

# Cooperative Networked Stabilisability of Linear Systems with Measurement Noise

Girish N. Nair and Robin J. Evans

**Abstract**— This paper investigates the problem of stabilising a linear, time-invariant plant with multiple controllers and noisy sensors over a digital network. A necessary and (almost) sufficient condition for determining networked uniform stabilisability, is derived, in terms of the feasibility of a set of linear inequalities involving the unstable eigenvalues of the plant and the various channel data rates. This provides a nearly exact characterisation, up to boundary points, of the region of all channel data rate combinations that permit uniform stability to be achieved. The auxiliary variables in this characterisation have a natural interpretation as the effective rates of information flow through the network, associated with each unstable mode. When channel rates are set to either zero or infinity, this agrees with a classical result on decentralised stabilisability under linear time-varying control.

## I. INTRODUCTION

The field of quantised or communication-limited control has developed rapidly in recent years, due to the emergence of various applications in which sensors and controllers communicate over channels with limited bit rates; see for instance [1] and the references therein. The data transmission limitations in these applications have a significant effect on achievable controller performance, for instance making closed-loop stability impossible if the feedback bit rate is too low. The analysis and design of such systems must thus consider the communication and the control aspects jointly, rather than in isolation.

With regard to the fundamental theory underlying these systems, advances have been made for plants with single controllers and sensors, in terms of determining minimum feedback data rates for stabilisability [2], [3], [4], [5], characterising general performance trade-offs [6], [7], [8], and designing schemes with guaranteed performance [9], [10], [11]. This paper attempts to extend the fundamental stabilisation limits for centralised plants to situations with multiple sensors and actuators. In particular, this question is posed: given a deterministic linear plant with multiple controllers and noisy sensors communicating over some digital network, how can one characterise all combinations of channel data rates which permit the existence of stabilising, decentralised coding and control schemes? This is the control-theoretic analogue of the problem of multiterminal data compression in information theory [12]. The case of noiseless plants with multiple sensors and one controller was considered in [13], in which a scheme based on *Slepian-Wolf* coding ideas [14] was proposed and shown to achieve stability under certain

conditions on the channel data rates. In [15], a set of data rate inequalities which were both necessary and sufficient for the stabilisability of such systems were subsequently derived.

In the paper [16], the case of multiple sensors and controllers was considered, leading to separate necessary and sufficient conditions for stabilisability, which coincided only under restrictive assumptions on the plant controllability and observability structures. The derivation of a tight condition for such systems is difficult because of the possibility of *indirect signalling* through the plant. An exact characterisation of the exponentially stabilising rate region for a noiseless plant with multiple sensors and controllers was presented in [17].

This paper extends these results to address the case of multiple controllers and sensors with bounded measurement noise. A necessary and almost sufficient condition for uniform stabilisability is derived, in terms of the feasibility of a finite number of linear inequalities involving the unstable open-loop eigenvalues of the plant, the various channel data rates and the graph cycles induced by the controllability and observability structure of the plant. This provides a nearly exact characterisation of the region of those channel data rate combinations that permit stability to be somehow achieved. The auxiliary variables introduced in the feasibility condition have a natural interpretation as the effective rates of information flow through the network, each associated with a plant dynamical mode. In the classical limit, with channel rates either zero or infinite, the condition reduces to a known result on decentralised stabilisability with linear time-varying controllers [18].

In the next section, the problem is formulated. The main result is given in section III, along with some graph theoretic notions. The remainder of this paper is a sketch of the proof of this result. Necessity is shown by analysing uncertainty volumes, while sufficiency is established by constructing a scheme which makes explicit the three-fold signalling, estimation and stabilisation tasks of the control policy.

## II. FORMULATION

Certain notational conventions are used in this paper. Sequences  $\{s(j)\}_{j=0}^k$  are written as  $\tilde{s}(k)$ , with  $\tilde{s}(-1)$  being the empty sequence. Positive integers are denoted by  $\mathbb{N}$ , nonnegative integers by  $\mathbb{Z}_{\geq 0}$  and the reals by  $\mathbb{R}$ . Lebesgue measure on  $\mathbb{R}^r$  is written  $\lambda_r$ . For a vector  $\mathbf{x} \in \mathbb{R}^n$ ,  $\|\mathbf{x}\|$  denotes its  $\infty$ -norm  $\max_{1 \leq h \leq n} |x_h|$ .

Consider a discrete-time, linear time-invariant (LTI) plant

G. N. Nair is the corresponding author. Both authors are with the Dept. Elect. & Electr. Eng., Uni. Melbourne, VIC 3010, Australia. E-mails: {g.nair, r.evans}@ee.unimelb.edu.au

with  $U$  actuators and overall state evolution given by

$$\mathbf{x}(k+1) = \mathbf{J}\mathbf{x}(k) + \sum_{i=1}^U \mathbf{B}_i \mathbf{u}_i(k) \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (1)$$

Each signal  $\mathbf{u}_i(k) \in \mathbb{R}^{m_i}$  is the input applied by the  $i$ -th actuator. The initial state  $\mathbf{x}(0)$  is not known *a priori*, but may take any values in a bounded set  $\mathcal{X}_0 \subset \mathbb{R}^n$  with nonzero Lebesgue measure. The outputs of the plant are observed at  $Y$  sensor nodes,

$$\mathbf{y}_j(k) = \mathbf{C}_j \mathbf{x}(k) \in \mathbb{R}^{p_j} + \mathbf{w}_j(k) \in \mathbb{R}^{p_j}, \quad \forall j \in [1, \dots, Y], k \in \mathbb{Z}_{\geq 0}, \quad (2)$$

where the measurement noise  $\mathbf{w}_j(k)$  can take any value in a bounded set  $\mathcal{W}_j \ni 0$ ,  $\forall k \in \mathbb{Z}_{\geq 0}$ ,  $j \in [1, \dots, Y]$ .

Note that the plant may be transformed via an appropriate similarity matrix to put  $\mathbf{J}$  into *real Jordan form* ([19]), without changing its inputs and outputs. Thus, *without loss of generality* (w.l.o.g.), it may be assumed that the state and model parameters of (1)–(2) are with respect to the transformed coordinates. It is assumed that the plant is reachable and observable with respect to the overall input matrix  $[\mathbf{B}_1, \dots, \mathbf{B}_U]$  and output matrix  $[\mathbf{C}_1^T, \dots, \mathbf{C}_Y^T]^T$ . For simplicity, it is also assumed that  $\mathbf{J}$  has real, distinct eigenvalues and hence is diagonal; the extension to more general Jordan forms is possible but not straightforward.

Suppose that plant information is processed by and exchanged between  $N$  geographically distributed *nodes*, consisting of  $Y$  sensors,  $U$  actuators and  $R$  relays. Relay nodes neither observe plant outputs nor apply control inputs, but simply process and pass data between other nodes. Note also that a sensor may also be an actuator, to allow for the possibility of local feedback. Each  $q$ -th node transmits data to every other  $r$ -th node via a uni-directional digital channel ( $q \rightarrow r$ ), onto which a symbol  $s_{q,r}(k)$ , from a finite alphabet  $\mathcal{S}_{q,r}(k)$  of time-varying cardinality  $|\mathcal{S}_{q,r}(k)| = \mu_{q,r}(k) \in \mathbb{N}$ , is transmitted at time  $k$ . It is assumed that each transmitted symbol is received without error, following the standard assumption in distributed source coding, and with unit propagation delay<sup>1</sup> The (*average*) *data rate* of the channel is defined as

$$R_{q,r} = \liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} \log_2 \mu_{q,r}(k) \text{ bits/sample}. \quad (3)$$

If no information is sent from the  $q$ -th to  $r$ -th node then  $\mu_{q,r}(k) = 1$  and  $R_{q,r} = 0$ , whilst perfect communication between a pair of nodes corresponds to  $\mu_{q,r}(k), R_{q,r} = \infty$ , i.e. unconstrained channel rates. Define the  $N \times N$  matrices of channel symbols and rates

$$\mathbf{S}(k) := [s_{q,r}(k)]_{1 \leq q,r \leq N}, \quad \mathbf{R} := [R_{q,r}]_{1 \leq q,r \leq N}$$

with the convention that diagonal elements  $s_{q,q}(k), R_{q,q} = 0$ .

The coding and control laws used by the nodes are now defined. In order to focus on the fundamental limitations on stabilisability arising from constrained data rates, no

<sup>1</sup>The formulation and results here can be extended to handle arbitrary, constant propagation delays.

structural or computational restrictions are imposed apart from causality. Thus if the  $q$ -th node is actually the  $j$ -th sensor, the symbol it transmits at time  $k$  to the  $r$ -th node may generally depend in a time-varying way on all past and present local measurements, and all past symbols it received,

$$s_{q,r}(k) \equiv \gamma_{q,r}(k, \tilde{\mathbf{y}}_j(k), \tilde{\mathbf{s}}_q(k-1)) \in \mathcal{S}_{q,r}(k), \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (4)$$

If the  $q$ -th node is a relay, then its coding rule for transmission to the  $r$ -th node simplifies to

$$s_{q,r}(k) = \gamma_{q,r}(k, \tilde{\mathbf{s}}_q(k-1)) \in \mathcal{S}_{q,r}(k). \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (5)$$

Similarly, if the  $r$ -th node is the  $i$ -th actuator, then the control signal it applies can depend in a time-varying way on all symbols it received up to time  $k$ ,

$$\mathbf{u}_i(k) \equiv \delta_r(k, \tilde{\mathbf{s}}_r(k-1)) \in \mathbb{R}^{m_i}, \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (6)$$

If the  $r$ -th node is also the  $j$ -th sensor, then an additional argument  $\tilde{\mathbf{y}}_j(k)$  should be incorporated on the right-hand side (RHS).

The *cooperative networked coder-controller (CoNC)* is defined as the time sequence  $\mathbf{C}$  of ordered tuples of all channel alphabets, encoder mapping sequences and controller mapping sequences. As mentioned in the introduction, the objective of this paper is to find the region  $\mathcal{R}$  of all channel rate matrices  $[R_{q,r}]_{1 \leq q,r \leq N}$  at which there exists a CoNC which stabilises the plant in the uniform bounded sense

$$\sup_{k \in \mathbb{Z}_{\geq 0}, \mathbf{x}(0) \in \mathcal{X}_0, \mathbf{w}_j(k) \in \mathcal{W}_j, j \in [1, \dots, Y]} \|\mathbf{x}(k)\| < \infty. \quad (7)$$

The main contribution of this paper is an exact characterisation of this region, up to boundary points.

### III. MAIN RESULT

Before presenting the main result, it is necessary to introduce certain concepts pertaining to the controllability and observability structure of the plant and the graph *cycles* it generates.

Recall from the previous section that the plant state dynamics are already in real Jordan form. Hence the open-loop *modes* of the plant are just the components  $x_h \in \mathbb{R}$ ,  $h = 1, \dots, n$  of the state vector, with dynamics governed by the corresponding eigenvalue  $\eta_h$  (or pair of complex conjugate eigenvalues  $\eta_h, \eta_h^*$ ). Previous literature on centralised coding and stabilisation (see, e.g. [4]) suggests that, in so far as stability is concerned, the dominant sources of information (or more precisely, uncertainty) are the unstable open-loop modes of the plant. Anticipating that the open-loop modes remain important in the networked scenario, define the indicators

$$d_{i,h} := \begin{cases} 1 & \text{if } h\text{th row of } \mathbf{B}_i \text{ is } \neq 0 \\ 0 & \text{otherwise} \end{cases}, \quad i \in [1, \dots, U], \quad (8)$$

$$e_{h,j} := \begin{cases} 1 & \text{if } h\text{th column of } \mathbf{C}_j \text{ is } \neq 0 \\ 0 & \text{otherwise} \end{cases}, \quad j \in [1, \dots, Y] \quad (9)$$

In other words,  $d_{i,h}$  indicates whether or not the  $i$ th input feeds into the  $h$ th open-loop mode and  $e_{h,j}$ , whether or not the  $h$ th mode affects the  $j$ th output. The reachability

and observability of the overall plant are equivalent to the existence, for each mode  $h$ , of input and output indices  $i, j$  such that  $d_{i,h}, e_{h,j} = 1$ .

Next, define an *irreducible*  $x_h$ -cycle  $c$  to be any finite sequence of nodes or modes such that

- 1) The first and last element is  $x_h$ , but every other element in the sequence occurs only once.
- 2) Any other dynamical mode  $x_f$ ,  $f \neq h$ , in the sequence must be followed by a sensor node which can observe it, i.e. with  $e_{f,j} = 1$ , and preceded by a controller node which can affect it, i.e. with  $d_{i,f} = 1$ .

Note that an  $x_h$ -cycle that passes through intermediate modes, not just network nodes, corresponds to a signalling path that transits through the plant. Denote the set of all irreducible  $x_h$ -cycles by  $\mathcal{C}_h$ . It is clear that  $|\mathcal{C}_h| < \infty$ , since the number of possible nodes and modes is finite and repetition within a cycle is not allowed. With some abuse of notation, the channel from the  $q$ -th to the  $r$ -th node is denoted by  $(q \rightarrow r)$ , and the statement  $(q \rightarrow r) \in c$  means that the channel occurs in cycle  $c$ . The main result of this paper can now be stated.

*Theorem 1:* Suppose a cooperative networked controller (CoNC) (4)–(6) stabilises the noisy plant (1)–(2) in the uniformly bounded sense (7) with channel data rates  $\{R_{q,r}\}$  (3). Then for every open-loop plant mode index  $h \in [1, \dots, n]$  and irreducible  $x_h$ -cycle  $c \in \mathcal{C}_h$ , there must exist  $\rho_{h,c} \geq 0$  such that

$$R_{q,r} \geq \sum_{h \in [1, \dots, n], c \in \mathcal{C}_h: (q \rightarrow r) \in c} \rho_{h,c}, \quad \forall (q \rightarrow r) \quad (10)$$

$$\sum_{c \in \mathcal{C}_h} \rho_{h,c} \geq \max\{\log_2 |\eta_h|, 0\}. \quad (11)$$

Conversely, for any channel rates  $\{R_{q,r}\}$  such that the inequalities (10)–(11) are feasible with strict inequality, it is possible to construct a CoNC that uniformly stabilises the plant.

This result is an exact characterisation, up to boundary points, of the region  $\mathcal{R}$  of channel rate matrices permitting stabilisability. The underlying intuition is natural. Each irreducible  $x_h$ -cycle  $c$  can be thought of as an independent pathway along which sensor information about that mode can be conveyed, through the system, to a controller that can help to stabilise it. Thus, each auxiliary variable  $\rho_{h,c}$  is in effect that portion of each channel rate  $R_{q,r}$  that carries data about  $x_h$  along cycle  $c \ni (q \rightarrow r)$ . This is also the reason for disallowing repetition of an element in  $c$ , since no extra information can be gained by repeating an element in the cycle. In addition, even though a CoNC may jointly encode two or more plant modes, stability is determined by the effective rate allocated to each *individual* mode. A similar feasibility-type condition was given in [16]. Apart from the lack of noise in the earlier work, the essential difference is that signalling paths through the plant, i.e. irreducible cycles over modes, were ignored. Consequently the condition there is sufficient but not necessary for stabilisability, except if the plant state is controllable from each input and observable at each output.

The result above reduces to the well-known *data rate theorem* of [3], [5], [20] for a centralised plant with a rate  $R$  channel from the sensor to the controller. In this case, there is only one irreducible  $x_h$ -cycle  $\forall h$ , so the feasibility condition (10)–(11) is equivalent to the existence of  $\rho_h \geq 0$  s.t.

$$R \geq \sum_{h \in [1, \dots, n]} \rho_h, \quad \rho_h \geq \max\{\log_2 |\eta_h|, 0\}.$$

This is possible iff  $R \geq \sum_{1 \leq h \leq n} \max\{\log_2 |\eta_h|, 0\}$ , which is precisely the centralised condition.<sup>2</sup>

It is also instructive to compare Theorem 1 with classical results on the decentralised stabilisation of linear time-invariant (LTI) plants with multiple sensors and controllers. In the classical scenario, each sensor is directly connected to a controller, with the channel rates between the various possible sensor-controller pairs effectively being either 0 (no direct communication) or  $\infty$  (perfect communication). If the controllers are also constrained to be LTI, it is well-known that stabilisability is not always possible and depends on concepts such as the existence of unstable *decentralised fixed modes* [21] and system completeness [22]. However, in [18] it is shown that decentralised stability can be recovered if the controllers are permitted to be time-varying, provided that the plant is centrally controllable and observable. It is briefly argued here that Theorem 1 is consistent with this result in the classical limit.

Central controllability and observability imply that for every plant mode  $x_h$ , there is at least one controller- $i$  that can directly affect it ( $d_{i,h} = 1$ ) and at least one sensor- $j$  that is directly influenced by it ( $e_{h,j} = 1$ ). As every sensor is connected by an infinite-rate channel to a controller and the number of modes and nodes in the system is finite, by tracing through all possible  $x_h$  cycles and excluding repetition it can be verified that for every mode  $x_h$  there exists at least one irreducible  $x_h$ -cycle which is *nontrivial*, i.e. that does not contain a nonexistent (0 rate) channel. Setting  $\rho_{h,c} = \infty$  for this  $x_h$ -cycle satisfies (11). As all irreducible  $x_h$ -cycles can only consist of channels  $(q \rightarrow r)$  with rate  $R_{q,r} = \infty$ , this also satisfies (10). Thus, the inequalities of Theorem 1 are feasible and the system is stabilisable. This agrees with [18], except that the controllers here are not constrained to be linear.

The remainder of this paper consists of an explanation of the necessity and (almost) sufficiency of (10)–(11). Due to space constraints, complete details are omitted.

#### IV. NECESSITY

In this section, the feasibility of the linear inequalities (11) is established as a necessary condition for the plant to be uniformly stabilisable. This is done using a variation of the concept from [23], [1] of maximum conditional state uncertainty volumes, which bound worst-case state norms from below. It is shown that any given cooperative networked coder-controller (CoNC) (4)–(6) with channel data rates  $\{R_{q,r}\}$  (3) can be replaced by one in which each mode

<sup>2</sup>Like the centralised result, Theorem 1 is independent of noise magnitude. As the noise magnitude increases, the worst-case state becomes larger, but remains bounded.

is encoded independently of the others, with negligible increase in maximum coordinate-wise conditional uncertainty volumes and channel rates. This directly induces a partition of information flows into “sub-channel” rates  $\{\rho_{h,c}\}$  along the irreducible cycles of each mode, such that the inequalities (10)–(11) are met.

Consider the plant (1)–(2) under any given CoNC  $\mathbf{C}$  and set the measurement noise to 0, which cannot increase the worst-case state norm (7). By hypothesis  $\lambda_n(\mathcal{X}_0)$  is strictly positive, so it can be verified that there must exist an  $n$ -dimensional hyper-cuboid  $\mathcal{X}'_0 \subseteq \mathcal{X}_0$ , with equal sides of length  $l$  oriented parallel to the coordinate  $x_h$  coordinate axes and with volume  $\lambda_n(\mathcal{X}'_0) = l^n > 0$ . Clearly, the worst case state norm cannot increase when the initial state is restricted to this subset. With minor notational abuse, for *any* system variables  $x \in \mathbb{R}$  and  $s$ , define the short-hand notation

$$\{x|s\} := \{x \in \mathbb{R} | s \text{ given}, \mathbf{x}(0) \in \mathcal{X}'_0\},$$

i.e. the set of all values that  $x$  can possibly take as  $\mathbf{x}(0)$  varies, under the condition that the variable  $s$  takes a given value. Note that all system variables are completely determined by the initial state  $\mathbf{x}(0)$ , since the measurement noise has been nulled. Let

$$v_h(k) := \max_{\tilde{\mathbf{S}}(k)} \lambda_1 \{x_h(0) | \tilde{\mathbf{S}}(k)\}, \quad (12)$$

the maximum uncertainty volume in the  $h$ -th initial state component, conditioned on all channel symbols up to time  $k$ . For any  $h \in [1, \dots, n]$  and  $k \in \mathbb{Z}_{\geq 0}$ ,

$$\begin{aligned} & \sup \{|x_h(k)|\} \\ & \geq 0.5 \lambda_1 \{x_h(k)\} \geq 0.5 \max_{\tilde{\mathbf{S}}(k)} \lambda_1 \{x_h(k) | \tilde{\mathbf{S}}(k)\} \quad (13) \\ & = 0.5 \max_{\tilde{\mathbf{S}}(k)} \lambda_1 \left\{ \eta_h^k x_h(0) + g_h(k, \tilde{\mathbf{S}}(k)) | \tilde{\mathbf{S}}(k) \right\} \\ & = 0.5 \max_{\tilde{\mathbf{S}}(k)} \lambda_1 \left\{ \eta_h^k x_h(0) | \tilde{\mathbf{S}}(k) \right\} \equiv 0.5 |\eta_h|^k v_h(k), \quad (14) \end{aligned}$$

where  $g_h(k, \tilde{\mathbf{S}}(k))$  is the accumulated control input. The bound (13) is due to the fact that the volume of a real set is at most twice its radius and cannot be enlarged by restriction to a subset, while (14) arises from the scaling property and translation invariance of Lebesgue measure. Thus, a uniformly bounded state implies uniformly bounded  $|\eta_h|^k v_h(k)$  over time  $k$ ,  $\forall h \in [1, \dots, n]$ .

In the following, it will be shown how, for a given terminal time  $k$ , the encoders  $\gamma_{q,r}$  for every channel can be replaced by independent scalar encoders of each initial state component, without increasing  $v_h(k)$  or the channel rates. Let a *generalised uniform scalar encoder (GUSE)* with  $M \in \mathbb{N}$  levels be any measurable function  $Q_M : [-l/2, l/2] \rightarrow [0, 1, \dots, M-1]$  such that the (possibly disconnected) coding regions have equal volume  $\lambda_1(Q_M^{-1}(s)) = \frac{l}{M}$ ,  $\forall s \in [0, 1, \dots, M-1]$ . Call the set of GUSEs  $Q_{M_1}, \dots, Q_{M_v}$  *mutually volume refining* if  $\forall s_i \in [0, \dots, M_i-1]$ ,  $i \in [1, \dots, v]$ ,

$$\lambda_1 \left( \bigcap_{i=1}^v Q_{M_i}^{-1}(s_i) \right) = \frac{l}{\prod_{i=1}^v M_i}. \quad (15)$$

The following lemma, given here without proof, states that any GUSE  $Q_M(\cdot)$  is exactly equivalent to a Cartesian product  $[Q_{M_1}(\cdot), \dots, Q_{M_v}(\cdot)]$  of mutually refinable GUSE “factors”.

*Lemma 1:* For any GUSE  $Q_M$  and factorisation  $M = \prod_{i=1}^v M_i$ , where  $M, M_1, \dots, M_v \in \mathbb{N}$ , there exists a set of mutually volume-refining GUSEs  $Q_{M_1}, \dots, Q_{M_v}$  such that each original coding region can be *uniquely* expressed as an intersection of the coding regions of the  $v$  mutually refining GUSEs. I.e.  $\forall s \in [0, \dots, M-1]$ ,  $\exists!(s_1, \dots, s_v) \in [0, \dots, M_1-1] \times \dots \times [0, \dots, M_v-1]$  s.t.

$$Q_M^{-1}(s) = \bigcap_{i=1}^v Q_{M_i}^{-1}(s_i). \quad (16)$$

A consequence of this lemma is that for any GUSE  $Q_M$  such that  $M = M_1 M_2$ ,  $M, M_1, M_2 \in \mathbb{N}$ , there exists a *coarsening* GUSE  $Q_{M_1}$  such that  $\forall s$ ,  $Q_M^{-1}(s)$  is contained in exactly one of the coarser coding regions  $Q_{M_1}^{-1}(s_1)$ .

Now, for any channel ( $q \rightarrow r$ ), define the *effective rate for mode  $h$*  to be

$$\rho_{h,q,r}(k) := \frac{-1}{k+1} \times \log_2 \left( \inf_{\mathcal{A} \subseteq \mathcal{X}'_0} \max_{\tilde{s}_{q,r}(k)} \frac{\lambda_1 \{x_h(0) | \tilde{s}_{q,r}(k) \& (\mathbf{x}(0) \in \mathcal{A})\}}{\lambda_1 \{x_h(0) | \mathbf{x}(0) \in \mathcal{A}\}} \right) \quad (17)$$

and let  $M_{h,q,r}(k) := \lceil 2^{(k+1)\rho_{h,q,r}(k)} \rceil$ . For every unstable mode  $h$ , consider the set of all sensors that are directly affected by it, i.e. with  $e_{h,j} = 1$ . Denote by  $\mathcal{L}_h$  the set of digital channels ( $q \rightarrow r$ ) that directly emanate from these sensors. Associate with the channels in  $\mathcal{L}_h$  an *arbitrary* set of mutually volume refining GUSEs  $Q_{M_{h,q,r}}^{q,r} : [-l/2, l/2] \rightarrow [0, \dots, M_{h,q,r}]$ , ( $q \rightarrow r$ )  $\in \mathcal{L}_h$ , as per the lemma above. These GUSEs cannot yet be fully applied to replace the encoders at each channel in  $\mathcal{L}_h$ , since some sensors may receive indirect information about other modes, via network nodes or controller signalling through observable modes. This is resolved by “propagating” the sensor GUSEs just constructed in a forward direction through the network, as described by the following algorithm.

Pick any irreducible cycle  $c \in \mathcal{C}_h$  and let  $q'$  be the element in  $c$  which succeeds the sensor  $q$  that observes  $x_h$ . If  $q'$  is a plant mode or another sensor which also observes  $x_h$  directly, skip it and move to another cycle in  $\mathcal{C}_h$ . However, if the next element is a node  $q'$  which does not observe  $x_h$  directly and its successor  $q''$  in the cycle  $c$  is not another plant mode, construct the  $x_h$  GUSE for the channel ( $q' \rightarrow q''$ ) along  $c$  as follows. First, calculate the effective mode  $h$  rate (17) for channel ( $q' \rightarrow r'$ ), and subtract the total rate allocated to other  $x_h$  GUSEs that may have already been propagated along other cycles in  $\mathcal{C}_h$  sharing the channel ( $q' \rightarrow q''$ ), to yield the *available rate*  $A_{h,q',q''}(k)$ . If this is greater than or equal to the rate for the “incoming” GUSE from channel ( $q \rightarrow q'$ ), propagate the incoming GUSE forward along ( $q' \rightarrow q''$ ) without change. However, if the available rate is less than the incoming GUSE rate  $\rho_{h,q,q'}(k)$ , Lemma 1 implies that a GUSE at rate  $\rho_{h,q,q'}(k) - A_{h,q',q''}(k)$  that *coarsens* all incoming ones can be constructed. Propagate this onward. Note that if the next element  $q''$  is another plant mode, the

incoming GUSE is propagated without change through to the sensor node  $q'''$  which observes it. Continue this process until the mode  $x_h$  is reached, and then repeat for every other cycle  $c \in C_h$  and mode  $h$ .

In this manner, it is clear that all channels in the network can eventually be endowed with mutually refinable Cartesian product GUSE's along each irreducible  $x_h$ -cycle, without changing the effective mode rates (17) on each channel. The original CoNC  $\mathbf{C}$  is now replaced with the CoNC  $\mathbf{C}'$  based on these GUSEs. Defining  $\rho_{h,c} := \liminf_{k \rightarrow \infty} \min_{(q \rightarrow r) \in c} \rho_{h,q,r}(k)$ , it can be verified from certain entropy-like properties of the effective mode rate definition, that for large  $k$  the channel rates are not exceeded, i.e.  $\sum_h \rho_{h,q,r}(k) \leq R_{q,r}$ , and furthermore that the scaled uncertainty volume (12) is not increased, with no increase in  $v_h(k)$ . The fact that (11) is met then follows from the fact that once the original CoNC  $\mathbf{C}$  has been replaced by the GUSE-based CoNC  $\mathbf{C}'$ , only the subchannel rates associated with  $x_h(0)$  along an irreducible  $x_h$ -cycle can contribute to keeping the scaled maximum coordinate-wise conditional uncertainty volume  $|\eta_h|^k v_h(k)$  uniformly bounded. By the mutual volume-refinability of the GUSEs for  $x_h$ , rates on different irreducible cycles add perfectly, leading to the necessity of (11) for volume stability, by standard results for centralised control [5], [1].

## V. SUFFICIENCY

In this section, it is shown that the strict feasibility of the linear inequalities (10)–(11) is sufficient for stabilisability. This is done by explicitly constructing a stabilising cooperative networked coder-controller (CoNC), using ideas based on uniform mutually refinability or *Slepian-Wolf* coding. The idea behind this technique is that each encoder transmits different, pre-agreed blocks in the binary expansion of all unstable modes that it can see, so that no redundant information is transmitted. The open-loop instability of the modes means that the effect of the additive bounded measurement noise is negligible in comparison. Upon receiving these blocks, each controller then independently applies an input sequence to correct only for those bits in the binary expansion of each mode which it can see, thereby avoiding redundant action. The linearity of the plant, combined with the additive formula for binary expansions, then guarantees that the *overall* control input compensates for the overall plant state, leading to uniform boundedness. An important characteristic of this scheme is that the state estimation, stabilisation and *signalling* tasks of the control policy are divided into three distinct phases. Due to space constraints, only the basic elements of the scheme are presented below. The analysis is omitted but follows similar lines to [16].

### A. Three-Phase Scheme

Suppose  $\exists \rho_{h,c} \geq 0$  such that (10)–(11) are met with strict inequality. Divide time  $k \in \mathbb{Z}_{\geq 0}$  into *coding cycles* of length  $T$  and at time  $j = IT$  let  $\chi_l > 0$  be a bound, known to all nodes, such that  $\|\mathbf{x}(IT)\| < \chi_l$ . The first  $2n$  instants in each cycle constitute the *plant signalling phase*, during

which controllers relay information to sensors about unstable modes not directly observable, by sending each controllable mode at time  $k = IT + n$  to prespecified discrete levels spaced sufficiently far apart. From time  $IT + n$  to  $IT + 2n$ , the controllers then undo the effect of the communication signals, to return each mode to the input-free trajectory  $\eta_h^{2n} x_h(IT)$ .

However, first consider the next  $d$  instants, from time  $IT + 2n$  till  $IT + 2n + d$ , which constitute the *state estimation phase* during which the sensors estimate unstable mode values at the start of the cycle. During this phase the plant inputs are set to zero, so that  $\forall t \in [2n + 1, \dots, 2n + d]$  each mode evolves according to

$$x_h(IT + t) = \eta_h x_h(IT + t - 1) = \eta_h^t x_h(IT).$$

For any  $t \in [2n, 2n + d]$ , the  $g$ th component of the  $j$ th sensor output is given by

$$y_j^{(g)}(IT + t) = \sum_{h: e_{h,j}=1} c_{j,h}^{(g)} \eta_h^t x_h(IT) + w^{(g)}(IT + t),$$

where the  $(g, h)$ th component  $c_{j,h}^{(g)}$  of the output matrix  $\mathbf{C}_j$  is nonzero for at least one  $h$ . As each  $\eta_h$  is distinct, it can be shown that, in the absence of noise, knowledge of  $\{y_j^{(g)}(IT + t)\}_{t=2n}^{3n}$  suffices to determine the values of all  $x_h(IT)$  such that  $c_{j,h}^{(g)} \neq 0$ .<sup>3</sup> Recalling that  $\mathbf{c}_{j,h} \neq \mathbf{0}$  if  $e_{j,h} = 1$ , each mode  $x_h(IT)$  such that  $e_{j,h} = 1$  influences at least one component of  $\mathbf{y}_j(IT + t)$ . In the presence of measurement noise, as time progresses the effect of the observable unstable modes is exponentially amplified and thus dominates over the bounded additive noise terms. Thus  $\forall \delta > 0$ , after a sufficiently large time  $d$  of no control, the  $j$ th sensor knows all unstable modes  $x_h(IT)$  for which  $e_{h,j} = 1$ , to within  $\pm \delta$ .

Let  $x_{h,j}^e(IT)$  denote this slightly erroneous estimate of  $x_h(IT)$  at the  $j$ th sensor. Expand  $0.5x_{h,j}^e(IT)/\chi_l + 0.5 \in [0, 1]$  as a unique sequence of binary digits and consider only the first  $B_h := \sum_{c \in C_h} \lceil T\rho_{h,c} + 1 \rceil$  bits in the expansion. Partition them into  $N_h := |C_h|$  contiguous blocks  $\mathcal{B}_{h,c}$ , with corresponding block length  $\lceil T\rho_{h,c} + 1 \rceil$ . At time  $k = IT + 2n + d$ , each  $j$ th sensor transmits onto each outgoing irreducible cycle  $c$  all bit blocks  $\mathcal{B}_{h,c}$  that it has calculated; no other transmissions are made during the cycle. The average number of bits transmitted on the  $(q \rightarrow r)$  channel per unit time is thus

$$\begin{aligned} & \frac{1}{T} \sum_{h,c:(q \rightarrow r) \in c \in C_h} \lceil T\rho_{h,c} + 1 \rceil \\ & \leq \frac{|C_h|}{T} + \sum_{h,c:(q \rightarrow r) \in c \in C_h} \rho_{h,c} < R_{q,r}, \end{aligned}$$

for sufficiently large  $T$ , so this is feasible.

As each channel has finite delay and there are only a finite number of irreducible cycles,  $\exists d' \in \mathbb{N}$  such that at time  $IT + 2n + d + d'$ , every controller has available all the bit blocks  $\mathcal{B}_{h,c}(IT)$  transmitted along irreducible  $x_h$ -cycles  $c$  to which it belongs. Times  $[IT + 2n + d + d', \dots, IT + 3n + d +$

<sup>3</sup>This is due to the invertibility of Vandermonde matrices; see e.g. [19].

$d'$ ] constitute the *stabilisation* phase of the cycle. Suppressing all inputs but the  $i$ th, the accumulated control action  $n$  steps into the future by the  $i$ th controller, on any  $h$ th mode such that  $d_{i,h} = 1$ , is

$$\sum_{q=0}^{n-1} \eta_h^{n-1-q} \mathbf{b}_{i,h}^T \mathbf{u}_i(IT + 2n + d + d' + q) \\ \equiv [\eta_h^{n-1} \mathbf{b}_{i,h}^T, \eta_h^{n-2} \mathbf{b}_{i,h}^T, \dots, \mathbf{b}_{i,h}^T] \mathbf{U}_i \equiv \bar{\mathbf{b}}_{i,h}^T \mathbf{U}_i,$$

where  $\mathbf{U}_i$  is the stacked control vector  $[\mathbf{u}_i(IT + 2n + d + d')^T, \dots, \mathbf{u}_i(IT + 2n + d + d' + n - 1)^T]^T$ . Define a matrix  $\bar{\mathbf{B}}_i$  with rows consisting of all  $\bar{\mathbf{b}}_{i,h}^T$  such that  $d_{h,i} = 1$ . As at least one component of each  $\bar{\mathbf{b}}_{i,h}^T$  is nonzero and each  $\eta_h$  is distinct, it can be established (again by the invertibility of Vandermonde matrices) that  $\bar{\mathbf{B}}_i \mathbf{U}_i$  can take any desired value by appropriate choice of the stacked controls  $\mathbf{U}_i$ . In particular, for every  $i$ th controller, apply a stacked control vector  $\mathbf{U}_i$  such that for any unstable  $h$ th mode with  $d_{i,h} = 1$ ,

$$\bar{\mathbf{B}}_i \mathbf{U}_i = -\eta_h^{3n+d+d'} \chi_l \left( \frac{1}{M_h 2^{B_h}} - \frac{1}{M_h} + 2 \sum_{j:e_{h,j}=1} \sum_{f=\omega_{h,c}}^{\omega_{h,c}+[T\rho_{h,c}]} \frac{\beta_{h,f,c}(IT)}{2^f} \right), \quad (18)$$

where  $\{\beta_{h,f,c}(IT)\}_{f=\omega_{h,c}}^{\omega_{h,c}+[T\rho_{h,c}]}$  is the bit block  $\mathcal{B}_{h,c}(IT)$  and  $M_h$  is the number of  $i$ th controllers such that  $d_{i,h} = 1$ . At all other times in each coding cycle, no control signals are applied.

Return now finally to the plant signalling phase. Recalling that  $\|\mathbf{x}(IT)\| \leq \chi_l$ , let  $D_l = 2\|\mathbf{C}\mathbf{J}^n\| \chi_l + 2 \max_{j,\mathbf{w} \in \mathcal{W}_j} \|\mathbf{w}\|$  and note that there are at most  $\sum_h B_h$  bits per coding cycle to be communicated to various sensors. It can be shown that, by setting  $2^{\sum_h B_h}$  different, pre-agreed possible combinations of the outputs  $\mathbf{y}_j(IT+n)$  such that the separation between permissible output vectors is greater than  $D_l$ , all signalled bits can be communicated across the plant in  $n$  steps, without being corrupted by the measurement noise and unknown plant state. From time  $IT+n$  to  $IT+2n$ , additional inputs are then applied to cancel the effect of the signalling inputs and return each mode to the open-loop trajectory  $\eta_h^{2n} x_h(IT)$ .

Note that, although the stable modes are not considered in the estimation or control phases, they do figure in the signalling phase. Details are omitted but it can be shown from an analysis similar to [16] that, for sufficiently large  $T$ , this three-fold scheme uniformly stabilises the plant for any choice of  $\{\rho_{h,c}\}$  satisfying (10)–(11) strictly.

## VI. CONCLUSION

In this paper, the problem of stabilising a linear plant with noisy outputs over a digital network was posed. A necessary and almost sufficient condition for determining networked uniform stabilisability, was derived, in terms of the feasibility of a set of linear inequalities involving the unstable open-loop eigenvalues, the channel data rates and the controllability and observability structure of the plant. The auxiliary variables

introduced in this condition have a natural interpretation as the effective rates of information flow associated with each unstable mode. Optimal performance and stabilisability with process and/or channel noise are currently being investigated.

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