STEADY FLOW IN SINUSOIDALLY VARYING CHANNELS

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Synopsis – The influence of channel irregularities on rating curves, estimation of resistance coefficients, and the propagation of flood waves suggests that a detailed study of such irregularities would be worth while; in this paper an initial attack is made on the problem by analysing steady flow in channels having irregularities of idealized periodic form. Two cases are treated: channels having sinusoidal variations in (a) bed level; (b) width. The former approximates to the pool- and-riffle formation characteristic of natural rivers. The flow may be termed "quasi-uniform" in the sense that it is not governed by any controls other than those imposed by the bed roughness and slope, i.e. there are no backwater or drawdown curves as those terms are usually understood.

The equation describing the flow is non-linear and non-elementary, but may be linearized and solved by assuming that the variation in width or bed level is small. For large variations, solutions were obtained by numerical methods with the help of a high-speed computer.

An interesting feature of the solutions is the existence of a phase difference between the surface wave form and the channel bed or width wave form. This phase difference is dependent on the discharge; the discharge-depth relation at a given section may therefore differ materially from the uniform-flow relation from which rating curves are normally synthesized.

Other questions resolved are that of the difference between water surface slope and energy slope (which may be considerable) and the criterion determining when, on a mild slope, the irregularities are no longer "drowned" but give rise to a series of critical-flow sections. Because of the phase difference, critical flow does not occur at the section of greatest constriction, but within a quarter-cycle of it.

LIST OF SYMBOLS

a — $2\pi y_0/S_0L$, Case II

b — Channel width

b_0 — Mean channel width

b_1 — Half amplitude of channel width variation

C	Chezy resistance coefficient
F	Froude number, $v/\sqrt{(gy)}$
F_0	Froude number in "mean" channel $= C\sqrt{(S_0/g)}$
g	Acceleration of gravity
H	Total energy of flow, $y+z+v^2/2g$
k	$2\pi y_0(1-F_0^2)/S_0L$, Case I
L	Wavelength of channel variation
Q	Total discharge
q	Discharge per unit width, Q/b
R	Hydraulic mean radius of cross-section
S_0	Bed slope
S_w	Water surface slope $= -d(y+z)/dx$
S_f	Friction slope $= v^2/C^2R$
v	Mean velocity at a section
x	Distance along the channel
y	Depth of water
y_0	Uniform depth in uniform channel
z	Height of bed above datum
z_0	Half amplitude of bed level variation
α_1	Phase angle, bed wave–depth wave, Case I
α_2	Phase angle, width wave–depth wave, Case II
β	Phase angle, bed wave–surface wave, Case I
ε_1	$2\pi z_0/S_0L$
ε_2	Ratio (half amplitude of depth wave)/y_0, Case I
ε_3	b_1/b_0
ε_4	Ratio (half amplitude of depth wave)y^0/, Case II
θ_1	$2\pi x/L-\alpha_1$
θ_2	$2\pi x/L-\alpha_2$

1. INTRODUCTION

Irregularities in natural river channels often introduce doubts about the proper use of uniform flow formulae in connection with such problems as the synthesis of rating curves, the estimation of flood discharge from high water marks, and the estimation of resistance coefficients such as the Manning n from measurements of discharge and slope, which is usually taken as the water surface slope.

When the irregularities take the form of variations in width, they are a major influence in the subsidence of flood waves, since they provide storage which makes the river equivalent to a chain of small lakes. Theoretical approaches have in fact been devised[1, 2] in which the equations of unsteady flow are rewritten as diffusion equations with a diffusion coefficient which depends on channel irregularities. However, this coefficient has not yet been quantitatively related to the geometry of the irregularities.

These applications suggest that a general contribution to the problems involved could be made by analysing steady flow in a channel having irregularities of some idealized, e.g. sinusoidal, form. The results will have immediate application to some steady flow situations occurring in practice, and will also form a preliminary study for a more general attack on the flood routing problem.

Two such cases will be dealt with: first, a channel having constant width and sinusoidal variations in bed level superimposed on a constant slope; and second, a channel of constant bed slope and sinusoidally varying width. The former approximates to the pool-and-riffle formation so characteristic of natural rivers, the latter to the irregularities which provide channel storage for floods. The flow considered is steady and "quasi-uniform" in the sense that it will not be governed by any controls upstream or downstream, but only by the bed roughness and slope. The variations in section are assumed to be gradual, i.e. form losses are negligilbe.

2. ANALYSIS

(a) *Case I. Sinusoidal Slope Variation*

We consider steady flow in a channel of very wide rectangular section, so that the hydraulic mean radius R is equal to the depth y, and of constant width. The longitudinal section is shown, and symbols are defined, in Fig. 1; the profile of the bed has the equation

$$z = -S_0x+z_0 \sin \frac{2\pi x}{L} \tag{1}$$

We use the Chezy resistance equation

$$v = C\sqrt{(RS_f)} \tag{2}$$

where S_f is the energy slope, equal to

$$-\frac{dH}{dx} = \frac{v^2}{C^2y} = -\frac{v}{g}\frac{dv}{dx}-\frac{dy}{dx}-\frac{dz}{dx} \tag{3}$$

Now from (1)

$$-\frac{dz}{dx} = S_0 - \frac{2\pi z_0}{L} \cos \frac{2\pi x}{L}$$

and from the continuity equation

$$\frac{dv}{v} + \frac{dy}{y} = 0$$

we have

$$-\frac{v}{g}\frac{dv}{dx} = \frac{v^2}{gy}\frac{dy}{dx} = F^2 \frac{dy}{dx}$$

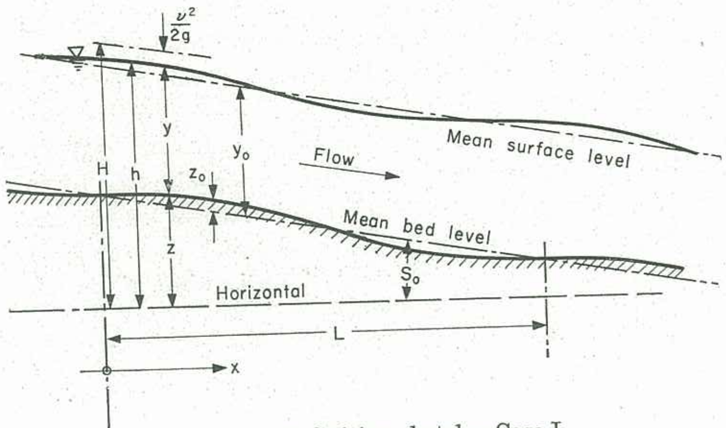

FIG. 1. Definition sketch—Case I.

where F is the Froude number, $v/\sqrt{(gy)}$. Substituting these results into (3), and writing $q = vy$, we obtain

$$\left(\frac{q^2}{C^2y^3} + \frac{2\pi z_0}{L} \cos \frac{2\pi x}{L} - S_0\right)dx + \left(1 - \frac{q^2}{gy^3}\right)dy = 0 \qquad (4)$$

an equation which is not linear and not soluble by elementary methods. However, an approximate solution can be found if we assume that the undulations in the bed are of small amplitude; in particular that the parameter

$$\varepsilon_1 = \frac{2\pi z_0}{S_0 L}$$

is small. The effect is to linearize Eq. (4). Since the flow is not modified by any upstream or downstream controls, i.e. may be described by the term "quasi-uniform" already introduced, there will be a cyclic variation in depth, of wavelength L, corresponding to the cyclic variation in bed level. Since the amplitude of this

oscillation will be small we can assume with sufficient accuracy that it will be sinusoidal, i.e. that

$$y = y_0\left\{1 + \varepsilon_2 \sin\left(\frac{2\pi x}{L} - \alpha_1\right)\right\} \qquad (5)$$

where y_0 is the uniform depth that would occur if the channel were uniform, i.e. $q^2 = C^2S_0y_0^3$. Squares and products of ε_1 and ε_2 will be neglected. We set $\theta_1 = (2\pi x/L - \alpha_1)$. Now

$$\frac{q^2}{C^2y^3} = \left(\frac{y_0}{y}\right)^3 S_0 = S_0(1 - 3\varepsilon_2 \sin \theta_1)$$

and

$$\frac{q^2}{gy^3} = \frac{C^2S_0}{g}\left(\frac{y_0}{y^3}\right)^3 = F_0^2(1 - 3\varepsilon_2 \sin \theta_1)$$

where $F_0 = C\sqrt{(S_0/g)}$ = the Froude number that would occur if there were uniform flow in a uniform channel. It will be assumed that C is constant, as this is consistent with the approximations already adopted; F_0 will therefore depend on channel properties alone, and not on the discharge.

Making the appropriate substitutions in (4), we obtain

$$\frac{dy}{dx} = \frac{S_0\{3\varepsilon_2 \sin \theta_1 - \varepsilon_1 \cos(\theta_1 + \alpha_1)\}}{1 - F_0^2(1 - 3\varepsilon_2 \sin \theta_1)}$$

But from (5), we have

$$\frac{dy}{dx} = \frac{2\pi\varepsilon_2 y_0}{L} \cos \theta_1$$

whence, neglecting the term in ε_2^2,

$$\varepsilon_1 \cos(\theta_1 + \alpha_1) = \frac{2\pi y_0(F_0^2 - 1)}{S_0 L}\varepsilon_2 \cos \theta_1 + 3\varepsilon_2 \sin \theta_1 \qquad (6)$$

Comparing coefficients of $\cos \theta_1$ and $\sin \theta_1$ in this identity, we obtain

$$\varepsilon_1 \cos \alpha_1 = \frac{2\pi y_0(F_0^2 - 1)}{S_0 L}$$

$$\varepsilon_1 \sin \alpha_1 = -3\varepsilon_2$$

whence

$$\tan \alpha_1 = \frac{3S_0 L}{2\pi y_0(1 - F_0^2)} = \frac{3}{k} \qquad (7)$$

and

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{-1}{\sqrt{(k^2+9)}} \tag{8}$$

where

$$k = \frac{2\pi y_0(1-F_0^2)}{S_0 L}$$

This parameter, k, may be used in two different ways as an indicator of the state of the flow. If the channel shape, slope, and

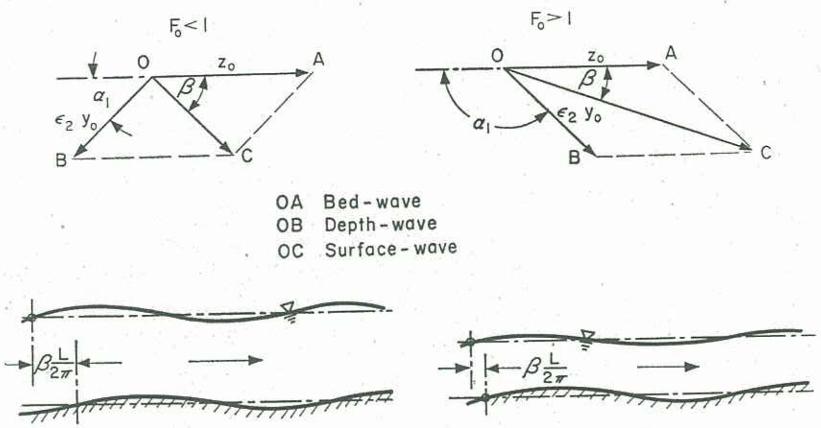

FIG. 2. The compounding of bed-wave and depth wave in Case I.

roughness are all fixed and the discharge is allowed to vary, then k increases with the discharge. Or if the shape, discharge and depth are all fixed and the mean slope S_0 and the roughness allowed to vary, then k becomes an inverse measure of the effects of resistance, becoming infinite when these effects vanish.

When $F_0<1$ (subcritical flow), α_1 is in the first quadrant, and when $F_0>1$ (supercritical flow), α_1 is in the second quadrant. A further phase difference of π is implied in the fact that $\varepsilon_2/\varepsilon_1$ is negative, so the "depth-wave" leads the "bed-wave" by a total phase angle of $(\pi+\alpha_1)$. The wave traced by the water surface can be determined by compounding the bed-wave and depth-wave as vectors, as shown in Fig. 2. The ratio between the half amplitudes $OB=|\varepsilon_2 y_0|$ and $OA=z_0$ is equal to

$$\frac{|\varepsilon_2 y_0|}{z_0} = -\frac{\varepsilon_2}{\varepsilon_1}\frac{2\pi y_0}{S_0 L}$$

$$= \frac{+1}{(1-F_0^2)\sqrt{(1+9/k^2)}}$$

$$= \frac{\cos\alpha_1}{1-F_0^2} \tag{9}$$

so that when resistance effects become very large, and $k\to 0$ so that $\alpha_1\to\pi/2$, the amplitude $|\varepsilon_2 y_0|$ tends to zero, β tends to zero, and surface wave and bed wave are in phase: this is true for both subcritical and supercritical flow. When resistance becomes small, then for $F_0<1$:

$$k\to+\infty, \qquad \alpha_1\to 0, \qquad \beta\to\pi \tag{10}$$

while for $F_0>1$:

$$k\to-\infty, \qquad \alpha_1\to\pi, \qquad \beta\to 0 \tag{11}$$

confirming the well-known results[3] that in flow of a perfect fluid over a sinusoidal bed, the surface wave and bed wave are 180 degrees out of phase for $F_0<1$, and exactly in phase for $F_0>1$. In the latter case, we see that $\beta\to 0$ both when resistance effects are very small and when they are very large: there is therefore a maximum value of β at some intermediate point. From (9) and from Fig. 2, it is readily shown that

$$\tan\beta = \frac{\cos\alpha_1 \sin\alpha_1}{1-F_0^2-\cos^2\alpha_1} \tag{12}$$

and that when $F_0>1$, $\tan\beta$ has a maximum value of

$$\frac{-1}{2F_0\sqrt{(F_0^2-1)}}$$

when $\tan\alpha_1 = -F_0/\sqrt{(F_0^2-1)}$. Even when F_0 exceeds 1 by only a small amount, this maximum value of β is quite close to π.

We can usefully think of the situation as one in which an oscillation in the bed level is forcing an oscillation in the depth; the effect of resistance is to reduce the amplitude of the induced oscillation, as shown by (9), and to introduce a phase shift, as shown by (7). Both these effects are well-known features of forced oscillations. However, this analogy does not extend to the resonance condition, which in the present case is determined by a property of mean flow ($F_0=1$), not by the matching of applied and natural frequencies.

(b) *Case II. Sinusoidal Width Variation*

We now consider steady flow in a channel of uniform slope S_0, and width b given by the equation

$$b = b_0 + b_1 \sin \frac{2\pi x}{L} \tag{13}$$

As before, it is assumed that the Chezy equation is true and the channel is wide and rectangular in section, so that $R = y$. The continuity equation this time takes the form

$$\frac{dv}{v} + \frac{dy}{y} + \frac{db}{b} = 0 \tag{14}$$

whence

$$-\frac{v}{g}\frac{dv}{dx} = \frac{v^2}{gy}\frac{dy}{dx} + \frac{v^2}{gb}\frac{2\pi b_1}{L} \cos \frac{2\pi x}{L}$$

Substituting these results into (3) we obtain

$$\left(\frac{v^2}{C^2 y} + \frac{v^2}{gb}\frac{2\pi b_1}{L} \cos \frac{2\pi x}{L} - S_0\right) dx + \left(1 - \frac{v^2}{gy}\right) dy = 0 \tag{15}$$

corresponding to (4). We set $\varepsilon_3 = b_1/b_0$, and assume as before that the depth y varies sinusoidally, i.e. that

$$y = y_0 \left\{1 + \varepsilon_4 \sin\left(\frac{2\pi x}{L} - \alpha_2\right)\right\} \tag{16}$$

where y_0 is the uniform depth for constant width b_0, i.e. $Q^2 = C^2 b_0^2 y_0^3 S_0$, where Q is the total discharge. If we set $\theta_2 = 2\pi x/L - \alpha_2$, then

$$\frac{v^2}{C^2 y} = \frac{Q^2}{C^2 b^2 y^3} = \frac{Q^2}{C^2 b_0^2 y_0^3}\{1 - 2\varepsilon_3 \sin(\theta_2 + \alpha_2) - 3\varepsilon_4 \sin \theta_2\}$$
$$= S_0\{1 - 2\varepsilon_3 \sin(\theta_2 + \alpha_2) - 3\varepsilon_4 \sin \theta_2\}$$

neglecting squares and products of ε_3 and ε_4. Similar expressions can be obtained for v^2/gb and v^2/gy; when these are substituted into (15) we obtain, setting $F_0^2 = C^2 S_0/g$ as before,

$$\frac{dy}{dx} = \frac{F_0^2(2\pi y \varepsilon_3/L)\cos(\theta_2 + \alpha_2) + S_0\{2\varepsilon_3 \sin(\theta_2 + \alpha_2) + 3\varepsilon_4 \sin \theta_2\}}{1 - F_0^2}$$
$$= \frac{2\pi y_0}{L} \varepsilon_4 \cos \theta_2$$

from (16). Setting $a = 2\pi y_0/S_0 L$, we obtain

$$\varepsilon_3\{F_0^2\, a \cos(\theta_2 + \alpha_2) + 2 \sin(\theta_2 + \alpha_2)\} = \varepsilon_4\{(1 - F_0^2)\, a \cos \theta_2 - 3 \sin \theta_2\} \tag{17}$$

Comparing coefficients of $\cos \theta_2$ and $\sin \theta_2$ leads finally to the results

$$\tan \alpha_2 = \frac{a(2 + F_0^2)}{F_0^2 a^2 (1 - F_0^2) - 6} \tag{18}$$

$$\frac{\varepsilon_4}{\varepsilon_3} = \frac{(F_0^4 a^2 + 4) \sin \alpha_2}{a(2 + F_0^2)} \tag{19}$$

The parameter a has the same general significance as k in Case I; when the effects of resistance become small, or the discharge becomes very great, a tends to infinity, and $\tan \alpha_2$ tends to zero (positive or negative, i.e. α_2 tends to 0 or π, according as F_0^2 is less or greater than unity). This argument verifies well-known results for zero resistance, just as in Case I. Similar deductions may be made about the variation of α_2 with resistance: when $F_0^2 < 1$, then α_2 goes from 0 at zero resistance to π at high resistance; when $F_0^2 > 1$, α_2 goes from π at zero resistance to a minimum of $\tan^{-1}\{-a(2 + F_0^2)/12\}$, and back to π at high resistance.

The angle α_2 thus always lies in the first or second quadrant, and $\varepsilon_4/\varepsilon_3$ is therefore always positive, so that α_2 is the entire phase difference by which the depth-wave leads the width-wave. In this case there is no bed wave to be compounded with the depth-wave, which therefore becomes the water surface wave.

(c) *The Water Surface Slope*

In Case I, this is equal to:

$$-\frac{d(y+z)}{dx} = S_0\left\{1 - \varepsilon_1 \cos \frac{2\pi x}{L} + \frac{\varepsilon_1}{1 - F_0^2} \cos \alpha_1 \cos\left(\frac{2\pi x}{L} - \alpha_1\right)\right\} \tag{20}$$

and in Case II, it is equal to:

$$S_0 - \frac{dy}{dx} = S_0\left\{1 - \frac{\varepsilon_3(F_0^4 a^2 + 4) \sin \alpha_2}{F_0^2 + 2}\right\} \tag{21}$$

In both cases the departure from S_0 is of the order of ε_1 or ε_3; a substantial departure from S_0 could no doubt be detected if the theory could be expanded to cover larger values of ε_1 and ε_3.

(d) *Rating Curves*

We recall that, as noted previously, k and a increase with discharge in a channel of fixed shape, slope, and roughness. This means that in Case I the phase angle β, by which the surface wave lags the bed wave, goes from 0 to π as the discharge goes from zero to infinity. The effect, as shown in Fig. 3, is that as the discharge increases the water surface may rise more slowly (as at section I) or more quickly (as at section II) than in a uniform channel, with a corresponding difference in the form of any rating curves taken at those sections.

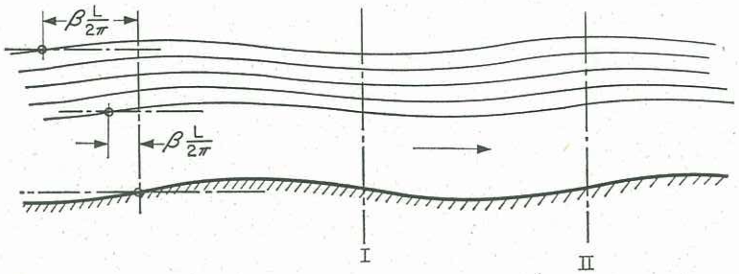

FIG. 3. Change of phase lag β with increasing stage, $F_0 < 1$, Case I.

The situation can be stated algebraically by combining Eqs. (5) and (8):

$$\frac{y}{y_0} = 1 - \frac{\varepsilon_1}{\sqrt{(k^2+9)}} \sin\left(\frac{2\pi x}{L} - \alpha_1\right) \tag{22}$$

The resulting equation indicates how, for a given x, the ratio y/y_0 varies with k and α_1, i.e. with the discharge. The corresponding equation for the width-variation case is:

$$\frac{y}{y_0} = 1 + \frac{\varepsilon_3(F_0^4 a^2+4)}{a(F_0^2+2)} \sin \alpha_2 \sin\left(\frac{2\pi x}{L} - \alpha_2\right) \tag{23}$$

and in both cases the departure of y from y_0 is of the order of ε_1 or ε_3. Again, if the theory is to predict substantial departures of y from y_0, i.e. substantial anomalies in the form of rating curves, it must be able to deal with larger values of ε_1 and ε_3.

(e) *"Drowning" of Irregularities in Subcritical Flow*

When $F_0^2 < 1$, as it usually is in natural rivers, it does not follow, either in Case I or Case II, that the flow will be subcritical throughout the whole channel. It may be that F_0^2 is large enough, or the variations in the cross-section strong enough, for critical flow to occur at one point in every cycle, with supercritical flow downstream followed by a hydraulic jump. In this case, one would expect the critical section to be near the section of greatest constriction.

Considering Case I first, we see from Eq. (4) that if there is a section where $F^2 = 1$ (and dy/dx finite) then at this section

$$\frac{q^2}{C^2 y^3} + \frac{2\pi z_0}{L} \cos \frac{2\pi z}{L} - S_0 = 0,$$

i.e.

$$\cos \frac{2\pi x}{L} = \frac{S_0 - q^2/C^2 y^3}{2\pi z_0/L} = \frac{1 - 1/F_0^2}{\varepsilon_1} \tag{24}$$

since $q^2/C^2 y^3 = g/C^2 = S_0/F_0^2$. For a solution of (24) to be possible it is necessary that

$$\varepsilon_1 \geq \left|1 - 1/F_0^2\right|$$

i.e.

$$F_0^2 \geq \frac{1}{1+\varepsilon_1}, \quad \text{or} \quad \varepsilon_1 \geq \frac{1-F_0^2}{F_0^2} \tag{25}$$

In the limiting case, where $F_0^2 = 1/(1+\varepsilon_1)$, then $2\pi x/L = \pi$, and the critical section is at I, Fig. 3, a quarter-cycle downstream from the section of maximum constriction.

In Case II, we obtain from (15) the equation

$$\frac{v^2}{C^2 y} + \frac{v^2}{gb} \frac{2\pi b_1}{L} \cos \frac{2\pi x}{L} - S_0 = 0$$

i.e.

$$\frac{1}{F_0^2} = 1 - \frac{2\pi y}{S_0 L} \frac{\varepsilon_3 \cos(2\pi x/L)}{1+\varepsilon_3 \sin(2\pi x/L)}$$

or

$$\frac{\cos(2\pi x/L)}{\{1+\varepsilon_3 \sin(2\pi x/L)\}^{5/3}} = \frac{F_0^2 - 1}{a\varepsilon_3 F_0^{8/3}} \tag{26}$$

The expression on the left of this equation must therefore be negative; it can be shown to possess a minimum value, $-M$, when

$$\sin \frac{2\pi x}{L} = \frac{3-\sqrt{(9+40\varepsilon_3^2)}}{4\varepsilon_3} \tag{27}$$

then for (26) to be soluble, it is necessary that

$$a\varepsilon_3 \geq \frac{1-F_0^2}{F_0^{8/3}M} \tag{28}$$

In the limiting case when $a\varepsilon_3 F_0^{8/3}M = 1-F_0^2$, then $2\pi x/L$ will be in the third quadrant, and the critical section will be somewhat upstream from the section of maximum constriction.

The inequalities (25) and (28) given the conditions which must be observed for critical flow to exist somewhere in the channel; otherwise the irregularities will be "drowned" and the flow will be subcritical throughout. If critical sections exist, their positions are given by (24) and (26); it is clear from these equations that as F_0^2 increases the critical sections move upstream in Case I and downstream in Case II until when $F_0^2=1$ they are at the section of maximum constriction.

The results of this section are true for all values of ε_1 and ε_3 (subject to the limitation that ε_3 cannot, physically, be greater than unity). They are independent of the results obtained from the approximate, or linear, theory applying to small values of ε_1 and ε_3.

3. NUMERICAL RESULTS AND APPLICATIONS

Solutions of (4) and (15) were obtained on a computer for a number of values of the parameters ε_1, ε_3, etc., in particular for the larger values which are not covered by the linear theory.

For Case I, the equations could be rewritten in terms of three dimensionless numbers—ε_1, y_0/z_0, and F_0^2. The parameter

$$k = \frac{2\pi y_0}{S_0 L}(1-F_0^2) = \frac{y_0}{z_0}\varepsilon_1(1-F_0^2)$$

is a particular combination of these three numbers which occurs only in the linear theory, and has no significance outside of that theory.

For Case II, the appropriate dimensionless numbers are ε_3, a, and F_0^2, which have already been defined.

Figure 4 shows typical Case I results—for $\varepsilon_1=1$ and $F_0^2=0{\cdot}2$. The value of ε_1 may be thought of as fairly large, implying as it does that the minimum value of the bed slope

$$-\frac{dz}{dx} = S_0\left(1-\varepsilon_1 \sin \frac{2\pi x}{L}\right)$$

is zero (when $x=0$). Larger values of ε_1 imply the existence of adverse

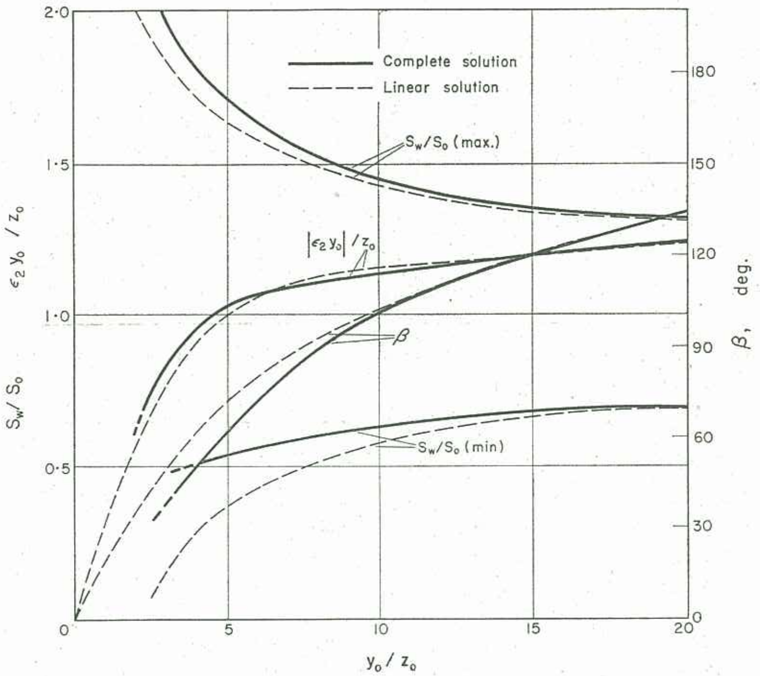

FIG. 4. Results of complete and linear solutions for Case I, $\varepsilon_1 = 1$, $F_0^2 = 0{\cdot}2$.

bed slopes at certain sections, i.e. of pronounced pool-and-riffle formation.

However, the value taken for ε_1 is large enough to be apparently beyond the scope of the linear theory; it is interesting therefore to note that the results of the complete theory (full lines) are not always greatly different from the results of the linear theory (broken lines). Four parameters are plotted against y_0/z_0—the maximum and minimum values of the ratio

$$\frac{\text{Water surface slope, } S_{w'} = -\dfrac{d(y+z)}{dx}}{\text{Bed slope, } S_0}$$

—the phase lag β (Fig. 2), and the amplitude ratio $|e_2 y_0|/z_0$.

Figure 5 shows the variation of depth over one cycle for a typical Case II result—$\varepsilon_1 = 0{\cdot}5$, $a = 3$, $F_0^2 = 0{\cdot}2$. Three curves are plotted—the complete theory, the linear theory, and the case where the resistance slope S_f is assumed to be equal to S_0.

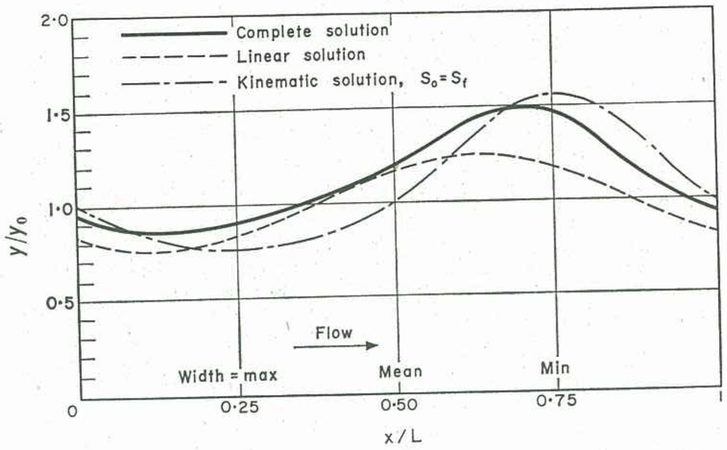

FIG. 5. Longitudinal profiles for Case II, $\varepsilon_3 = 0{\cdot}5$, $a = 3$, $F_0^2 = 0{\cdot}2$.

(a) *Flood Routing*

The case $S_0 = S_f$ is included because of the relevance of the situation to the flood routing problem: in the treatment of this problem it is common to adopt the approximation $S_f = S_0$ if the slope is steep enough, and the corresponding flood wave has been termed a "kinematic" wave. The approximation is valid if the other slope terms in the equations—of which dy/dx is typical—are small compared with S_0, and we may accordingly use the size of the ratio $(dy/dx)/S_0$ as a criterion for the validity of the approximation.

Contributions to the term dy/dx are made by the flood wave itself; however, we are concerned here with the contributions made by channel irregularities. The curves of S_w/S_0 in Fig. 4 indicate that dy/dx may be of considerable size in Case I; the same is true of Case II, as may be shown by examining the linear solution for

the case shown in Fig. 5. The maximum value of dy/dx is equal to $2\pi y_0 \varepsilon_4 / L$, so that

$$\frac{(dy/dx)\text{ max}}{S_0} = a\varepsilon_4$$

In this case $a = 3$, $\varepsilon_4 = 0{\cdot}31$, so that

$$\frac{(dy/dx)}{S_0} = 0{\cdot}93 \qquad (29)$$

a value which seems large enough to invalidate the "kinematic" approximation. However, it is interesting to note from Fig. 5 that the kinematic curve is closer to that of the complete solution than is the curve of the linear solution.

The question of real interest for the flood routing problem is how the amount of water stored in a given reach varies with the discharge, and it is in terms of that question that the differences between the curves of Fig. 5 are significant. These differences have implications of some interest in flood routing theory, but they are beyond the scope of this paper.

(b) *Rating Curves; Water Surface and Energy Slopes*

These questions can be discussed in the light of the results set out in Table 1 for two Case II situations, showing the variation of depth and slopes over a complete cycle.

The first question to be considered is the effect on the rating curve which is indicated in Fig. 3 for Case I. The figures in Table 1 indicate that this effect is slight: at the section $x = 8L/12$, where the effect is most pronounced, the ratio y/y_0 drops only from 1·458 to 1·411 as a (proportional to y_0) rises from 3 to 5. This means that $y/y_0 \propto y^{-0{\cdot}054}$, i.e. that

$$Q \propto y_0^{1{\cdot}5} \propto y^{1{\cdot}58} \qquad (30)$$

The slope of the rating curve therefore differs very little from the value it would have in a uniform channel; examination of the computed results for Case I, and for other values of the parameters in Case II, leads to the same conclusion.

The variation of the water surface slope, however, is a much more pronounced effect. There are large variations along the channel at the same discharge, and with varying discharge at the same section.

TABLE 1. RESULTS FOR CASE II (WIDTH VARIATION); $\varepsilon_3 = 0{\cdot}5$, $F_0^2 = 0{\cdot}2$.

		$a = 3$				$a = 5$			
Width	$12\frac{x}{L}$	$\frac{y}{y_0}$	$\frac{S_w}{S_0}$	$\frac{S_f}{S_0}$	$\frac{S_f}{S_w}$	$\frac{y}{y_0}$	$\frac{S_w}{S_0}$	$\frac{S_f}{S_0}$	$\frac{S_f}{S_w}$
Mean	0	0·929				0·957			
			1·487	1·154	0·776		1·419	0·990	0·698
	1	0·844				0·913			
			1·052	0·949	0·902		0·843	0·723	0·858
	2	0·835				0·930			
			0·702	0·736	1·049		0·516	0·539	1·045
Max	3	0·887				0·980			
			0·473	0·577	1·220		0·339	0·446	1·316
	4	0·979				1·050			
			0·345	0·504	1·461		0·238	0·432	1·816
	5	1·094				1·129			
			0·286	0·521	1·822		0·164	0·499	3·043
Mean	6	1·218				1·217			
			0·268	0·641	2·392		0·075	0·689	9·21
	7	1·346				1·314			
			0·359	0·882	2·457		0·070	0·945	13·55
	8	1·458				1·411			
			0·902	1·125	1·247		0·781	1·231	1·576
Min	9	1·475				1·434			
			1·923	1·309	0·681		2·409	1·408	0·585
	10	1·314				1·287			
			2·263	1·365	0·603		2·901	1·422	0·490
	11	1·094				1·088			
			1·941	1·302	0·670		2·245	1·262	0·562
Mean	12	0·929				0·957			

The same is true of the energy slope $S_f = v^2/C^2 y$, and the slope ratio S_f/S_w; the conclusion suggested is that the water surface slope S_w and bed slope S_0 would be very poor measures of the energy slope for the purpose of, say, deducing a value of Manning's n.

The example chosen is an extremely variable channel—in that the maximum width is three times the minimum—and it may be thought unlikely that any reach in such a channel would be seriously considered as a gauging site. However there are in mountainous rivers many examples of short straight gorges having much wider sections upstream and downstream, and such gorges are often chosen as gauging sites—because of easy access, for example.

However, these gorges would have flow characteristics broadly similar to the constricted section in Case II, and here, as Table 1 shows, S_w and S_0 are widely at variance with S_f and with each other. In particular, Table 1 shows that as the flow increases the point of minimum S_w moves downstream into the section of minimum width: the consequent reduction in water surface slope with increasing discharge can in fact be observed in narrow gorges of the type mentioned above.

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