side of the wake similar to the 2D case. However, the foci on the same side of the wake at $x_3 = 0$ and $x_3 = L_3/2$ are spiraling in opposite directions. The velocity vectors projected onto these two planes are shown in Figure 4. Note that in both planes the velocity vectors are moving in a clockwise direction in the top half of the wake and in a counterclockwise direction in the bottom half of the wake. Therefore, the streamlines are spiraling outward on both sides of the wake at $x_3 = 0$ and inward at $x_3 = L_3/2$. In Figure 3a, note that at the center of each foci there is a single streamline that leaves the symmetry plane and connects the foci at $x_3 = 0$ and $x_3 = L_3/2$ on the same side of the wake. In principle, these connections should not exist since the spanwise velocity component, u_3 should be identically zero on the symmetry planes. However, since the velocity vector is very nearly singular near the center of the foci, any slight perturbation in our computation of the streamlines due to numerical round-off error could cause them to deviate from the plane of symmetry. Since the spanwise velocity is nonzero on either side of the symmetry planes, a very small distance away from the symmetry planes, the streamlines near the center of the rollers have a strong spanwise component since the streamwise and transverse velocities, u_1 and u_2, are very small.

At $t = 25$, rake points are initialized in the symmetry planes $x_3 = 0$ and $x_3 = L_3/2$ and a similar pattern of oppositely spiraling foci appear. The saddle points at $x_3 = 0$ appear to connect foci on opposite sides of the wake, whereas the saddle points at $x_3 = L_3/2$ still connect foci on the same side of the wake. This is an indication that the wake rolls-up at different rates along the span.

If rake points are initialized in between the planes of symmetry, then a "limit cycle " emerges (see Figure 5). Along the inner radius (r_1) the streamlines are spiraling toward $x_3 = 0$ in a helical manner; as they approach $x_3 = 0$ they return toward $x_3 = L_3/2$ on the outer radius r_2 where $r_2 > r_1$, forming a closed cylindrical cell.

Based on the streamline patterns at $t = 0$ and $t = 25$ there appear to be three different types of critical points present: 1) compressing unstable foci at $x_3 = 0$, 2) stretching stable foci at $x_3 = L_3/2$, and 3) saddle points in between the foci in each plane. We compute the three invariants of the velocity gradient tensor and classify the critical points using the method of Chong, Perry, & Cantwell [1988]. The critical points are displayed on a P-Q diagram shown in Figure 6. The topology of the velocity field indicates that there are spanwise regions of localized vortex compression alternating with regions of vortex stretching in what will eventually develop into the spanwise rollers. A comparison of the vorticity and strain rate fields with the initial streamline pattern will show that some of the nonlinear aspects of the wake are topologically equivalent to the initial linear behavior.

4. VORTICITY FIELD

During the early stages of roll-up the behavior of the streamwise vorticity in the saddle region is consistent with the initial 3D streamline pattern (see Figure 7a). Note that the streamwise vorticity is stretched by the strain field generated by spanwise vorticity of the same sign.

The streamwise vorticity at $t = 18$ resembles the "ribs" seen in mixing layers. At a subsequent time $t = 40$ a comparison of the streamline pattern and the streamwise vorticity indicates that the strain field is between adjacent spanwise rollers of opposite sign (Figure 7b). Namely, the topology has changed such that the saddle points now connect foci on opposite sides of the wake.

While vortex stretching accounts for the intensification of the streamwise vorticity in the saddle point region, it also plays a significant role in the development of the spanwise rollers. Spanwise alternating regions of localized vortex compression and stretching were found to exist near the center of the spanwise rollers during the nonlinear stages (Chen, Mansour, & Cantwell [1989]). Strong negative strain rates at $x_3 = 0$ result in vortex compression removing spanwise vorticity (ω_3) in the center of an otherwise uniform roller. The vortex stretching term responsible for the production of spanwise vorticity, $S_{33}\omega_3$, along with ω_3 are shown in Figure 8. Note that the spanwise vorticity is removed in the center of the core leaving behind a concentration of vorticity on the perimeter of the roller resembling a "hoop". The existence of stretching stable foci (at $x_3 = L_3/2$) and compressing unstable foci (at $x_3 = 0$) in the initial topology of the velocity field may explain the spanwise nonuniformity in the developed rollers.

5. CONCLUSIONS

In this work the topology of a compressible plane wake was studied using a generalized three-dimensional critical point theory applied to both 2D and 3D velocity fields. It was found that initially the wake behaves as two independent mixing layers separated by an alleyway. After the fluid in the alleyway is entrained into the rollers, the topology changes such that the saddle points no longer connect foci on the same side of the wake, but instead, connect foci on opposite sides of the wake.

The initial topology of the 3D velocity field was found to consist of: 1) stretching stable foci, 2) compressing unstable foci, 3) saddle points, and 4) limit cycles. The developed spanwise vorticity field was found to be topologically equivalent to the initial 3D velocity field. Namely, spanwise alternating regions of vortex stretching and vortex compression in the developed vorticity field are an extension of the critical points of the initial linear eigenfunctions.

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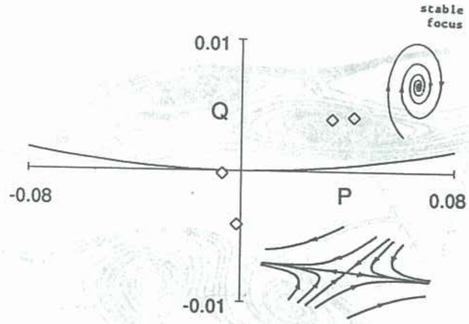


FIGURE 1. CLASSIFICATION OF CRITICAL POINTS ON A P-Q DIAGRAM FOR A 2D WAKE AT $Re = 300$, $M = 1.0$.

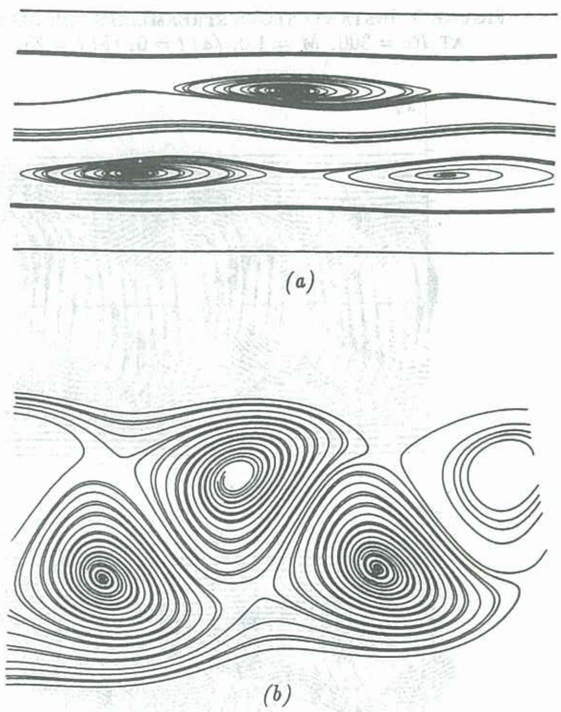
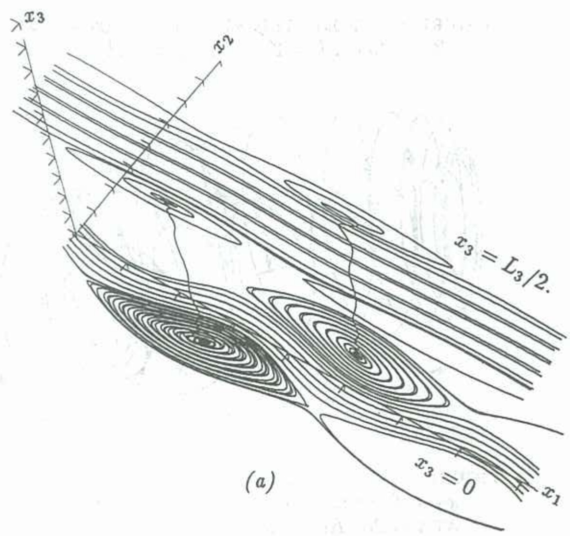


FIGURE 2. INSTANTANEOUS STREAMLINES FOR 2D WAKE AT $Re = 300$, $M = 1.0$ *(a)* $t = 0$, *(b)* $t = 35$.



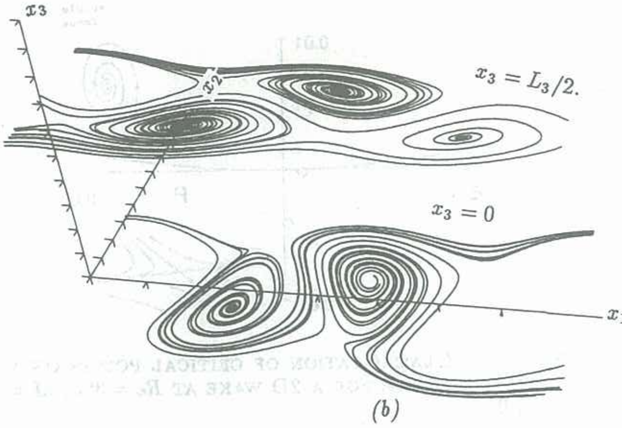


FIGURE 3. INSTANTANEOUS STREAMLINES FOR 3D WAKE AT $Re = 300$, $M = 1.0$. *(a)* $t = 0$, *(b)* $t = 25$.

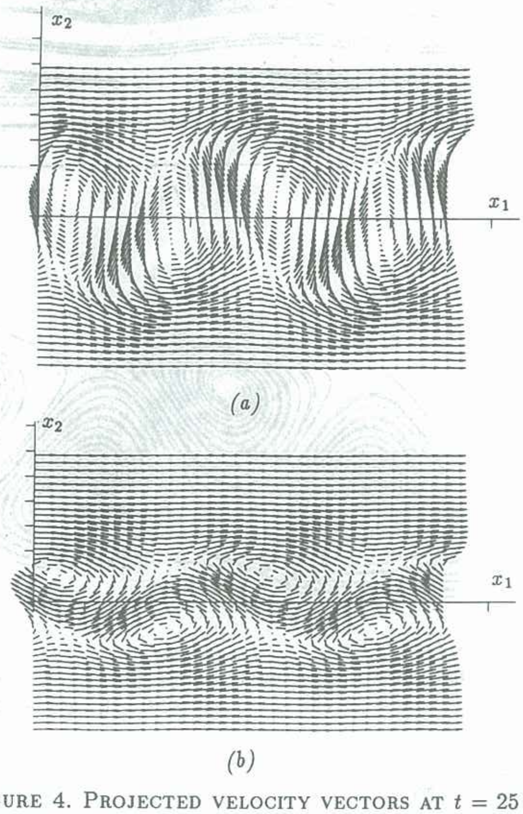


FIGURE 4. PROJECTED VELOCITY VECTORS AT $t = 25$ $Re = 300$, $M = 1.0$. *(a)* $x_3 = 0$; *(b)* $x_3 = L_3/2$.

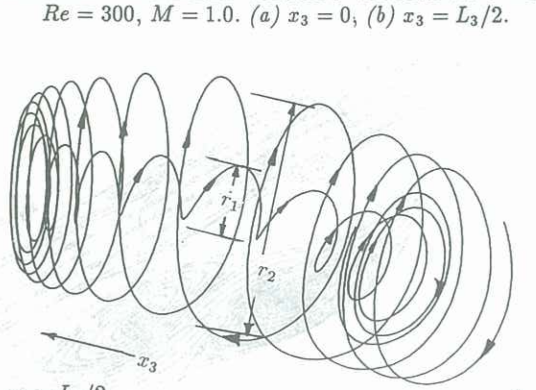


FIGURE 5. INSTANTANEOUS STREAMLINE INITIALIZED AT $x_3 = L_3/8$ ILLUSTRATING "LIMIT CYCLE" BEHAVIOR AT $t = 25$, $Re = 300$, $M = 1.0$.

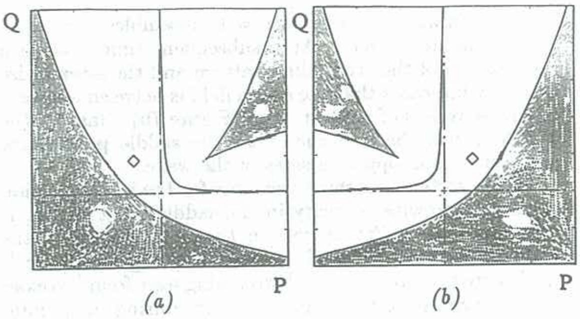


FIGURE 6. CLASSIFICATION OF CRITICAL POINTS ON A *(a)*P-Q DIAGRAM (R>0), AND *(b)*P-Q DIAGRAM FOR (R<0) FOR A 3D WAKE AT $Re = 300$, $M = 1.0$.

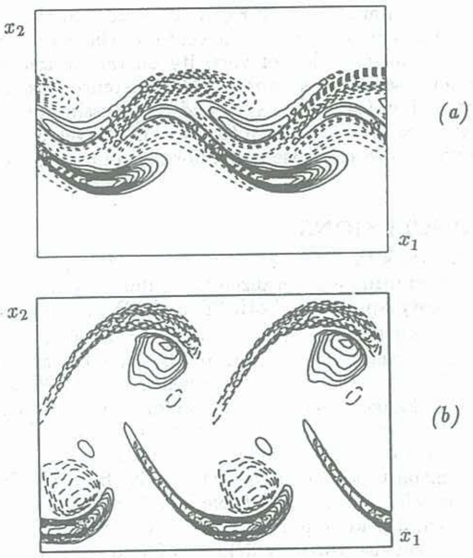


FIGURE 7. STREAMWISE VORTICITY (ω_1) IN A 3D WAKE AT $Re = 300$, $M = 1.0$. *(a)* $t = 18.0$, *(b)* $t = 40.0$.

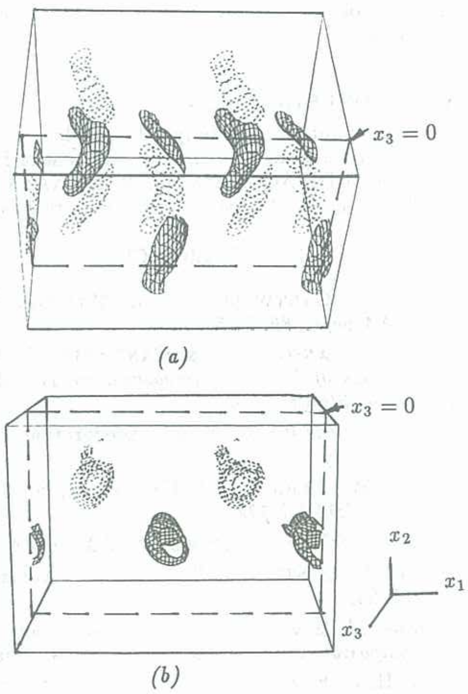


FIGURE 8. *(a)* VORTEX STRETCHING TERM, $S_{33}\omega_3$, AT 40% OF PEAK VALUE, *(b)* SPANWISE VORTICITY MAGNITUDE AT 50% OF PEAK VALUE SHOWING "HOOP-LIKE" STRUCTURE.