Using Piecewise-Constant Congestion Taxing Policy in Repeated Routing Games

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In 1963

- William S. Vickrey, a Nobel Laureate in Economics, started his paper on resource allocation in transportation with

  “proposition that in no other major area are pricing practices so irrational, so out of date, and so conductive to waste as in urban transportation”

- He argued, this is because of

  “absence of off-peak differentials ”
  “underpricing of some modes relative to the others”

- He proposed that we use

  “detection and billing method” based on “electronic identifier units carried in each vehicle, which would activate recording devices in or on the road.”

- A lot has happened in the past 50 years (Civil Rights Act of 1964, moon landing, Star Wars, fall of Berlin war, launch of the Hubble Telescope, …)
Advances in transportation engineering

- Many local governments have introduced congestion taxing systems in, e.g., Stockholm, London, San Francisco, and Singapore

> “Since the traffic flow increases due to external factors, primarily increasing population in the county, the charge must, however, increase to keep the traffic flow at the present level” (Börjesson, et al, 2012)

> Worse, these systems do not respond to temporary traffic changes and are designed based on the average behaviour of the travellers

Dynamic congestion pricing techniques are implemented, e.g., San Diego I-15 High-Occupancy Toll (HOT) Lanes

> Dynamic congestion pricing only to a limited number of vehicles (i.e., single occupancy) in a subset of the lanes (i.e., HOT lanes)

Imposing taxes on all the lanes is certainly controversial because of the sense of the entitlement and political issues (not our job as an engineer)

the difficulty to respond to time-varying taxes at the same time as driving
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- the sense of the entitlement and political issues (not our job as an engineer)
- the difficulty to respond to time-varying taxes at the same time as driving
The central planner sets the congestion taxes for wide windows of iterations in advance and announces the taxes publicly for those days.

This introduces delay in the system!
A directed graph $G = (V, E)$ models the transportation network.

A set of commodities $\{(s_k, t_k)\}_{k=0}^K$
A directed graph $G = (\mathcal{V}, \mathcal{E})$ models the transportation network.

- A set of commodities $\{(s_k, t_k)\}_{k=0}^{K}$
- $\mathcal{P}_k$ denotes the set of all admissible paths connecting $s_k \in \mathcal{V}$ to $t_k \in \mathcal{V}$
- We use the notation $\mathcal{P} = \bigcup_{k=1}^{K} \mathcal{P}_k$
Routing game

\( \phi(e) = \sum_{p \in P : e \in p} f_p \) denotes the flow of drivers on edge \( e \in E \)

\( f_p \in \mathbb{R}_{\geq 0} \) denotes the flow of players over path \( p \in P \)

}\[ \phi(4, 5) = f_{p1} + f_{p1}' + f_{p1}'' = F_1 \]

\( f_{p1} \in \mathbb{R} \geq 0 \) denotes the flow of players over path \( p \in P \)

Commodity 1 \( \leq k \leq K \) transfers a flow equal to \( F_k \in \mathbb{R} \geq 0 \)

Feasibility

A flow vector \( f = (f_p)_{p \in P} \in \mathbb{R}^{|P|} \) is feasible if

\[ \sum_{p \in P} k f_p = F_k \text{ for all } 1 \leq k \leq K \]
Routing game

\[ f_{p_1} + f_{p_2} + f_{p_3} = F_1 \]

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- \( \phi_e = \sum_{p \in \mathcal{P} : e \in p} f_p \) denotes the flow of drivers on edge \( e \in \mathcal{E} \)
Routing game

\[ \tilde{\ell}(4,5)(\phi(4,5)) \]

A driver that travels along an edge \( e \in \mathcal{E} \) experiences a cost equal to \( \tilde{\ell}_e(\phi_e) \)
Routing game

- A driver that travels along an edge $e \in \mathcal{E}$ experiences a cost equal to $\tilde{\ell}(\phi_e)$.
- A driver from commodity $1 \leq k \leq K$ that uses path $p \in \mathcal{P}_k$ experiences a total cost of $\ell_p(f) = \sum_{e \in p} \tilde{\ell}(\phi_e)$.

\[
\ell_p(f) = \tilde{\ell}_{(0,4)}(\phi_{(0,4)}) \nonumber \\
+ \tilde{\ell}_{(4,5)}(\phi_{(4,5)}) \nonumber \\
+ \tilde{\ell}_{(5,1)}(\phi_{(5,1)})
\]
Routing game

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- A driver from commodity \( 1 \leq k \leq K \) that uses path \( p \in \mathcal{P}_k \) experiences a total cost of \( \ell_p(f) = \sum_{e \in p} \tilde{\ell}_e(\phi_e) \)
- A driver is in control of an infinitesimal part of the flow that strategically tries to minimize its own cost for using the road \( \min_{p \in \mathcal{P}_k} \ell_p(f) \)

Wardrop Equilibrium

A flow vector \( f \) is a Nash equilibrium if for all \( 1 \leq k \leq K \), \( f_p > 0 \) for a path \( p \in \mathcal{P}_k \) implies that \( \ell_p(f) \leq \ell_{p'}(f) \) for all \( p' \in \mathcal{P}_k \)
Standing assumptions

Feasibility
The set of feasible flows $F((F_k)_{k \in K})$ is nonempty.

Existence
For each $e \in E$, the mapping $\tilde{\ell}_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is

(i) twice continuously differentiable,
(ii) convex,
(iii) non-decreasing.
Repeated routing game

- Repeated games have been an interest of the economists as a way to enforce cooperation through repetition and reputation
- Various learning dynamics
  - Best response dynamics (Axelrod, 1984)
  - Fictitious play (Brown, 1951)
  - Logit-response dynamics (Alós-Ferrer & Netzer, 2010)
  - No regret learning (Arora, et al, 2012)
  - Cournot dynamics (Seade, 1980)
- Evolutionary game theory in routing games
  - Population dynamics (Fischer & Vöcking, 2004)
- Day-to-day traffic assignment
  - Simplex gravity flow dynamics (Smith, 1983)
  - Proportional-switch adjustment (Smith, 1984)
  - network tâtonnement process (Friesz, et al, 1994)
  - projected dynamical system (Zhang & Nagurney, 1996)
Repeated routing game

1: Initialize \( w_p[1] = 1, \forall p \in \mathcal{P}_k, \forall k \in \{1, \ldots, K\} \)
2: Initialize \( \tilde{\tau}_e[n'] = 0, \forall e \in \mathcal{E}, \forall n' \in \{1, \ldots, D\} \)
3: \textbf{for} \( n = 1, 2, \ldots \) \textbf{do}
4: \quad Calculate the flows
\[
    f_p[n] = F_k \frac{w_p[n]}{\sum_{p' \in \mathcal{P}_k} w_{p'}[n]}, \forall p \in \mathcal{P}_k, \forall k \in \{1, \ldots, K\}
\]
5: \quad Update the weights
\[
    w_p[n+1] = w_p[n] \exp \left( -\frac{\epsilon[n]}{\rho_1 + \rho_2} \sum_{e \in p} \left( \tilde{\ell}_e(f[n]) + \tilde{\tau}_e[n] \right) \right)
\]
6: \quad \textbf{if} \( n \equiv 0 \pmod{D} \) \textbf{then}
7: \quad \quad Set the tolls for the next \( D \) days
\[
    \tilde{\tau}_e[n'] = \left[ \phi_e \frac{d\tilde{\ell}_e(\phi_e)}{d\phi_e} \right] \bigg|_{\phi_e = \phi_e[n]}, \forall e \in \mathcal{E}, \forall n' \in \mathbb{N} : n' - n \in \{1, \ldots, D\}
\]
8: \quad \textbf{end if}
9: \textbf{end for}
Repeated routing game

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   \[ w_p[n + 1] = w_p[n] \exp \left( -\frac{\epsilon[n]}{\rho_1 + \rho_2} \sum_{e \in p} (\tilde{\ell}_e(f[n]) + \tilde{\tau}_e[n]) \right) \]
6: if $n \equiv 0 \pmod{D}$ then
7: Set the tolls for the next $D$ days
   \[ \tilde{\tau}_e[n'] = \left[ \phi_e \frac{d \tilde{\ell}_e(\phi_e)}{d \phi_e} \right]_{\phi_e = \phi_e[n]}, \forall e \in E, \forall n' \in \mathbb{N} : n' - n \in \{1, \ldots, D\} \]
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Convergence

- The social cost function $C(f) = \sum_{p \in P} f_p \ell_p(f)$ is defined to be the total time that the society wastes in traffic.
- Define $\mathcal{S} = \text{arg min}_{f \in \mathcal{F}} C(f)$ to be the set of socially optimal flows.
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**Theorem**

If $\epsilon[n] = \alpha/(n + \beta)$, $\forall n \in \mathbb{N}$, with $\alpha, \beta \in \mathbb{R}_{>0}$, then $\lim_{n \to \infty} \text{dist}(S, (f_p[n])_{p \in P}) = 0$. 

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- The algorithm cannot adapt itself fast enough (especially, after many steps because the step size is very small).

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When \( \epsilon \) is large enough, the algorithm can adapt rapidly to the changes in the parameters of the routing game, however, the solution can potentially be far from the socially optimal flow.
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The Bureau of Public Roads model (Singh & Dowling, 2002) for the delay on each edge

\[ \tilde{\ell}_e(\phi_e) = \frac{d_e}{v_{e,\text{max}}} \left[ 1 + 0.15 \left( \frac{\phi_e}{c_e} \right)^4 \right] \]

- \( d_e \) is the length of the road
- \( v_{e,\text{max}} = 70 \text{ km/h} \) is the speed limit
- \( c_e (\sim 2000 \text{ vec/h/lane}) \) is the capacity of the road (Roess & McShane, 1987)
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- \( F_1 = 8000 \text{ vehicle/h for } (s_1, t_1) = (0, 1) \),
- \( F_2 = 3000 \text{ vehicle/h for } (s_2, t_2) = (7, 3) \), and
- \( F_3 = 4000 \text{ vehicle/h for } (s_3, t_3) = (0, 8) \)
Numerical example (fixed demand)

- The flows of vehicles over various paths for all source–destination pairs versus the iteration number
- The flows for most of the paths settle very rapidly
Numerical example (fixed demand)

- The social cost function versus the iteration number
- The social cost of the extracted flows approaches the cost of the socially optimal flow rapidly
Numerical example (fixed demand)

- The congestion taxes over various edges versus the iteration number
- The drivers on highly congested roads, e.g., (0, 1), pay much more to be persuaded to use less-congested alternatives
Numerical example (fixed demand)

- The delays over the roads with and without congestion taxes
- 4.6% improvement in the social cost function
Numerical example (varying demand)

Select a constant step size $\epsilon[n] = 5 \times 10^{-2}$ for all $n \in \mathbb{N}$
Numerical example (varying demand)

- Define \( f^*[n] \) to be the socially optimal flow for demands \( (F_k[n])_{k=1}^3 \)
- The smaller \( C(f[n])/C(f^*[n]) - 1 \) is, the closer the social cost of the generated flow is to the cost of the socially optimal flow at time \( n \)
- The algorithm closely follows the socially optimal flow
Conclusions and future work

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- Devised a rule to construct piecewise-constant congestion
- Guarantee the convergence of the flows to the socially optimal solution with vanishing step sizes
- Ensure convergence to a neighborhood of the socially optimal solution with constant step size

Future work
- Multi-class traffic to understand the influence of the drivers value-of-time
- Devising piecewise-constant congestion charges policies for only a subset of the edges in the transportation network
- Proving the convergence for a larger class of update equations

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