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THE SMALL-SCALE STRUCTURE OF ACCELERATION CORRELATIONS AND ITS ROLE IN THE STATISTICAL THEORY OF TURBULENT DISPERSION.

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ABSTRACT

Formal asymptotic expansions are used to describe the behaviour of Lagrangian turbulence statistics in the inertial sub-range. This approach is necessary in order to correctly develop the statistical theory of turbulent dispersion. Contrary to conventional ideas, the analysis indicates that the relative dispersion of two particles is not simply related to conditioned dispersion of one-particle (say by fixing its initial velocity). The actual structure of relative dispersion is developed.

STATISTICAL THEORY

For a general theory of turbulent small-scale processes it is appropriate to consider the velocity and displacement *difference* variables:

$$\underline{u}(t) = \underline{u}(t) - \underline{u}(t_0) \qquad (1)$$

and

$$\underline{x}(t) = \underline{x}(t) - (t-t_0)\underline{u}(t_0) - \underline{x}(t_0) \qquad (2)$$

respectively. Batchelor (1950) indicates how to analyze dispersion in terms of difference variables. Next we assume that the small-scale structure of the turbulence is isotropic, homogeneous and stationary. Then the mean values of the variables (1) and (2) vanish and the first significant statistical information is held by the covariances. We write

$$\langle u_i^{(1)}(t_1)u_j^{(1)}(t_2)\rangle = D^*_{ij}(t_1,t_2) \qquad (3)$$

for the velocity-difference covariance where the one in parenthesis indicates the Lagrangian variable for particle one. Similarly, the two-particle velocity-difference covariance is written

$$\langle u_i^{(1)}(t_1)u_j^{(2)}(t_2)\rangle = D^*_{2ij}(t_1,t_2) \qquad (4)$$

and from these the relative-velocity covariance, for the variable $\underline{u}^{(r)} = \underline{u}^{(2)} - \underline{u}^{(1)}$, is determined as

$$D^{*(r)}_{ij} = 2D^*_{ij} - 2D^*_{2ij} \,. \qquad (5)$$

Note that while (3) is independent of the *fixed* initial particle location, (4) and thus (5) depends upon the initial *fixed* separation of the pair of particles.

The difference variables are simply related to the particle accelerations (Monin & Yaglom, 1975). In terms of the relative-acceleration covariance, $R^{*(r)}$, we have

$$D^{*(r)}_{ij} = \int_0^{t_1}\int_0^{t_2} R^{*(r)}_{ij}(\tau_1,\tau_2)d\tau_1 d\tau_2 \qquad (6)$$

which may be expressed in its one- and two-particle constituent forms using (5) and the equivalent equation for the acceleration covariance:

$$R^{*(r)}_{ij} = 2R^*_{ij} - 2R^*_{2ij} \,. \qquad (7)$$

At this level of description, the task of describing relative dispersion reduces to the analysis of the one- and two-particle acceleration covariance. Monin & Yaglom (1975) estimate these in terms of the mean energy dissipation in the flow, $\bar{\varepsilon}$, the fluid viscosity, ν, and the initial pair separation $\underline{\Delta}_0$. Their crude estimate gives the initial ratio of R^*_{2ii}/R^*_{ii} as proportional to

$$\frac{\nu^{1/2}}{L(\bar{\varepsilon}L)^{1/6}}\left(\frac{L}{\Delta}\right)^{2/3} \quad (= r)$$

where $\Delta = |\underline{\Delta}_0|$ and L is some outer length scale for the turbulence. With the outer velocity fluctuation, σ^2, being given by $(\bar{\varepsilon}L)^{2/3}$ a Reynolds number may be defined from the outer scales: $\mathcal{R} = \nu^{-1}L(\bar{\varepsilon}L)^{1/3}$. Then r is proportional to $(L/\Delta)^{2/3}\mathcal{R}^{-1/2}$. Now the most important practical case to consider is when $\mathcal{R} \gg 1$. Also the small-scale spatial structure of interest corresponds to cases when $\Delta \ll L$. However, it is not meaningful to consider initial separations which depend upon $\mathcal{R}$. Thus in the limit $\mathcal{R} \to \infty$ the ratio r tends to zero. The smallness of r suggests neglecting the two-particle acceleration covariance relative to the one-particle one. Monin & Yaglom (1975) do indeed do this and it is then evident from (6) and (7) that the relative dispersion is effectively equivalent to the one-particle dispersion. The flaw in the argument, however, is that although r is small it is only so for a fraction of the two-time integration domain in (6). To more precisely account for the relative dispersion process we need to more carefully examine the structure of the one- and two-particle covariances.

ONE-PARTICLE DISPERSION

For the special case we are considering the one-particle acceleration covariance depends on a single scalar function of the two-time time difference,

$$R^*_{ij} = R^*(\tau)\delta_{ij} \qquad (8)$$

where $\tau = |t_2 - t_1|$ and δ_{ij} is the Kronecker delta. The initial covariance may be estimated, as above, by dimensional arguments; thus we write

$$R^*(\tau) = \bar{\varepsilon}t_\eta^{-1}R\left(\frac{\tau}{t_\eta}\right) \text{ where } t_\eta = \left(\frac{\nu}{\bar{\varepsilon}}\right)^{1/2} \qquad (9)$$

Here we have introduced the Kolmogorov time scale, t_η, which is a time scale supposed typical of the actual dissipation processes. The dispersion possesses a second time scale, t_L, which is typical of the outer flow and is approximately L/σ. Thus $t_L = \mathcal{R}^{1/2} t_\eta$ and so the outer time scale is much larger than the inner or dissipation time scale. Because of the two-scale nature (9) is an inadequate representation and it is instead appropriate to interpret R^* in terms of matched asymptotic expansions. Thus for $\tau \approx t_\eta$ the inner expansion of R^* consists of a sequence of terms of decreasing order of magnitude:

$$R^* = \bar{\varepsilon} t_\eta^{-1}\left(\mathcal{G}_0\left(\frac{\tau}{t_\eta}\right) + \delta_1(\mathcal{R})\mathcal{G}_1\left(\frac{\tau}{t_\eta}\right) + \ldots\right) \qquad (10)$$

where the first term is independent of the outer scales but the remaining terms represent corrections due to the outer flow and are hence $\mathcal{R}$ dependent. Of course $\delta_1 \ll 1$. Similarly, an outer expansion may be written down to describe to covariance for $\tau \approx t_L$ (large lags) say

$$R^* = \bar{\varepsilon} t_L^{-1}\left(\bar{\mathcal{G}}_0\left(\frac{\tau}{t_L}\right) + \bar{\delta}_1(\mathcal{R})\bar{\mathcal{G}}_1\left(\frac{\tau}{t_L}\right) + \ldots\right) \qquad (11)$$

where in this case the first term is independent of the small scales but the higher-order terms represent corrections due to them. These expansions are required to match when $t_\eta \ll \tau \ll t_L$, i.e. the inner expansion as $\tau/t_\eta \to \infty$ matches with the outer expansion as $\tau/t_L \to 0$ in the sense of van Dyke (1975). A detailed examination of this proposed behaviour is given by Borgas & Sawford (1989). Here we will restrict attention to the leading-order terms.

Supposing that, firstly, some algebraic power law holds in the matching region and, secondly, the first terms in (10) and (11) match at leading order we have that

$$\bar{\varepsilon} t_\eta^{-1}\mathcal{G}_0\left(\frac{\tau}{t_\eta}\right) \approx \bar{\varepsilon} t_L^{-1}\bar{\mathcal{G}}_0\left(\frac{\tau}{t_L}\right) \text{ when } t_\eta \ll \tau \ll t_L$$

and

$$\bar{\varepsilon} t_\eta^{-1}\left(\frac{\tau}{t_\eta}\right)^\gamma \approx \bar{\varepsilon} t_L^{-1}\left(\frac{\tau}{t_L}\right)^\gamma$$

is only possible for $\gamma = -1$. Thus $\mathcal{G}_0(\xi) = \mathcal{C}\xi^{-1}$ as $\xi \to \infty$ and $R^* \approx \mathcal{C}/\tau$ for $t_\eta \ll \tau \ll t_L$ ($\mathcal{C}$ is some $O(1)$ constant). Clearly the process is equivalent to Kolmogorov's hypothesis at this order. However, we will now show that this particular prediction is kinematically inconsistent.

A useful covariance is the two-time velocity covariance with $t_1 = t_2 = \tau$, i.e. $\bar{D}^*(\tau) = D^*(\tau,\tau)$. Then

$$R^* = \frac{1}{2}\frac{d^2}{d\tau^2}\bar{D}^*$$

or equivalently

$$\bar{D}^*(\tau) = 2\int_0^\tau \left(\tau - t\right)R^*(t)dt \ . \qquad (12)$$

Since $\bar{D}^* \to 2\sigma^2$ as $\tau \to \infty$ it follows that

$$\int_0^\infty R^*(\tau)d\tau = 0 \quad \text{and} \quad \int_0^\infty \tau R^*(\tau)d\tau = -\sigma^2 \qquad (13)$$

are two kinematic constraints which must independently be consistent with the expansions used. In order to assess this possibility a uniformly valid approximation to R^* over the whole-time lag domain is needed. When given the inverse-lag matching above the composite expansion method (van Dyke, 1975) gives

$$R^* \approx \bar{\varepsilon} t_\eta^{-1}\mathcal{G}_0\left(\frac{\tau}{t_\eta}\right) + \bar{\varepsilon} t_L^{-1}\bar{\mathcal{G}}_0\left(\frac{\tau}{t_L}\right) - \mathcal{C}\tau^{-1} \ . \qquad (14)$$

Using (14) in the first integral in (13) gives

$$\int_0^\infty R^*(\tau)d\tau = \mathcal{C}\bar{\varepsilon}\log\left(\frac{t_L}{t_\eta}\right) + O(\bar{\varepsilon}) \gg \bar{\varepsilon}$$

where $\bar{\mathcal{G}}_0$ is assumed to vanish exponentially fast when $\tau/t_L \to \infty$. Therefore according to (13) $\mathcal{C}$ must vanish. What this means is that the expansions are not uniform. $\mathcal{G}_0$ decreases faster that inversely and so for large lags is not a very large fraction of the covariance. Higher-order terms, such as $\delta_1\mathcal{G}_1$ have *overtaken* it. Similarly, $\bar{\mathcal{G}}_0$ does not grow inversely with small lags and higher-order terms dominate it in the inner region. However, because $\mathcal{G}_0(\xi)$ is integrable over $[0,\infty)$ the velocity covariance from (12) behaves like

$$\bar{D}^*(\tau) \approx \mathcal{C}_0\tau \text{ when } t_\eta \ll \tau \ll t_L$$

where $\mathcal{C}_0$ is an $O(1)$ universal constant given by

$$\mathcal{C}_0 = 2\int_0^\infty \mathcal{G}_0(\xi)d\xi \ . \qquad (15)$$

Thus the velocity covariance has a well defined inertial sub-range but where the universal constant is associated with the dissipation-range acceleration covariance and not an inertial sub-range acceleration property. The latter is trivial in the sense that it is always a function of both τ and $\mathcal{R}$ such that it remains much smaller than $\bar{\varepsilon}\tau^{-1}$. Note that a matched asymptotic expansion representation of $\bar{D}^*$ is not non-uniform in the manner of the acceleration covariance and the leading-order composite approximation is a good representation of it.

The outer expansion for $\bar{D}^*$ has the same form as (11) except that it is proportional to $\bar{\varepsilon} t_L$ and the functions are denoted by say $\mathcal{D}_i$'s. A good representation of $\mathcal{D}_0$ according to Hinze (1975) and Deardorff & Peskin (1970) is

$$\mathcal{D}_0\left(\frac{\tau}{t_L}\right) = 2\left(1 - \exp\left(-\frac{1}{2}\mathcal{C}_0\frac{|\tau|}{t_L}\right)\right) \qquad (16)$$

The literal accuracy of (16) is not important but it shows that there are significant $O(\bar{\varepsilon} t_L^{-1})$ acceleration covariances (by differentiating twice and remembering (12)) and it gives the correct inertial sub-range behaviour as $\tau \to 0$. Note that the linear variation with τ of the velocity covariance in the inertial sub-range when differentiated twice gives a null result, again suggesting the rather special nature of acceleration inertial sub-range.

The limit as $\nu \to 0$ is very illustrative. Then for $\tau \geq 0$

$$R^*(\tau) = \mathcal{C}_0\bar{\varepsilon} t_L^{-1}\delta\left(\frac{\tau}{t_L}\right) + \frac{1}{2}\bar{\varepsilon} t_L^{-1}\mathcal{D}_0''\left(\frac{\tau}{t_L}\right) \qquad (17)$$

where $\delta(\xi)$ is a generalised δ function (Lighthill, 1958). This result is arrived at because $\mathcal{G}_0$ is integrable. (17) is the required form of acceleration covariance when turbulent dispersion is modelled using a Langevin equation (Novikov, 1963).

Figure 1 shows schematically the acceleration covariance and various approximations in the inner and outer regions. Matching is illustrated by the fact that

$$\lim_{\tau/t_\eta \to \infty} \bar{\varepsilon} t_\eta^{-1}\delta_1\mathcal{G}_1 = \lim_{\tau/t_L \to 0} \bar{\varepsilon} t_L^{-1}\mathcal{D}_0''$$

and the non-uniformity by the fact that $\bar{\varepsilon} t_\eta^{-1}\mathcal{G}_0$ clearly underpredicts the covariance for $\tau \gg t_\eta$.

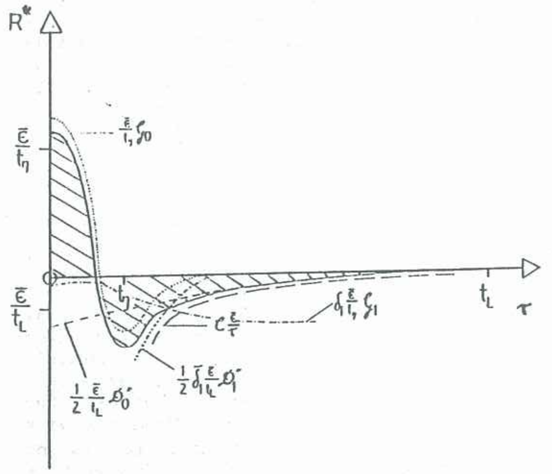

Figure 1. Asymptotic structure of the one-particle acceleration covariance.

Finally, the analysis extends easily to particle displacement statistics and there are no other important ramifications of the kinematic constraints.

TWO-PARTICLE DISPERSION

The cross-acceleration covariance, say R^*_{2ij}, is the relevant generator of, say, the cross-velocity covariance. However, these quantities depend on initial separation which implies two things: firstly, that the acceleration process is not stationary (the particles are less well correlated after being dispersed than initially) and secondly there is a directional bias imposed by the vector $\underline{\Delta}_0$. The latter problem is overcome by considering scalar products of acceleration, velocity, displacement *et cetera*. Then all two-time covariances are functions of t_1 and t_2 as well as the parameters t_η, t_L and $t_0 = (\Delta^2/\bar{\varepsilon})^{1/3}$. t_0 reflects the time-scale for the rate of decorrelation of the cross-statistics. Provided that Δ is much greater than the Kolmogorov length scale, $\eta = (\nu^3/\bar{\varepsilon})^{1/4} = \mathcal{R}^{-3/4}L$, the influence upon the cross-covariances of the parameter t_η is trivial. Now the time-scale t_0 plays the role of the inner scale, while the outer scale is still t_L. Monin & Yaglom (1975) argue that the magnitude of elements of R^*_{2ij} is $O(\bar{\varepsilon}t_0^{-1})$. Thus we can write an inner expansion for R^*_{2ii},

$$R^*_{2ii} = \bar{\varepsilon}t_0^{-1}\mathcal{B}_0\left(\frac{t_1}{t_0},\frac{t_2}{t_0},\frac{t_L}{t_0}\right) + o(\bar{\varepsilon}t_0^{-1}) \qquad (18)$$

where higher-order corrections (including $\mathcal{R}$-dependent terms) are not explicitly considered. Figure 2 shows a schematic of the behaviour represented by (18). An analogous outer expansion could also be written down. However, for the simple goal of showing the importance of the two-particle cross covariance it is sufficient to solely consider the matching region. Note that the standard inertial sub-range has two disjoint domains for two-particle statistics. There is the behaviour for "small" times, $t_1, t_2 \ll t_0$, and "intermediate" times, $t_0 \ll t_1, t_2 \ll t_L$. The intermediate sub-range is our present concern as this represents potentially universal small-scale behaviour independent of the initial conditions. The matching behaviour in the intermediate sub-range requires R^*_{2ii} to have the form

$$R^*_{2ii} \approx \bar{\varepsilon}t_1^{-1}\mathcal{B}\left(\frac{t_1}{t_2}\right) \quad \text{when } t_0 \ll t_1, t_2 \ll t_L \qquad (19)$$

where $\mathcal{B}(\xi)$ is a smooth function such that $\mathcal{B}(\xi) = \xi\mathcal{B}(\xi^{-1})$, the latter for symmetry in the time variables t_1 and t_2. Similarly the mean velocity cross-product, from (4), has a matching form

$$D^*_{2ii} \approx \bar{\varepsilon}t_1\mathcal{D}_2\left(\frac{t_1}{t_2}\right) \quad \text{when } t_0 \ll t_1, t_2 \ll t_L \qquad (20)$$

where $\mathcal{D}_2(\xi)$ is a smooth function such that $\mathcal{D}_2(\xi) = \xi^{-1}\mathcal{D}_2(\xi^{-1})$.

The relationship between the velocity and acceleration (scalar-product) covariance is

$$R^*_{2ii} = \frac{\partial^2}{\partial t_1 \partial t_2} D^*_{2ii} \qquad (21)$$

and substitution of (19) and (20) into (21) gives

$$\mathcal{B}(\xi) = -\left(2\xi^2\frac{d}{d\xi} + \xi^3\frac{d^2}{d\xi^2}\right)\mathcal{D}_2(\xi) \ . \qquad (22)$$

Now in the one-particle case the equivalent substitution of naive matching forms resulted in an inconsistency (the linear term in τ vanished upon twice differentiating). However, there is no such problem evident for two-particle statistics. This is reinforced when (22) is solved for $\mathcal{D}_2$, assuming that $\mathcal{B}$ is given. Then

$$\mathcal{D}_2(\xi) = -\int_0^\xi \left(\chi^{-2} - \xi^{-1}\chi^{-1}\right)\mathcal{B}(\chi)\,d\chi \qquad (23)$$

with the relevant details given by Borgas & Sawford (1989). For $\mathcal{D}_2$ to be well defined by (23) requires that $\mathcal{B}(\xi) \approx \xi^{1+\lambda}$ for $\xi \to 0$, where $\lambda > 0$ is some constant. This result is equivalent to $\mathcal{B}^* \ll \bar{\varepsilon}\tau^{-1}$ as $\tau \to \infty$ and is the analogue of the exclusion of inverse lags from the one-particle acceleration covariance matching structure. However, it is evident from (23) that there is more structure in the intermediate sub-range other than for small ξ and $\mathcal{D}_2$ is generally a significant

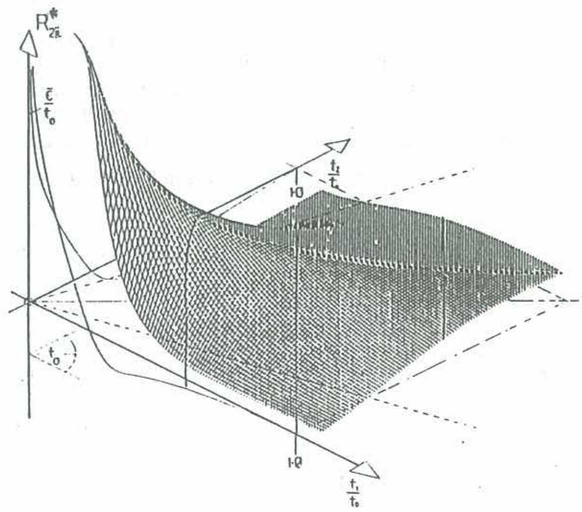

Figure 2. Schematic of the two-particle acceleration covariance.

non-trivial representation of the velocity covariance in the intermediate sub-range. Moreover, in the context of matched expansions, the leading-order matching behaviour is well defined and the expansions for acceleration covariance are uniform (in the sense as above).

In a similar way we may calculate the cross-displacement covariance and the results are summarised below for the particular case of $t_1 = t_2$ thus describing mean-square difference quantities. For velocities

$$\langle u_i^{(1)}(t) u_i^{(2)}(t) \rangle \approx 2\bar{\mathfrak{D}}_0 \bar{\varepsilon} t \,, \quad t_0 \ll t \ll t_L \tag{24}$$

while for displacements

$$\langle x_i^{(1)}(t) x_i^{(2)}(t) \rangle \approx \left(\bar{\mathfrak{D}}_0 - \frac{1}{3}\bar{\mathfrak{D}}_1 \right) \bar{\varepsilon} t^3, \quad t_0 \ll t \ll t_L \tag{25}$$

where $\bar{\mathfrak{D}}_0 = \int_0^1 \xi^{-1} \mathfrak{B}(\xi) d\xi$ and $\bar{\mathfrak{D}}_1 = \int_0^1 \mathfrak{B}(\xi) d\xi$ are both O(1) universal constants. Note that the equivalent one-particle results are $3\mathcal{C}_0 \bar{\varepsilon} t$ and $\mathcal{C}_0 \bar{\varepsilon} t^3$ for velocity and displacement respectively, but where the universal constant $\mathcal{C}_0$ is related to the inner dissipation region (through (15)) and is not an acceleration inertial-range property as are $\bar{\mathfrak{D}}_0$ and $\bar{\mathfrak{D}}_1$. Nevertheless, despite the different interpretations of the universal constants and the vastly different scales of the two acceleration processes the estimates of the intermediate sub-range dispersion are of equal order-of-magnitude in both cases. Therefore, the two-particle contribution must be considered in any account of relative dispersion.

RELATIVE DISPERSION

From (6) and (7) it is evident that the one- and two-particle dispersion effects are simply additive for the relative-dispersion covariance. Here again we only consider inertial sub-range behaviour. From the previous sections it follows that

$$\langle u_i^{(r)}(t)^2 \rangle \approx \begin{cases} 6\mathcal{C}_0 \bar{\varepsilon} t \,, & t_\eta \ll t \ll t_0 \\ (6\mathcal{C}_0 - 4\bar{\mathfrak{D}}_0) \bar{\varepsilon} t, & t_0 \ll t \ll t_L \end{cases} \tag{26}$$

and

$$\langle x_i^{(r)}(t)^2 \rangle \approx \begin{cases} 2\mathcal{C}_0 \bar{\varepsilon} t^3, & t_\eta \ll t \ll t_0 \\ 2(\mathcal{C}_0 - \bar{\mathfrak{D}}_0 + \frac{1}{3}\bar{\mathfrak{D}}_1) \bar{\varepsilon} t^3, & t_0 \ll t \ll t_L \end{cases} \tag{27}$$

where in the inner range, $t_\eta \ll t \ll t_0$ two-particle effects can be shown to be truly negligible. The analysis of Monin & Yaglom (1975) proposes that the initial small-time sub-range extends over the entire inertial range, which is equivalent to the constants $\bar{\mathfrak{D}}_0$ and $\bar{\mathfrak{D}}_1$ both vanishing, i.e. ignoring the two-particle acceleration covariance. However, we believe that this assumption is unwarranted and is not supported by any evidence that we know of. Unfortunately, the simple analysis followed here cannot determine the numerical value of the universal constants (and therefore show that they do not vanish); only some calculation based on the Navier-Stokes equations could accomplish that.

CONCLUSION

We have shown the importance of two-particle effects in relative dispersion. The conditioned one-particle statistics can no longer be thought to adequately represent two-particle statistics, at least in the intermediate part of the inertial sub-range. Thomson (1989) inferred as much from his Langevin-equation modelling of two-particle dispersion and also predicted this as an actual physical property of turbulent dispersion. Our more elaborate analysis supports Thomson's view and lends overall support to the Langevin-equation technique. An important inference of this work is that, since the two-particle acceleration covariance is important for dispersion, the Langevin models should in future adequately model them.

REFERENCES

BATCHELOR, G.K. (1950) The application of the similarity theory of turbulence to atmospheric diffusion. Quart. J. Roy. Meteor. Soc. 76, 133-146.

BORGAS, M.S. & SAWFORD, B.L. (1989) The small-scale structure of acceleration correlations and its role in the statistical theory of turbulent dispersion. Submitted to J. Fluid Mech.

DEARDORFF, J.W. and PESKIN, R.L. (1970) Lagrangian statistics from numerically integrated turbulent shear flow. Phys. Fluids 13, 584-595.

VAN DYKE, M. (1975) Perturbation methods in fluid mechanics. Parabolic Press.

HINZE, J.O. (1975) Turbulence. McGraw-Hill.

LIGHTHILL, M.J. (1958) Introduction to Fourier analysis and generalized functions. CUP.

MONIN, A.S. & YAGLOM, A.M. (1975) Statistical fluid mechanics: Mechanics of turbulence, Vol. 2, MIT Press.

NOVIKOV, E.A. (1963) Random force method in turbulence theory. Soviet Physics JETP, 17, 1449-1454.

THOMSON, D.J. (1989) A stochastic model for the motion of particle pairs in isotropic high Reynolds number turbulence, and its application to the problem of concentration variance. J. Fluid Mech. (In press).