Constraints on Capacity in a Multi-user Channel

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I. INTRODUCTION

In multi-user Shannon theory constraints are imposed on the sum of the bit rates of users. For each subset of users, there is an associated constraint. We define a notion of tightest constraints and provide results about the structure of the sets of users that correspond to tightest constraints. We consider an application to mobile radio and power control.

II. THE RESULTS

Consider a discrete-time, memoryless multi-user channel with \( M \) users,

\[ (X_1, X_2, \ldots, X_M) \rightarrow Y. \]

If user \( i \) achieves \( R_i \) bits/\( s \) and \( \mathcal{L} \) is a subset of users, then multi-user Shannon theory ([1], [4]) imposes the bound

\[
\sum_{i \in \mathcal{L}} R_i \leq I[(X_i)_{i \in \mathcal{L}}; Y | (X_j)_{j \in \mathcal{L}'}],
\]

where \( I \) is mutual information. Here, we assume a particular choice \( Q_1, Q_2, \ldots, Q_M \) of input distributions for the independent variables \( X_1, X_2, \ldots, X_M \).

Assume the rate vector \( R \) satisfies the constraints given in (1). We say that a constraint corresponding to a subset \( \mathcal{L} \) of users is tight, and that \( \mathcal{L} \) is a tight constraint set of users, if there is inequality in (1) for that choice of \( \mathcal{L} \).

Theorem 1 Suppose that \( \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_J \) are tightest constraint sets given a fixed vector of rate requirements \( R = (R_1, R_2, \ldots, R_M) \). Let \( \mathcal{L} \) be defined by \( \mathcal{L} \equiv \bigcup_{j=1}^{J} \mathcal{L}_j \). Then for any subset \( S \) of \( \{1, 2, \ldots, J\} \) we have that

(i) \( \bigcup_{j \in S} \mathcal{L}_j \) is a tightest constraint set,

(ii) \( \bigcap_{j \in S} \mathcal{L}_j \) is either \( \emptyset \) or is a tightest constraint set.

Proof See [3]. The theorem is stated less generally in [3] but the proof is valid for the result stated above.

We remark that similar arguments to those given in [3] are employed in [5] to show that the set \( \{1, 2, \ldots, M\} \) is a tight constraint in a permutation invariant symmetric channel.

The following theorem applies to an additive real-valued channel, and provides more structure than does Theorem 1. We have only considered the additive model (2), but the result may hold more generally.

Theorem 2 Suppose that \( X_i \) has strictly positive density function \( q_i \) for \( i = 1, 2, \ldots, M \) (i.e. \( q_i(x) > 0 \) for all \( x \in \mathbb{R} \)) and \( Z \) has a strictly positive density. Let \( \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_J \) be tightest constraint sets for the model

\[ Y = \sum_{i=1}^{M} X_i + Z \]

given a fixed vector of rate requirements \( R = (R_1, R_2, \ldots, R_M) \). Then there exists a permutation \( (n_j)_{j=1}^{J} \) of \( \{1, 2, \ldots, J\} \) such that

\[ \mathcal{L}_{n_1} \subseteq \mathcal{L}_{n_2} \subseteq \ldots \subseteq \mathcal{L}_{n_J}. \]

A counter-example in [3] shows that the nesting result is not true for all multi-user channels (the counter-example involves discrete alphabets).

III. AN APPLICATION TO MOBILE RADIO

Power constrained Gaussian channels are of the most interest, since the region (1) defined with Gaussian \( Q_i \) is the set of all achievable rates for the channel, if \( Q_i \) has the maximum variance (power) allowed for user \( i \). A Gaussian, multi-user, multi-receiver model of mobile radio is considered in [2], [3] and [6]. We use Theorem 1 to show that if the network is operated efficiently then the overall constraint set \( \{1, 2, \ldots, M\} \) must be tight. If a user is not in a tightest constraint set then it can reduce power without sacrificing capacity. Theorem 1 shows that the overall constraint is tight when all users are in a tightest constraint set. This result illustrates the fact that power control is fundamental to the efficient operation of a cellular network, and provides a necessary condition for optimality.

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