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THE ONSET OF THERMAL CONVECTION IN A RECTANGULAR DUCT FLOW

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ABSTRACT

This study concerns the onset of thermal convection in a viscous fluid subject to flow in a rectangular duct. The onset of thermal convection is analyzed by means of a linear stability analysis in which the flow field perturbations are expanded in sets of complete orthonormal functions satisfying the boundary conditions of the flow field. It was found that if the width and depth of the duct are of the same order of magnitude, then the side walls delay the onset of thermal convection in planes perpendicular to the unperturbed flow velocity vector. Their presence may cause the vertical plane of the unperturbed flow velocity vector to be the most susceptible to the development of perturbations in the flow field. In such a case the onset of thermal convection is represented by overstability. Finite rolls may develop in planes parallel or perpendicular to the unperturbed velocity vector according to the values assumed by the Reynolds number of the unperturbed flow and the ratio between the width and depth of the duct cross section.

INTRODUCTION

This study stems from an interest in simulating the performance of various devices as heat exchangers, mechanical tools, and others, whose flow field is represented as a viscous fluid subject to flow in a rectangular duct, heated from below.

The conventional Rayleigh-Jeffreys problem [Nield, 1967; Veronis, 1968] is schematized by a heated infinite horizontal fluid layer, for which the linear stability analysis provides values of the critical wave number K but does not identify the exact shape of the convection cells. In the present study the Rayleigh-Jeffreys problem represents a particular case of thermal convection in a duct of infinite width when there is no flow.

Several studies [Linden, 1974; Legros et. al., 1977; et al.] investigated both theoretically and experimentally the effect to thermal instability phenomena in an infinite fluid layer subject to shear flow. Such phenomena are typical to various oceanographic and engineering problems, where the infinite viscous layer assumption is valid. The previous studies mentioned above proved, by a linear stability analysis, that all modes normal to the shear direction are stabilized by the shear, and consequentely the preferred mode of instability is that of two dimensional sheets aligned with the shear direction.

However, various mechanical devices often require the consideration of the reference to the lateral confining walls of the duct which may significantly affect the flow field and determine the shape of the convection rolls. Some previous studies, [Davis, 1967; Magen et al., 1975; Edwards, 1988]

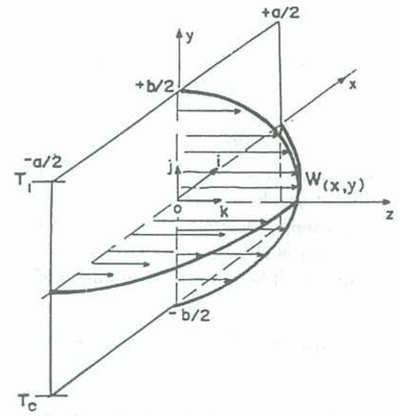

Fig. 1: *Schematic description of the rectangular duct flow*

concerned the analysis of thermal convection in a rectangular box or other types of containers. They referred to a fluid body which is originally at rest. The present study refers to the onset of thermal convection in a rectangular duct flow whose schematic description is given in Figure 1. A finite duct, where the depth and width are of the same magnitude, is considered. We employ the Galerkin method, in which the velocity perturbations are expanded in a set of complete orthonormal functions, satisfying the boundary conditions of the flow field.

THE BASIC EQUATIONS

The basic equations used for the analysis of the flow field instability phenomena are the equations of continuity, motion, heat transfer, and the equation of state, relating temperature variations with density changes. The flow field coordinates and variables are nondimensionalized as follows

$$x^+ = x/b; \quad y^+ = y/b; \quad z^+ = z/b; \quad \mathbf{u}^+ = \mathbf{u}b/\nu;$$

$$T^+ = \frac{(T - T_0)}{\beta_T b}; \qquad \rho_0 = \frac{1}{2}(\rho_b + \rho_t)$$

$$t^+ = t\frac{\nu}{b^2}; \qquad P^+ = (P + \rho_0 g y)\frac{b^2}{\nu^2 \rho_0} \qquad (1)$$

where x is the lateral horizontal coordinate; y is the vertical coordinate; z is the horizontal coordinate in the unperturbed flow direction; b is the depth of the duct; $\mathbf{u}$ is the velocity vector; T is the temperature; ρ is the density; ν is the kinematic viscosity; P is the pressure; g is the gravitational acceleration; t is the time; ρ_0 and T_0 are density and temperature of reference, respectively; ρ_b and ρ_t are the fluid densities at the bottom and top of the duct, respectively; β_T is the temperature gradient.

We assume that the temperatures at the rigid bottom and top of the rectangular duct have constant values T_b and T_t, respectively. The side walls of the duct are rigid and insulated. The duct boundaries determine the temperature and velocity distribution under steady state unperturbed conditions as follows

$$T(y) = T_0 - \beta_T y \; ; \qquad T_0 = \frac{1}{2}(T_b + T_t)$$

$$\mathbf{u} = W(x,y)\mathbf{k} \qquad (2)$$

The velocity distribution $W(x,y)$ is schematically shown in Figure 1.

By applying the Boussinesq approximation, introducing (1) and omitting the plus superscripts, the governing equations are obtained as follows

$$\nabla \cdot \mathbf{u} = 0 \qquad (3)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + RaT\mathbf{j} + \nabla^2\mathbf{u} \qquad (4)$$

$$Pr[\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T] = \nabla^2 T \qquad (5)$$

where $\mathbf{i}$ is a unit vector in the x direction; $\mathbf{j}$ is a unit vector in the y direction; $\mathbf{k}$ is a unit vector in the z direction; ∇ and ∇^2 represent the gradient vector and the Laplacian, respectively; Ra is the thermal Rayleigh number; Pr is the Prandtl number. These parameters are defined as follows

$$Ra = \frac{\alpha_T \beta_T g b^4}{\kappa_T \nu} \; ; \qquad Pr = \frac{\nu}{\kappa_T} \qquad (6)$$

where a is the width of the duct; α_T is the volumetric thermal expansion coefficient linearly relating density with temperature; κ_T is the thermal diffusivity.

Referring to rigid boundaries of the flow field we obtain the following approximate expression for the velocity field in a rectangular duct [Berker, 1963]

$$W(x,y) = 18Re(1+\frac{1}{\tau})[0.25 - \frac{x^2}{\tau^2}][0.25 - y^2] \qquad (7)$$

where $\tau = a/b$ is the ratio between width and depth, respectively; $Re = 4VR_h/\nu = 2Q/(a+b)\nu$ is the Reynolds number of the flow; V is the average flow velocity; R_h is the hydraulic radius of the duct; Q is the duct discharge. This equation is applicable provided that the ratio between a and b is $O(1)$.

THE LINEAR STABILITY PROBLEM

According to the linear stability analysis the flow field in the duct is subject to small disturbances in the velocity $\mathbf{v}$, temperature θ and pressure p. The velocity perturbation is represented as follows

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad (8)$$

Introducing the flow field disturbances into (3)–(5), neglecting second order terms, and eliminating by simple mathematical operations p and w, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}[W\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)] + \frac{\partial W}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial W}{\partial y}\frac{\partial u}{\partial z} = Ra\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial x}\nabla^2 v - \frac{\partial}{\partial y}\nabla^2 u \qquad (9)$$

$$\frac{\partial}{\partial t}\nabla^2 v + \frac{\partial}{\partial z}\left(W\frac{\partial^2 v}{\partial z^2}\right) + \frac{\partial v}{\partial z}\left(\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2}\right) + 2\frac{\partial W}{\partial x}\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial W}{\partial y}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - 2\frac{\partial^2 W}{\partial x \partial y}\frac{\partial u}{\partial z} = \nabla_1^2(Ra\theta - Rs\gamma) + \nabla^4 v \qquad (10)$$

$$Pr[\frac{\partial \theta}{\partial t} + v\frac{\partial T}{\partial y}] + W\frac{\partial \theta}{\partial z} = \nabla^2\theta \qquad (11)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \; ; \qquad \nabla^4 = \nabla^2 \cdot \nabla^2 \qquad (12)$$

Assuming that the appearance of two dimensional rolls represents the onset of thermal and thermohaline convection we follow Davis [1967] and consider that such rolls may develop in the x–y or the y–z plane. Criteria of stability for the x–y plane in the present study are eventually covered by Davis' study, as this plane is not affected by the unperturbed flow. Therefore we refer to the development of convection cells in the y–z plane. Provided that convection rolls are developed in the y–z plane, the flow field perturbations can be expressed by the following normal mode expansion

$$[\,u, v, \theta\,]\,(x,y,z,t) = [\,\bar{u}(x,y), \bar{v}(x,y), \bar{\theta}(x,y)\,]\; e^{(\sigma t + iK_z z)} \qquad (13)$$

where $K_z = K$ is the wave number in the z direction; σ is a complex quantity expressing amplification and oscillations of the flow field perturbations; $\bar{u}(x,y)$, $\bar{v}(x,y)$, $\bar{\theta}(x,y)$ are complex quantities.

Introducing (13) into (10)–(12) and omitting the bar sign from $\bar{u}$, $\bar{v}$, $\bar{\theta}$ we obtain

$$L_0(u) + L_1(v) = RaD_x\theta$$
$$P_0(u) + P_1(v) = Ra\nabla_1^2\theta$$
$$R_0(\theta) = -v \qquad (14)$$

where

$$L_0(u) = [(\nabla^2 - \sigma - iKW)D_y - iKW_y]u$$
$$L_1(v) = [(-\nabla^2 + \sigma + iKW)D_x + iKW_x]v$$
$$P_0(u) = [-2iKW_{xy} - 2iKW_xD_y + iKW_yD_x]u$$
$$P_1(v) = [(\sigma + iKW)\nabla^2 - \nabla^4 + iK(2W_xD_x - W_yD_y + W_{xx} - W_{yy})]v$$
$$R_0(\theta) = [\nabla^2 - Pr(\sigma + iKW)]\theta \qquad (15)$$

where the subscripts $_x$ and $_y$ refer to partial derivatives with regard to x and y, respectively; D_x and D_y are operators defined as $D_x = \partial/\partial x$ and $D_y = \partial/\partial y$, respectively

At the rigid boundaries of the rectangular duct, the constant temperature distribution, the no-slip condition, and the continuity imply, respectively

$$\theta = v = Dv = 0 \quad \text{at} \quad y = \pm 0.5 \text{ and } x = \pm 0.5\tau \qquad (16)$$

We utilize the Galerkin method in order to obtain approximate solutions of the system (14), subject to the boundary conditions (16). Solutions are obtained by expanding $\theta(x,y)$, $u(x,y)$ and $v(x,y)$ in sets of complete orthonormal functions, satisfying the boundary conditions (16). We assume that such functions are represented by Fourier series as follows

$$u(x,y) = \sum_{m=1}^{\infty} U_m \tau^{-1/2} C_m(\frac{x}{\tau})C_m(y);$$
$$v(x,y) = \sum_{m=1}^{\infty} V_m \tau^{-1/2} C_m(\frac{x}{\tau})C_m(y) \qquad (17)$$
$$\theta(x,y) = \sum_{m=1}^{\infty} T_m \tau^{-1/2} E_m(\frac{x}{\tau})E_m(y)$$

where U_m, V_m, T_m, G_m are complex coefficients; $E_m(r)$, $C_m(r)$ are even orthonormal functions defined as follow

$$E_m(r) = \sqrt{2}\cos(2m-1)\pi r$$

$$C_m(r) = \frac{\cosh(\lambda_m r)}{\cosh(\lambda_m/2)} - \frac{\cos(\lambda_m r)}{\cos(\lambda_m/2)} \quad (18)$$

where λ_m are the positive roots of the equation

$$\tanh(\frac{\lambda}{2}) + \tan(\frac{\lambda}{2}) = 0 \quad (19)$$

r is a dummy variable.

Introducing (17) into (14), multiplying these equations by

$\hat{C}_n(x, y) = \tau^{-1/2} C_n(x/\tau) C_n(y)$

and by

$\hat{E}_n(x, y) = \tau^{-1/2} E_n(x/\tau) E_n(y)$,

and integrating over the duct cross section area, we obtain by simple mathematical operations the following matricial expressions of order N, where N is a truncation number

$$[\mathbf{a}] \cdot [\mathbf{U}] + [\mathbf{b}] \cdot [\mathbf{V}] = Ra[\mathbf{c}] \cdot [\mathbf{T}]$$
$$[\mathbf{d}] \cdot [\mathbf{U}] + [\mathbf{e}] \cdot [\mathbf{V}] = Ra[\mathbf{f}] \cdot [\mathbf{T}]$$
$$[\mathbf{g}] \cdot [\mathbf{T}] = [\mathbf{h}] \cdot [\mathbf{V}] \quad (20)$$

Here [**a**], [**b**], [**c**], [**d**], [**e**], [**f**], [**g**] and [**h**] are complex matrices; [**U**], [**V**] and [**T**] are complex vectors.

By some matricial operations the set of equations (20) is transformed to the following characteristic equation of the stability problem

$$[\alpha] \cdot [\mathbf{T}] = Ra[\mathbf{T}] \quad (21)$$

where

$$[\alpha] = [\mathbf{f}]^{-1} \cdot [\mathbf{e}] \cdot [\mathbf{h}]^{-1} \cdot [\mathbf{g}] \quad (22)$$

Equation (21) represents the characteristic equation of onset of thermal convection in the rectangular duct flow.

RESULTS AND DISCUSSION

The critical Rayleigh number is the parameter which determines the onset of thermal convection in the flow field. This parameter depends on the Reynolds number Re, the Prandtl number Pr and the ratio between the width and depth of the rectangular duct τ. For each set of the independent parameters we calculate the eigenvalues of (21) for various values of the wave number K. The minimum eigenvalue obtained in such a calculation is the critical Rayleigh number which is associated with the critical wave number. The latter determines the size of the convection cells.

Verification of the applicability of our approach for the determination of the flow field stability was obtained by compairing our results referring to $Re = 0$ to those of Davis; they concerned thermal convection in a box, provided that one side of the box has an infinite extent. Our results were usually in good agreement with those obtained by Davis' approach.

Figure 2 refers to the onset of thermal convection in the y–z plane when the fluid is originally subject to flow at $Re = 10$. All curves representing the dependence between Ra and K are associated with $\omega \neq 0$, namely overstable motions exist in the flow field which is subject to thermal convection. Such motions are probably attributed to momentum difference between the different fluid layers. The unperturbed flow velocity increases the value of the critical Rayleigh number and decreases the critical wave number. The phenomenon of horizontal section of the curve Ra versus K in the region $0 < K < K_c$ occurs at $\tau \leq 1$.

Figure 3 concerns the effect of an increase in the Reynolds number on the thermal convection in the y–z plane. This Figure indicates that such an increase leads to an increase of the critical Rayleigh number and a decrease of the critical wave number.

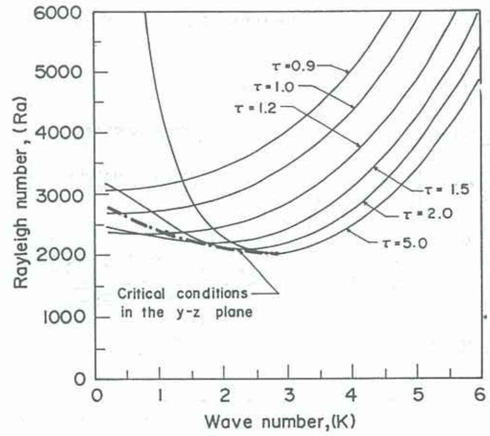

Fig. 2: *The Rayleigh number versus the wave number for various values of* τ *[*$Re = 10$, $Pr = 10$*]*

We also introduce into Figure 3 the value of the critical Rayleigh number in the x–y plane referring to the development of two convection cells in this plane. Comparing critical values of Rayleigh number in the y–z plane and and x–y plane, we can conclude that for $\tau = 2$ and $Re \geq 15$ thermal covection takes place in the x–y plane.

Figure 4 provides some more information concerning the shift of thermal convection from the y–z plane to the x–y plane. This Figure indicates that for originally stagnant fluid, thermal convection always develops in the y–z plane. For large values of τ and sufficiently large Reynolds number the unperturbed flow causes the x–y plane to be the most susceptible to the development of thermal convection.

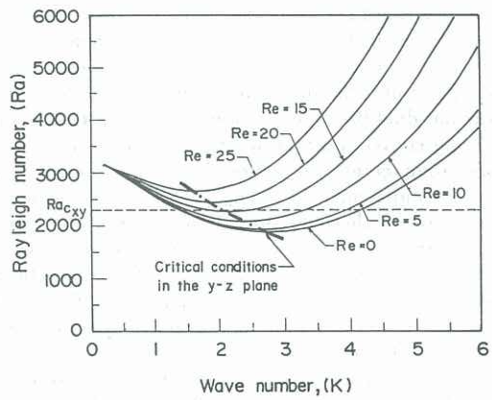

Fig. 3: *The Rayleigh number versus the wave number for various values of* Re *[*$Pr = 10$, $\tau = 2$*]*

The larger is the value of τ the smaller is the value of Reynolds number needed for shifting thermal convection from the y–z plane to the x–y plane. However at values of τ smaller than unity the shifting effect of the Reynolds number vanishes. The increase of Reynolds number decreases the value of the critical wave number, and above a certain value of the Reynolds number the critical wave number eventually vanishes. As shown in Figure 4, the phenomenon of thermal convection in the y–z plane with vanishing values of the wave number may take place provided that τ is smaller than 1.5.

The results represented in the preceding paragraphs imply that the unperturbed flow velocity stabilizes the flow field, and delays the onset of thermal convection in the y-z plane. That flow velocity may cause the convection cells to develop in the x-y plane instead of the y-z plane.

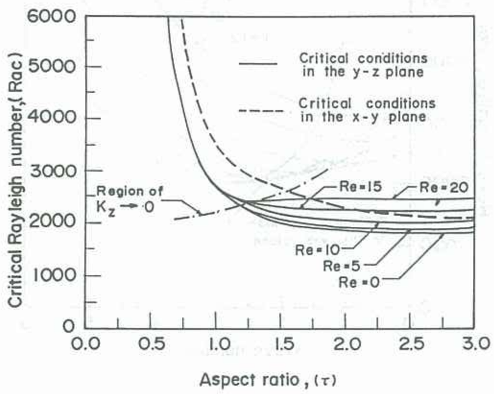

Fig. 4: *The critical Rayleigh number versus τ for various values of Re $[Pr = 10]$*

The larger is the value of τ the smaller is the value of Re needed to cause the shift from thermal convection in the y-z plane to thermal convection in the x-y plane. If $Re \neq 0$ then convection motions developed in the y-z plane are always associated with overstable motions. Such motions are generated by the difference between the momentum typical to different levels at the fluid body. In such a case an increase of Prandtl number means faster dissipation of the momentum perturbation, and stabilization of the flow field.

SUMMARY AND CONCLUSIONS

The onset of thermal convection in a rectangular duct flow was investigated by a linear stability analysis. The analysis led to a system of differential equations which define the point of instability as an eigenvalue problem. In the rectangular duct flow the side walls may suppress the onset of thermal convection according to the mode described in the preceding section, and determine the vertical plane parallel to the unperturbed velocity vector to be most susceptible to onset of thermal convection. Some of the features typical to the phenomenon of onset of thermal convection in a duct flow were determined by calculations of stability referring to that plane. Such calculations indicated that an increase of Reynolds number stabilizes the flow in the narrow duct. Overstability always characterizes the point of instability. Overstable motions are attributed to the differences in momentum between different fluid layers.

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