A unification of time-scale methods for systems with disturbances

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Abstract— **This paper develops a unified framework for studying robustness of the input-to-state stability (ISS) property and presents new results on robustness of ISS to slowly-varying parameters, to highly oscillatory signals, and to generalized singular perturbations. The common feature in these problems is a time-scale separation between slow and fast variables which permits the definition of a boundary layer system like in classical singular perturbation theory. To address various robustness problems simultaneously, the asymptotic behavior of the boundary layer is allowed to be complex and it generates an average for the derivative of the slow state variables. The main results establish that if the boundary layer and averaged systems are ISS then the ISS bounds also hold for the actual system with an offset that converges to zero with the parameter that characterizes the separation of time-scales. This result is then applied to various classical robustness problems.**

Keywords— **Singular perturbations, averaging, slowly-varying systems, input-to-state stability.**

I. Introduction

One of the strongest threads in the fabric of nonlinear stability theory is the inherent robustness of uniform asymptotic stability to regular perturbations, slowly varying parameters, highly oscillatory signals and fast unmodeled dynamics. For a uniformly asymptotically stable equilibrium, these results are classical. Robustness to regular perturbations, also called "total stability" or "stability under constantly acting perturbations", dates back to the 1940's in the work of Malkin [40], Goršin [19], Vrkoč [70], as summarized in [71, Chapter VI] and [22, Section 56]. Robustness to slowly-varying parameters derives from the work of Hoppensteadt [25] which establishes robustness of uniform asymptotic stability to singular perturbations and extends to the infinite interval the classical results due to Tikhonov [67]. (Additional statements on approximating solutions of singularly perturbed systems on the infinite interval are given in [51]. For further references, consult [29].) Robustness of asymptotic stability to highly oscillatory signals, found in the "averaging" literature, has a rich history summarized in [52, Section 8.1]. Pioneering contributions to averaging theory were made by Bogoliubov and coauthors in [32], [12]. Additional averaging-based robustness results can be found in [52], [23], [69], [42], [54], [41], [24], [34], [47], [63], and, in discrete-time, [55, Chapter 7].

Averaging and singular perturbation techniques have been combined, especially in the identification and adaptive control literature, under the title "two time-scale averaging". In this setting the boundary layer system, obtained in the singular perturbation approach by setting the derivative of the slow state variables to zero, is time-varying and possesses a time-varying integral manifold on which the derivative of the slow state variables can be averaged. This approach is taken in [50], [11], and [24, Section V.3], and, in the context of adaptive control and identification, in [39], [1], [49], [16], [53], and [55, Chapter 8].

Averaging can be useful in singular perturbation studies even when the boundary layer system is time-invariant. Instead of insisting that the trajectories of the boundary layer system converge to an equilibrium manifold, as in classical singular perturbation theory (see [29]), or a time-varying integral manifold like in certain adaptive control problems, the "steady-state" behavior of the boundary layer may be complex. For example, the trajectories of the boundary layer may converge to a family of limit cycles, or some other more complicated family of attractors, parameterized by the slow state variables. The steady-state behavior can then be used to average the derivative of the slow state variables. This idea can be found in the work of Anosov [4], more recently in the optimal control results of Gaitsgory [17], [18], in the work of Grammel [20], [21] and in the elegant, pioneering formulation of Artstein and coauthors [9], [8], [5], [6], [7], where the averaging is done using invariant measures (see [45]) and the reduced system is typically a differential inclusion. Except for the results on near asymptotic stability of the origin in [5], these papers focus on approximating trajectories on compact time intervals.

Many recent robustness studies have focused on systems with exogenous disturbances where uniform asymptotic stability is replaced by the input-to-state stability (ISS) property, introduced by Sontag [56]. (ISS has been the basis for global stabilization algorithms [56], robust output or partial state feedback designs [48], [26], and modular nonlinear adaptive control [31, Chapters 5,6,9]. Generalized versions of ISS, where distance of the state to a point has been replaced by distance of the state to an attractor, have been considered in [37], [60].) Robustness of ISS has been established for systems with small time delays [62], for singularly perturbed systems [13], and for systems having an average [46], [44]. Most of these results rely on the existence of a converse Lyapunov function for ISS, which was established in [58] (see also [38], [59], [61] and [66]).

The first author was supported in part by the AFOSR under grant F49620-00-1-0106 and the NSF under grant ECS-9988813. CCEC, Electrical and Computer Eng. Dept., University of California, Santa Barbara, CA, 93106-9560, USA

Second author supported by BOF grant 011D0696 of the Ghent University. This paper presents research results of the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors. SYSTeMS, Universiteit Gent, Technologiepark 9, 9052, Zwijnaarde, Belgium

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In this paper, we develop a unified framework for studying robustness of general ISS properties and presents new results on robustness of ISS to slowly-varying parameters, to highly oscillatory signals, and to generalized singular perturbations. The common feature in these problems is a time-scale separation between slow and fast variables which permits the definition of a boundary layer system like in classical singular perturbation theory. To address the various robustness problems simultaneously, the asymptotic behavior of the boundary layer is allowed to be complex, and it generates an average for the derivative of the slow state variables, like in the seminal work of Artstein [5]. The main results establish that if the boundary layer and averaged systems are ISS then the ISS bounds also hold for the actual system with an offset that converges to zero with the parameter that characterizes the separation of time-scales. Our result relies on the proof technique we introduced in [44] which enables capturing the behavior of the system on the infinite time interval even though the actual system's solutions are typically not close to the simplified systems' solutions on this interval due to exogenous disturbances and the general ISS properties considered.

Our results cover well-known facts from the literature on averaging, regular perturbations, and singular perturbations. When the fast state variable is time we recover results in classical averaging, partial averaging ([42],[54]), averaging with fast and slow disturbances [46], and averaging for pulse-width modulated control systems [34]. When the fast state variables include time and the boundary layer system contains a time-varying integral manifold we obtain the "two time-scale averaging" results from adaptive control and identification. When the slow state variables are not present we produce results on robustness to slowly varying parameters and regular perturbations. In another special case, we recover results on asymptotic stability for weakly nonlinear oscillators. We also address standard singular perturbations and unconventional situations like when the boundary layer has an unstable equilibrium manifold [5, Remark 5.1].

We present these results as follows: In Section III we give an example that illustrates the main concept we will be developing in more generality in the core of the paper. We study the robustness of generalized ISS to generalized singular perturbations in Section IV where our main assumptions and result are given. In Section V we present sufficient conditions for the main assumptions of Section IV to hold. In Section VI we apply these sufficient conditions to cover the special cases of robustness mentioned above. Technical proofs are given in Section VII. We summarize our work in Section VIII.

II. Notation

A. Notation

- We will often write (x_s, x_f) in place of $(x_s^T, x_f^T)^T$.
- A function is said to belong to class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is class-$\mathcal{K}$ in its first argument and decreasing to zero in its second argument.
- $\mathcal{B}$ denotes a closed unit ball, $\rho\mathcal{B}$ a closed ball of radius $\rho > 0$, and $\mathcal{X} + \rho\mathcal{B}$ the set obtained by taking a closed ball of radius ρ around every point in the set $\mathcal{X}$.
- For a function $d(\cdot)$ belonging to a set of functions $\mathcal{D}$ that take values in $\mathcal{V}$, we will use both $d \in \mathcal{D}$ and $d \in \mathcal{V}$.

III. A motivational example

In a wide variety of industrial applications, the control action is due not to the instantaneous motion of the actuator but rather to some average effect of this motion. The most common example is the actuation of electric motors via pulse-width modulating (PWM) power electronic circuitry. Inspired by such applications, we construct our example with a van der Pol oscillator as a prototype vibrating actuator and an RL circuit as an elementary linear plant. The voltage and current in the van der Pol circuit oscillate rapidly but the shape of the oscillation can be adjusted by varying the circuit's capacitance. The control algorithm will adjust the capacitance based on the error between the voltage across the resistor in the RL plant and its reference value.

We combine the RL circuit input-output equations

$$\begin{array}{rcl} \dot{v}_s &=& -v_s + u \\ y &=& v_s \ , \end{array} \tag{1}$$

where v_s is the voltage across the resistance, with those for the van der Pol circuit

$$\begin{array}{rcl} \varepsilon\ \dot{v}_f &=& \exp(u_c)\left(-\frac{1}{3}v_f^3 + v_f - I_f\right) \\ \varepsilon\ \dot{I}_f &=& v_f \\ y_c &=& K|v_f| - v_{dc} \end{array} \tag{2}$$

where v_f is the voltage across the capacitor and i_f is the current through the inductor. Equations (1) and (2) are in normalized units. The capacitance C is sum of a nominal value $C_\circ$ and an adjustable value $\widetilde{C}$ related to u_c by

$$\exp(-u_c) = \frac{C_\circ + \widetilde{C}}{C_\circ}\ . \tag{3}$$

We have normalized $\sqrt{L/C_\circ}$ to one and defined $\varepsilon := \sqrt{LC_\circ}$. Central to our control algorithm derivation will be the assumption that ε is small.

The output equation for the van der Pol circuit can be realized with the combination of a rectifier and an operational amplifier, where v_{dc} is a constant bias voltage.

One interconnection condition that we impose is

$$y_c = u \tag{4}$$

which indicates that the output of the van der Pol circuit is the input voltage to the RL circuit.

The second interconnection condition will include the control law to be inserted between the measurement of the RL circuit voltage, v_s, and the adjustable capacitor, $\widetilde{C}$ or u_c, in the van der Pol circuit. To derive a control algorithm we exploit our assumption that ε is small which will make the van der Pol circuit oscillations fast compared to the dynamics in the RL circuit. Because of this time-scale separation property, we will rely on the static mapping

$$\begin{aligned} u_c \mapsto f(u_c) &:= \qquad\qquad\qquad\qquad\qquad\qquad (5) \\ &\lim_{T\to\infty} \frac{1}{T}\int_0^T K|v_f(t,v_f(0),I_f(0),u,\varepsilon)|dt - v_{dc} \\ &(v_f(0),I_f(0)) \in \mathbb{R}^2 \setminus \{0\} \ , \quad \varepsilon > 0 \end{aligned}$$

to approximate the effect of the oscillating actuator. This mapping is well-defined and locally Lipschitz continuous.[1] Restricting our attention for this function to the domain $[-1,1]$ (a numerical approximation with $K = 29.63$ and $v_{dc} = 38.926$ is shown in Figure 1), the simplified control

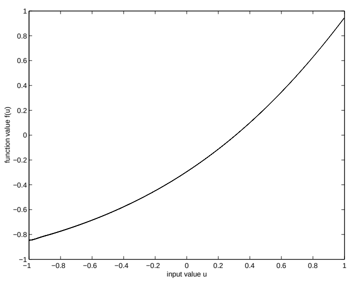

Fig. 1. Approximate plot of the map $u_c \mapsto f(u_c)$.

system becomes

$$\dot v_s = -v_s + f\left(\mathrm{sat}(u_c)\right) \ . \qquad (6)$$

To allow the values of the nonlinear mapping (5) to be uncertain, we employ an integral controller

$$\begin{array}{rcl} \dot\xi & = & r - v_s \\ u_c & = & \xi \end{array} \qquad (7)$$

so that our simplified closed-loop system is

$$\begin{array}{rcl} \dot v_s & = & -v_s + f(\mathrm{sat}(\xi)) \\ \dot\xi & = & r - v_s \ . \end{array} \qquad (8)$$

When the nonlinearity is monotone and r is strictly in the range of $f(\cdot)$ so that $f^{-1}(r)$ exists, the point $(v_s,\xi) = (r, f^{-1}(r))$ is globally asymptotically stable. Let $\mathcal{A}_f(u_c)$ be the set of points on the limit cycle of the van der Pol equation corresponding to the constant input u_c. Then, for the complete closed-loop system (1)-(4), (7), we can make a set arbitrarily close to the set

$$\left\{(v_s,\xi,v_f,I_f) : (v_s,\xi) = (r, f^{-1}(r)) \ , \ (v_f,I_f) \in \mathcal{A}_f(\xi)\right\}$$

asymptotically stable with basin of attraction arbitrarily close to the set $\mathbb{R}^2 \times \left(\mathbb{R}^2 \setminus \{0\}\right)$ by choosing ε sufficiently small. A similar statement can be made even if f is not monotone. All of these statements are now demonstrated.

Suppose $f(\cdot)$ and r are such that

$$f(-1) < r < f(1) \ . \qquad (9)$$

Let ξ_r be any value such that $f(\mathrm{sat}(\xi_r)) = r$. Define the Lyapunov function

$$V(v_s,\xi) := \frac{1}{2}(v_s - r)^2 + \int_{\xi_r}^{\xi} \left[f\left(\mathrm{sat}(\zeta)\right) - r\right] d\zeta \ . \qquad (10)$$

It follows from (9) that the sublevel sets of V are compact. The derivative of V along the solutions of simplified closed-loop (8) is negative semidefinite, ensuring for (8) global stability of the set

$$\mathcal{A}_s(c) := \{(v_s,\xi) : V(v_s,\xi) \le c\} \qquad (11)$$

for each c such that $\mathcal{A}_s(c)$ is nonempty. In fact, the time derivative of $V(v_s(t),\xi(t))$ is equal to $-(v_s(t) - r)^2$ so that the Krasovskii/LaSalle invariance principle ([30, Theorem 14.1], [35]) guarantees global convergence to a point of the form $(r, \widetilde{\xi}_r)$ where $f(\mathrm{sat}(\widetilde{\xi}_r)) = r$. We conclude that, with

$$c^*(r) := \sup_{\{\widetilde{\xi}_r \ : \ f(\mathrm{sat}(\widetilde{\xi}_r)) = r\}} V(r,\widetilde{\xi}_r) \ , \qquad (12)$$

the set $\mathcal{A}_s(c^*(r))$ is globally asymptotically stable. Of course, when $\widetilde{\xi}_r$ is unique and thus equal to ξ_r then $c^*(r) = 0$ and $\mathcal{A}_s(c^*(r))$ is the point (r,ξ_r). If the map $u_c \mapsto f(u_c)$ is monotone, this applies to every value of r considered.

Now we ask: to what extent does this asymptotic stability property also hold for the complete closed-loop system (1)-(4), (7), at least for $\varepsilon > 0$ sufficiently small? The answer is "semiglobally, practically". By this we mean:

Proposition 1: Let $f(\cdot)$ and r satisfy (9). Let r generate the set $\mathcal{A}(c^*(r))$ via the equations (10)-(12). For each compact subset Ω of $\mathbb{R}^2 \times \left(\mathbb{R}^2 \setminus \{0\}\right)$ and each neighborhood $\mathcal{N}$ of the compact set

$$\{(v_s,\xi,v_f,I_f) : (v_s,\xi) \in \mathcal{A}_s(c^*(r)) \ , \ (v_f,I_f) \in \mathcal{A}_f(\xi)\} \qquad (13)$$

there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0,\varepsilon^*]$ there is a set in $\mathcal{N}$ that is asymptotically stable with basin of attraction containing Ω. ■

Proof. The result follows as a special case of our general results presented below. See, especially, Section VI-A. ■

IV. Unified Framework

A. Assumptions

We consider nonlinear dynamical systems with state x decomposed into a "slow" state x_s and a "fast" state x_f, and driven by a "slow" disturbance d_s and a "fast" disturbance d_f. The governing differential equation has the form

$$\begin{array}{rcl} \dot x_s & = & F_s(x_s,x_f,d_s(t),d_f(t),\varepsilon) \\ \dot x_f & = & F_f(x_s,x_f,d_s(t),d_f(t),\varepsilon) \end{array} \qquad (14)$$

[1] Local Lipschitz continuity follows from the local Lipschitz continuity of the right-hand side of the van der Pol equation together with the fact that $f(u_c)$ can be determined by considering one initial condition in the limit cycle corresponding to u_c and integrating over one period of the limit cycle, since the basin of attraction for this limit cycle is $\mathbb{R}^2 \setminus \{0\}$.

where ε is a small, positive parameter. To fit time-varying systems into the form (14), we augment the state-space with the equations $\dot{t} = 1$ and/or $\dot{t} = \varepsilon$. The equation is assumed to have at least one solution, locally in time, for each initial condition and disturbance of interest. The functions d_s and d_f belong to the sets $\mathcal{D}_{s,\varepsilon}$ and $\mathcal{D}_f$ of measurable functions taking values in subsets of Euclidean space $\mathcal{V}_s$ and $\mathcal{V}_f$, respectively. The disturbance sets are such that, for each $\varepsilon > 0$, the solution set to (14) is time-invariant:

Assumption 1: The sets $\mathcal{D}_{s,\varepsilon}$ and $\mathcal{D}_f$ are shift invariant, i.e., if $t \mapsto d_s(t)$ is belongs to $\mathcal{D}_{s,\varepsilon}$ then $t \mapsto d_s(t + t_\circ)$ belongs to $\mathcal{D}_{s,\varepsilon}$ for all $t_\circ$, and similarly for $\mathcal{D}_f$. ■

The "slow" (and, by extension, "fast") terminology we are using is justified by the following assumption which is geared toward ensuring that, when ε is small, the rates of change of $t \mapsto x_s(t)$ and $t \mapsto d_s(t)$ are small.

Assumption 2: The following conditions hold:
1. $F_s(x_s, x_f, d_s, d_f, 0) = 0$ for all $(x_s, x_f, d_s, d_f) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f} \times \mathcal{V}_s \times \mathcal{V}_f$.
2. For each $T > 0$ and $\rho > 0$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$d_s \in \mathcal{D}_{s,\varepsilon} \quad \Longrightarrow \quad |d_s(t) - d_s(0)| \le \rho \quad \forall t \in [0, T] \ . \qquad (15)$$

■

Remark IV.1: An example of a class of functions $\mathcal{D}_{s,\varepsilon}$ that satisfies Assumption 2 is the class of functions given by $t \to d_s(\varepsilon t)$ where $d_s(\cdot)$ belongs to a class of uniformly equi-continuous functions.

In general, the condition on the set $\mathcal{D}_{s,\varepsilon}$ does not require that the time derivative of the slowly-varying input $d_s(\cdot)$ exists or, if it exists, that it is small when ε is small. (For the class mentioned above, equi-continuity of $d_s(\cdot)$ does not necessarily imply that the derivative of $t \mapsto d_s(\varepsilon t)$ exists or is small when ε is small.) In this way, our statements will be related to "slowly-varying" results where the derivative of the disturbance is only required to be small in the mean, like in [36] and [27]. ■

Remark IV.2: In our motivational example, the state of the plant and controller, v_s and ξ, corresponds to the slow state variables while the states of the oscillating van der Pol circuit, v_f and I_f, correspond to the fast state variables. The motivational example can be put into a form where Assumption 2 holds by changing from the original time-scale t to a new time-scale t/ε. ■

Remark IV.3: The fast disturbance d_f can represent environmental influences and/or can be used to realize the behavior of a differential inclusion with a nonempty, compact, convex right-hand side. There are several ways to express a set-valued map in terms of a function with a parameter ranging over a unit ball, in this case d_f ranging over the unit ball in $\mathbb{R}^{n_s} \times \mathbb{R}^{n_f} =: \mathcal{V}_f$. The parameterization based on the Steiner selection is one the most appealing because the resulting function inherits the continuity properties of the set-valued map it is parameterizing. See [10, Chapter 9]. For related remarks, see Section VI-B. ■

For the system (14), we are interested in the infinite time interval input-to-state behavior resulting from stability assumptions on two simplified systems that arise from (14). The first simplified system that we consider, which is obtained from (14) by setting $\varepsilon = 0$ and using Assumption 2, is the system

$$\begin{aligned} \dot{x}_s &= 0 \\ \dot{x}_f &= F_f(x_s, x_f, d_s, d_f(t), 0) \\ \dot{d}_s &= 0 \ . \end{aligned} \qquad (16)$$

We will refer to this system as the *boundary layer* system so that our terminology is consistent with the classical singular perturbation literature. We will use z_{bl} to denote the composite state for this system, i.e., $z_{bl} := (x_s, x_f, d_s)$.

We express our stability assumption on the boundary layer system in terms of two "measuring functions", an output measuring function $z_{bl} \mapsto \omega_{f,o}(z_{bl})$ and an input measuring function $d_f \mapsto \omega_{f,i}(d_f)$. The measuring functions are not required to be continuous, *a priori*, and they are allowed to take values in $\mathbb{R}_{\ge 0} \bigcup \{\infty\}$. A common example of an output measuring function is the norm, i.e., $z_{bl} \mapsto |z_{bl}|$, or the distance to a closed set $\mathcal{A}$, i.e., $z_{bl} \mapsto |z_{bl}|_{\mathcal{A}} := \inf_{\zeta \in \mathcal{A}} |\zeta - z_{bl}|$. A common example of an input measuring function is a class-$\mathcal{K}$ function γ of the norm, i.e., $d_f \mapsto \gamma(|d_f|)$. Another common output measuring function, relevant for the case where d_f is used to realize the behavior of a differential inclusion, is the identically zero function. The set $\mathcal{H}_f$ denotes a set of initial conditions for the system (16).

Assumption 3: There exist a class-$\mathcal{KL}$ function β_f such that, for all initial conditions $z_{bl}(0) \in \mathcal{H}_f$ and all disturbances $d_f \in \mathcal{D}_f$, the solutions of (16) exist and satisfy

$$\omega_{f,o}(z_{bl}(t)) \le \max\{\ \beta_f(\omega_{f,o}(z_{bl}(0)), t)\ ,\ ||\omega_{f,i}(d_f)||_\infty\ \} \qquad (17)$$

for all $t \ge 0$. ■

Remark IV.4: For our motivational example, we can take $\omega_{f,i}$ identically zero (we considered no disturbances), and $\mathcal{H}_f = \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})$. Then, following the lead of Kurzweil [33], and also of [66, Section 3.3], we can pick the function $\omega_{f,o}$ in the following way: Define the set (recall that $\mathcal{A}_f(u)$ is the set of points on the limit cycle associated with u)

$$\widetilde{\mathcal{A}}_f := \{(v_s, \xi, v_f, I_f) : (v_f, I_f) \in \mathcal{A}_f(\mathrm{sat}(\xi))\} \qquad (18)$$

and let $\delta > 0$ be such that the (closure of the) set

$$\mathcal{P} = \left\{(v_f, I_f) : \exists (v_s, \xi) \text{ s.t. } |(v_s, \xi, v_f, I_f)|_{\widetilde{\mathcal{A}}_f} \le \delta\right\} \qquad (19)$$

does not contain the origin. Such a δ exists since there is positive distance from the origin to $\bigcup_{u \in [-1,1]} \mathcal{A}_f(u)$. Define

$$\kappa := \inf_{(v_f, I_f) \in \mathcal{P}} |(v_f, I_f)| \qquad (20)$$

and pick $\omega_{f,o}$ to be the function

$$\omega_{f,o}(z) := \max\left\{ |(v_s, \xi, v_f, I_f)|_{\widetilde{\mathcal{A}}_f}\ ,\ \frac{1}{|(v_f, I_f)|} - \frac{2}{\kappa} \right\} \qquad (21)$$

where, in this definition, $z = (v_s, \xi, v_f, I_f)$. It follows immediately that

$$|z|_{\widetilde{\mathcal{A}}_f} \le \omega_{f,o}(z) \qquad (22)$$

and, with the definition of δ, that

$$\omega_{f,o}(z) \le \delta \quad \Longrightarrow \omega_{f,o}(z) = |z|_{\widetilde{\mathcal{A}}_f} \tag{23}$$

and $\omega_{f,o}$ is unbounded as (v_f, I_f) approaches the origin. The fact that Assumption 3 holds with this choice for $\omega_{f,o}$ follows from the reasoning used to establish [66, Proposition 3].[2] ■

Remark IV.5: While it is the $\mathcal{L}_\infty$ norm of the measuring function of the inputs that shows up explicitly in the stability bound (17), other norms are easily addressed in our framework. For example, let $\alpha : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a given continuous function and suppose $\mathcal{D}_f$ is the set of functions of the form $d_f = (d_{1_f}, d_{2_f})$ where $d_{1_f}(\cdot)$ is any function such that $\alpha(|d_{1_f}(\cdot)|)$ is forward integrable and $d_{2_f}(t) := \int_0^t \alpha(|d_{1_f}(\tau)|)d\tau$. If we take $\omega_{f,i}(d_f) = d_{2_f}$ then

$$||\omega_{f,i}(d)||_\infty = \int_0^\infty \alpha(|d_{1_f}(t)|)dt \ . \tag{24}$$

The given set $\mathcal{D}_f$ doesn't formally satisfy Assumption 1 but if the differential equation (14) is independent of d_{2_f} then the solution set will be time-invariant, which is the main reason for Assumption 1. In this case, the stability bound (17) relates the output measuring function to the integral of the actual disturbances d_{1_f}. In this way we recover the "integral input-to-state stability" property introduced in [57] and studied in [2] and [3]. ■

In the next assumption, we define the right-hand side of a *reduced* system using the average effect of the boundary layer's asymptotic behavior on F_s. In what follows $\mathcal{S}_{bl}(z_{bl}, d_f)$ represents all solutions of the boundary layer system (16) starting at z_{bl} under the influence of the function d_f. The set $\mathcal{R}_s$ represents a set over which we expect the x_s component of the solution to (14) to range. The set $\mathcal{K}_f$ is a set that we expect to be recurrent for the fast dynamics. The function $\omega_{av,i}$ is another measuring function.

Assumption 4: There exist an integer m, a function $F_{av} : \mathbb{R}^{n_s} \times \mathcal{V}_s \times \mathcal{V}_f \times \mathbb{R}^m$, and, for each $\rho > 0$, there exist $T^* > 0$ and $\varepsilon^* > 0$ such that for each

$$\left.\begin{aligned} T &\ge T^* \\ \varepsilon &\in (0, \varepsilon^*] \\ (x_s, d_s, d_f) &\in \mathcal{R}_s \times \mathcal{V}_s \times \mathcal{D}_f \\ z_{bl} = (x_s, x_f, d_s) &\in \mathcal{K}_f \\ \omega_{f,o}(z_{bl}) &\le ||\omega_{f,i}(d_f)||_\infty \\ \phi_{bl} &\in \mathcal{S}_{bl}(z_{bl}, d_f) \end{aligned}\right\} \tag{25}$$

there exists a measurable function $e : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ satisfying

$$||e||_\infty \le ||\omega_{av,i}(d_f)||_\infty \tag{26}$$

[2]The cited proposition requires $\widetilde{\mathcal{A}}_f$ to be compact, which it is not. However, since neither the boundary layer dynamics nor the set $\widetilde{\mathcal{A}}_f$ depend explicitly on v_s, that component of z can be ignored. Moreover, since the dependence on ξ in the boundary layer dynamics and the set $\widetilde{\mathcal{A}}_f$ is restricted to the range $[-1, 1]$, the compact set arguments used in the reference are still appropriate.

such that

$$\left| \int_0^T \left[F_s(x_s, \phi_{f_{bl}}(t), d_s, d_f(t), \varepsilon) - \varepsilon F_{av}(x_s, d_s, d_f(t), e(t)) \right] dt \right| \le T\rho\varepsilon \ . \tag{27}$$

■

Remark IV.6: For our motivational example, we can take $m = 0$, F_{av} to be the right-hand side of the reduced system (8), $\omega_{av,i}$ identically zero, $\mathcal{R}_s = \mathbb{R}^2$ and $\mathcal{K}_f = \mathbb{R}^4$. Indeed, with these choices, Assumption 4 is satisfied if (recall the time-scale change indicated in Remark IV.2): for each $\rho > 0$ there exists $T^* > 0$ such that, for all $T \ge T^*$ and all $(v_f(0), I_f(0)) \in \mathcal{A}_f(\mathrm{sat}(\xi))$ (see also the next remark for further clarification),

$$\left| \frac{1}{T} \int_0^T \left[K|v_{f_{bl}}(t)| - v_{dc} - f(\mathrm{sat}(\xi)) \right] dt \right| \le \rho \tag{28}$$

where $v_{f_{bl}}(\cdot)$ represents the evolution of the voltage in the van der Pol circuit with $u_c = \mathrm{sat}(\xi)$ and $\varepsilon = 1$. For each value of ξ, the integrand is periodic with zero mean. Moreover, there is an upper bound on the period and a bound on the integrand. Thus, the integral is periodic in T with a known bound and so, for the given ρ, it is easy to pick $T^* > 0$ so that (28) holds for all $T \ge T^*$. Note that ε^* can be anything since the integral does not depend on ε. ■

Remark IV.7: The way we restrict our attention to asymptotic trajectories of the boundary layer is by considering only those initial conditions $z_{bl} \in \mathcal{K}_f$ of the boundary layer (16) satisfying

$$\omega_{f,o}(z_{bl}) \le ||\omega_{f,i}(d_f)||_\infty$$

(compare with (17) letting $t \to \infty$). For the motivational example, this corresponds to only considering initial conditions of the boundary layer on a limit cycle, since $\omega_{f,i} \equiv 0$ and $\omega_{f,o}$ is zero only on a limit cycle. ■

Assumption 4 is used to generate the *reduced* system

$$\dot{x}_s = \varepsilon F_{av}(x_s, d_s(t), d_f(t), e(t)) \ . \tag{29}$$

The role of $e(\cdot)$ in the definition of the average is to allow the possibility of an ensemble of solutions for the average system corresponding to multiple steady-state solutions of the boundary layer system. The average system proposed in [5], expressed in terms of invariant measures and pertaining to the case where d_s and d_f are not present, is typically a differential inclusion with a nonempty, compact, convex right-hand side because the steady-state behavior of the boundary layer system is often different for different initial conditions. A differential inclusion can be recovered through e by taking the measuring function $\omega_{av,i}$ to be identically equal to one and using the idea in Remark IV.3. For example, when F_s depends on (x_s, x_f, ε), $x \mapsto \widetilde{F}_{av}(x_s)$ is a set-valued map with nonempty, compact, convex values and $F_{av}(x_s, \mathcal{B}) = \widetilde{F}_{av}(x_s)$ for all x_s, then Assumption 4 holds with $F_{av}(x_s, e)$ and $\omega_{av,i} \equiv 1$ if and only if (using

[14, §5, Lemma 12]) for each $\rho > 0$ there exists $T^* > 0$ and $\varepsilon^* > 0$ such that, whenever (25) is satisfied, we have

$$\frac{1}{T}\int_0^T \frac{F_s(x_s, \phi_{f_{bl}}(t), \varepsilon)}{\varepsilon} dt \subseteq \widetilde{F}_{av}(x_s) + \rho\mathcal{B} \ . \tag{30}$$

When we talk about a differential inclusion satisfying Assumption 4 in Sections VI-B and VI-I, this is what we mean. Artstein has shown [9, Proposition 3.5], [5, Proposition 4.5], [7, Theorem 4.5] that averages defined in terms of invariant measures have the property corresponding to (30).

Another situation where e allows an ensemble of solutions is when d_f is present and an equilibrium manifold for the boundary layer system is ISS. This situation is discussed in more detail in Section VI-E.1. In the case where $\dot{x}_f = 1$ and $\omega_{av,i}$ is identically zero, Assumption 5 asks for a classical average, or the weak/strong average introduced in [46]. See Section VI-G.

We make the following stability assumption for (29):

Assumption 5: There exists a class-$\mathcal{KL}$ function β_s such that, for all $\varepsilon > 0$, all initial conditions $x_s(0) \in \mathcal{H}_s$, all disturbances $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$, and all e satisfying $||e||_\infty \leq ||\omega_{av,i}(d_f)||_\infty$, the solutions of (29) exist and satisfy

$$\omega_{s,o}(x_s(t)) \leq \max\left\{\beta_s(\omega_{s,o}(x_s(0)), \varepsilon t), ||\omega_{s,i}(d_s, d_f)||_\infty\right\} \tag{31}$$

for all $t \geq 0$. ■

Remark IV.8: For our motivational example, we can take $\mathcal{H}_s = \mathbb{R}^2$, $\omega_{s,o}(v_s, \xi) = \max\{0, V(v_s, \xi) - c^*(r)\}$, and $\omega_{s,i}$ identically zero. The function $\omega_{s,o}$ is positive definite and radially unbounded with respect to the compact set $\mathcal{A}_s(c^*(r))$. (For the definition of this set, see equation (11)-(12) and the surrounding discussion.) ■

Now we want to pass, at least approximately, from the stability bounds on the two simplified systems (see Assumptions 3 and 5) to the corresponding bounds on the original system (14). Assumptions 6 and 7, which follow, will make this possible. We will later guarantee Assumptions 6 and 7 by joining Assumptions 2-5 with regularity assumptions on the functions characterizing the problem.

In Assumptions 6 and 7, $\mathcal{K}_s$ and $\mathcal{K}_f$ are sets of initial conditions from which we want the stability bounds to apply. The set $\mathcal{K}_f$ is the same one considered in Assumption 4. The first of the final two assumptions asks that the solutions of (14) be close, in an appropriate sense, to the solutions of the boundary layer on compact time intervals.

Assumption 6: The following hold:
1. $\sup_{z \in \mathcal{K}_f} \omega_{f,o}(z) < \infty$;
2. There exists $T^* > 0$ and for each $T \geq T^*$ and $\delta > 0$ there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, each $(x_s(0), x_f(0), d_s(0)) \in \mathcal{K}_f$, each $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$, and each solution $(x_s(\cdot), x_f(\cdot))$ of (14), there exists $z_{bl}(0) \in \mathcal{H}_f$ and a solution $z_{bl}(\cdot)$ of (16) such that, with $z(t) := (x_s(t), x_f(t), d_s(t))$,

$$|\omega_{f,o}(z(t)) - \omega_{f,o}(z_{bl}(t))| \leq \delta \qquad \forall t \in [0, T] \tag{32}$$

and

$$z(t) \in \mathcal{K}_f \qquad \forall t \in [T^*, T] \ . \tag{33}$$

■

The last assumption asks that the x_s component of the solutions to (14) be close, in an appropriate sense, to the solutions of the reduced system, on compact time intervals of length proportional to $1/\varepsilon$.

Assumption 7: The following hold:
1. $\sup_{x_s \in \mathcal{K}_s} \omega_{s,o}(x_s) < \infty$;
2. There exists $T^* > 0$ and for each $T \geq T^*$ and each $\delta > 0$ there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, each $x_s(0) \in \mathcal{K}_s$, each $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$, and each solution $x_s(\cdot)$ of (14), there exists $x_{s,av}(0) \in \mathcal{H}_s$ and a solution $x_{s,av}(\cdot)$ of (29) such that

$$|\omega_{s,o}(x_s(t)) - \omega_{s,o}(x_{s,av}(t))| \leq \delta \qquad \forall t \in [0, T/\varepsilon] \tag{34}$$

and

$$x_s(t) \in \mathcal{K}_s \qquad \forall t \in [T^*/\varepsilon, T/\varepsilon] \ . \tag{35}$$

■

Remark IV.9: For our motivational example, we can take $\mathcal{K}_s$ to be any compact subset of $\mathbb{R}^2$ that contains a neighborhood of $\mathcal{A}_s(c^*(r))$, and we can take $\mathcal{K}_f$ to be any sufficiently large compact subset of $\mathbb{R}^2 \times \left(\mathbb{R}^2 \setminus \{0\}\right)$. In particular $\mathcal{K}_f$ should contain a neighborhood of the family of attractors and its projection onto the (v_s, ξ) coordinates should contain a compact set that is determined by $\mathcal{K}_s$. For more details, see Remark V.1.

To make a convincing argument that Assumptions 6 and 7 hold for our motivational example, we will need to wait until we specify sufficient conditions for these assumptions in Section V. See the discussion in Section VI-A. ■

B. General Result

We are now prepared to state and prove the result that follows from the assumptions we have made. In the next section, we will give sufficient conditions for the functions defining the problem to guarantee that Assumptions 6 and 7 hold. After that, we will investigate the meaning of our assumptions for special cases corresponding to weakly nonlinear oscillators, singular perturbations with an equilibrium manifold, regular and slowly varying perturbations, classical averaging and partial averaging.

Theorem 1: If Assumptions 1, 3, and 5-7 hold then for each $\delta > 0$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, all $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$ and all initial conditions such that $x_s(0) \in \mathcal{K}_s$ and $z(0) := (x_s(0), x_f(0), d_s(0)) \in \mathcal{K}_f$, the solutions of (14) exist and satisfy

$$\begin{aligned}&\omega_{s,o}(x_s(t)) \leq \\ &\qquad \max\{\ \beta_s(\omega_{s,o}(x_s(0)), \varepsilon t)\ ,\ ||\omega_{s,i}(d_s, d_f)||_\infty\ \} + \delta\end{aligned} \tag{36}$$

and, with $z(t) := (x_s(t), x_f(t), d_s(t))$,

$$\begin{aligned}&\omega_{f,o}(z(t)) \leq \\ &\qquad \max\{\ \beta_f(\omega_{f,o}(z(0)), t)\ ,\ ||\omega_{f,i}(d_f)||_\infty\ \} + \delta\end{aligned} \tag{37}$$

for all $t \geq 0$.

Proof. Define

$$\begin{aligned} c_s &:= \sup_{x_s \in \mathcal{K}_s} \omega_{s,o}(x_s) \\ c_f &:= \sup_{z \in \mathcal{K}_f} \omega_{f,o}(z) \ . \end{aligned} \tag{38}$$

These values are finite according to the first items of Assumptions 6 and 7. Let $\delta > 0$ be given and let $\widetilde{\delta} > 0$ be such that

$$\begin{aligned} \sup_{r\in[0,c_s],t\in[0,\infty)} \left[\beta_s(r+\widetilde{\delta},t) - \beta_s(r,t)\right] + \widetilde{\delta} &\le \frac{\delta}{2} \\ \sup_{r\in[0,c_f],t\in[0,\infty)} \left[\beta_f(r+\widetilde{\delta},t) - \beta_f(r,t)\right] + \widetilde{\delta} &\le \frac{\delta}{2} \ . \end{aligned} \tag{39}$$

The existence of $\widetilde{\delta}$ follows from the properties of class-$\mathcal{KL}$ functions. Let $T_6^* > 0$ and $T_7^* > 0$ come from Assumptions 6 and 7, respectively. Then let $T \ge \max\{T_6^*, T_7^*\}$ be large enough so that

$$\begin{aligned} \beta_s(c_s,\tau) &\le \frac{\delta}{2} \qquad \forall \tau \in [T,\infty) \\ \beta_f(c_f,\tau) &\le \frac{\delta}{2} \qquad \forall \tau \in [T,\infty) \ . \end{aligned} \tag{40}$$

The existence of T follows from the properties of class-$\mathcal{KL}$ functions. Let $\varepsilon_6^* > 0$ and $\varepsilon_7^* > 0$ come from Assumptions 6 and 7, respectively, for the pair $(2T, \widetilde{\delta})$. Let $z(\cdot)$ be a solution of the original system (14) starting in $\mathcal{K}_f$ and let $z_{bl}(\cdot)$ be the corresponding solution of the boundary layer system (16) given by Assumption 6. Then, from Assumptions 3 and 6 and using (39), we have that, for all $t \in [0, 2T]$,

$$\begin{aligned} &\omega_{f,o}(z(t)) \\ &\quad \le \omega_{f,o}(z_{bl}(t)) + \widetilde{\delta} \\ &\quad \le \max\{\beta_f(\omega_{f,o}(z_{bl}(0)),t) \ , \ \|\omega_{f,i}(d_f)\|_\infty\} + \widetilde{\delta} \\ &\quad \le \max\left\{\beta_f(\omega_{f,o}(z(0)) + \widetilde{\delta},t) \ , \ \|\omega_{f,i}(d_f)\|_\infty\right\} + \widetilde{\delta} \\ &\quad \le \max\{\beta_f(\omega_{f,o}(z(0)),t) \ , \ \|\omega_{f,i}(d_f)\|_\infty\} + \frac{\delta}{2} \ . \end{aligned} \tag{41}$$

Using (40) it also follows from (41) that for $t \in [T, 2T]$,

$$\omega_{f,o}(z(t)) \le \|\omega_{f,i}(d_f)\|_\infty + \delta \ . \tag{42}$$

Finally, since $T \ge T_6^*$ it follows that $z(T) \in \mathcal{K}_f$. Now, using the shift invariance of $\mathcal{D}_{s,\varepsilon}$ and $\mathcal{D}_f$ provided by Assumption 1, the argument can be applied repeatedly to obtain

$$\omega_{f,o}(z(t)) \le \|\omega_{f,i}(d_f)\|_\infty + \delta \qquad \forall t \in [T,\infty) \ . \tag{43}$$

The conclusion (37) then follows from (41) and (43).

The conclusion (36) is obtained with a calculation like (41)-(43) using Assumptions 5 and 7. ■

V. Conditions for close solutions

In this section we present sufficient conditions on the measuring functions in Assumptions 3 and 5 and the functions on right-hand side of the actual system (14) and the average system (29) that guarantee Assumptions 6 and 7. Our first assumption concerns the measuring functions. This assumption is made to guarantee the recurrence of the sets $\mathcal{K}_f$ and $\mathcal{K}_s$ as in Assumptions 6 and 7.

Assumption 8: The following conditions hold:
1. $\mathcal{K}_s \subseteq \mathcal{H}_s$,
2. $\mathcal{K}_f \subseteq \mathcal{H}_f$,
3. $$\sup_{z \in \mathcal{K}_f} \omega_{f,o}(z) =: c_{f,o} < \infty \tag{44}$$
4. $$\sup_{d_f \in \mathcal{V}_f} \omega_{f,i}(d_f) =: c_{f,i} < \infty \tag{45}$$
5. $$\sup_{x_s \in \mathcal{K}_s} \omega_{s,o}(x_s) =: c_{s,o} < \infty \tag{46}$$
6. $$\sup_{d_s,d_f \in \mathcal{V}_s \times \mathcal{V}_f} \omega_{s,i}(d_s,d_f) =: c_{s,i} < \infty \tag{47}$$
7. There exists $\rho > 0$ such that
$$\{x_s : \omega_{s,o}(x_s) \le c_{s,i} + \rho\} \subseteq \mathcal{K}_s \tag{48}$$
and
$$\begin{aligned} \{z = (x_s,x_f,d_s) : d_s \in \mathcal{V}_s \ , \ \omega_{f,o}(z) \le c_{f,i} + \rho \ , \\ \omega_{s,o}(x_s) \le \max\{\beta_s(c_{s,o},0) \ , \ c_{s,i}\} + \rho\} \quad \subseteq \quad \mathcal{K}_f \ . \end{aligned} \tag{49}$$

Remark V.1: For our motivational example, we can take $\mathcal{K}_s$ to be an arbitrary compact subset of $\mathbb{R}^2$ that contains a neighborhood of the set where $\omega_{s,o}$ is zero (since $c_{s,i} = 0$). Then we can take $\mathcal{K}_f$ to be a large enough compact subset of $\mathbb{R}^2 \times \left(\mathbb{R}^2 \setminus \{0\}\right)$ to satisfy (49). This remark adds more detail to Remark IV.9. ■

The last assumption we make is on the continuity of the functions that define the problem on the sets of interest.

Assumption 9: There exist $L > 0$, $M > 0$ and $\sigma > 0$ and for each $\rho > 0$ there exists $\varepsilon^* > 0$ such that, defining

$$\mathcal{X}_s := \{x_s : \omega_{s,o}(x_s) \le \max\{\beta_s(c_{s,o},0) \ , \ c_{s,i}\} \} \tag{50}$$

and

$$\mathcal{Z}_f := \{z : \ \omega_{f,o}(z) \le \max\{\beta_f(c_{f,o},0) \ , \ c_{f,i}\} \} \tag{51}$$

and

$$\begin{aligned} \mathcal{U}_f(\sigma) := \{z = (x_s,x_f,d_s) : \\ x_s \in \mathcal{X}_s + \sigma\mathcal{B} \quad , \quad z \in \mathcal{Z}_f + \sigma\mathcal{B} \ , \ d_s \in \mathcal{V}_s \ \} \end{aligned} \tag{52}$$

we have:
1. $\omega_{s,o}$ is uniformly continuous on $\mathcal{X}_s + \sigma\mathcal{B}$,
2. $\mathcal{X}_s + \sigma\mathcal{B} \subseteq \mathcal{R}_s$,[3]
3. $\omega_{f,o}$ is uniformly continuous on $\mathcal{U}_f(\sigma)$,
4. for each $c \in [0, c_{f,i}]$, if $\omega_{f,o}(z) \le c + \varepsilon^*$ then there exists z_c such that $\omega_{f,o}(z_c) \le c$ and $|z - z_c| \le \rho$,

[3]Recall that the set $\mathcal{R}_s$ comes from Assumption 4 and characterizes a region where the integral of F_s is approximately equal to the integral of εF_{av}.

5. for all $d_f \in \mathcal{V}_f$, $(x_s, x_f, d_s), (y_s, y_f, w_s) \in \mathcal{U}_f(\sigma)$, $|d_s - w_s| \le \varepsilon^*$, $|e| \le \sup_{d_f \in \mathcal{V}_f} \omega_{av,i}(d_f)$, $\varepsilon \in (0, \varepsilon^*]$,

$$|F_s(x_s, x_f, d_s, d_f, \varepsilon)| \le \varepsilon M \quad (53)$$

$$|F_{av}(x_s, d_s, d_f, e)| \le M \quad (54)$$

$$\max\{|x_s - y_s|\,,\ |x_f - y_f|\} \le \varepsilon^* \Longrightarrow$$

$$|F_s(x_s, x_f, d_s, d_f, \varepsilon) - F_s(y_s, y_f, w_s, d_f, \varepsilon)| \le \varepsilon\rho \quad (55)$$

$$|x_s - y_s| \le \varepsilon^* \Longrightarrow$$

$$|F_f(x_s, x_f, d_s, d_f, \varepsilon) - F_f(y_s, y_f, w_s, d_f, 0)| \le L\left(|x_f - y_f| + \rho\right) \quad (56)$$

$$|F_{av}(x_s, d_s, d_f, e) - F_{av}(y_s, d_s, d_f, e)| \le L|x_s - y_s|\ . \quad (57)$$

Remark V.2: The purpose of item 4 is to guarantee that if $\omega_{f,o}(z)$ is close to c then z is close to the set where $\omega_{f,o}(\zeta) \le c$. Consider the special case where $c_{f,i} = 0$. In this case, if it is possible to find $\eta > 0$ and a class-$\mathcal{K}$ function α such that, with $\mathcal{A}_f := \{z : \omega_{f,o}(z) = 0\}$,

$$\omega_{f,o}(z) \le \eta \qquad \Longrightarrow \qquad \alpha(|z|_{\mathcal{A}_f}) \le \omega_{f,o}(z) \quad (58)$$

then item 4 is satisfied for any $\varepsilon^* \le \min\{\eta, \alpha(\rho)\}$). As established in Remark IV.4, specifically by (22), the condition (58) is satisfied for our motivational example. ■

Remark V.3: For our motivational example, the underlying sets restrict our attention to a compact subset of $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})$. The right-hand side of our example is locally Lipschitz and our measuring functions are continuous on this set. Combining this observation with the previous remark establishes that our motivational example, in the appropriate time-scale, satisfies Assumption 9. ■

The main result of this section is that, under Assumptions 2-5, we can guarantee Assumptions 6 and 7, which are assumptions about trajectories, by replacing them with Assumptions 8 and 9 which are assumptions about functions.

Proposition 2: If Assumptions 2 - 5, 8, and 9 hold then Assumptions 6 and 7 hold.

Proof. See Section VII-A. ■

Remark V.4: The uniform (over $\mathcal{U}_f(\sigma)$) Lipschitz continuity of F_f with respect to x_f, respectively, F_s with respect to x_s, can be relaxed to mere continuity when all of the underlying sets are compact. However, the proof technique for such results is significantly different than the technique used here. The alternative approach is based on classical results on continuity of solutions on compact time intervals as can be found in, for example, [14, §8], and is used extensively in the work of Artstein [9], [5], [6], [7]. In Sections VI-B and VI-I we illustrate how, in various situations, continuity can be converted to Lipschitz continuity without difficulty. ■

VI. Applications

We now discuss how our general result applies to several situations where F_s and/or F_f have special structure that corresponds to classical robustness problems.

A. Motivational example

Following Remarks IV.2, IV.4, IV.6, IV.8, IV.9, V.1, V.2, and V.3, we find that our motivational example satisfies all of the assumptions of Proposition 2. It then follows from the combination of Proposition 2 and Theorem 1 that the trajectories can be made to converge in finite time from an arbitrarily large compact subset Ω of $\mathbb{R}^2 \times (\mathbb{R}^2 \times \setminus \{0\})$ to an arbitrarily small neighborhood of the set defined in (13). According to [66, Proposition 4], the reachable set from this small neighborhood, which is again an arbitrarily small neighborhood, is asymptotically stable with basin of attraction containing Ω.

B. Reduced system is an upper semicontinuous inclusion

As mentioned below (29), the approach in [9], [5] often produces a reduced system (29) that is a differential inclusion even though the original system (14) has no exogenous disturbances and its right-hand side is locally Lipschitz, so that solutions are unique. This is because the steady-state behavior of the boundary layer system is often different for different initial conditions. Reduced systems that are differential inclusions with compact, convex right-hand sides can be realized in our framework by taking $\omega_{av,i}$ identically equal to one in Assumption 4 and exploiting $e(\cdot)$ to parameterize the set-valued map of the inclusion. However, the resulting function F_{av} does not automatically satisfy (57). We now show how an inclusion can be converted to a function depending on e satisfying (57) without significantly changing the stability property of the reduced system.

Suppose, like in [9], [5], the reduced system (29) corresponds to a differential inclusion

$$\dot{x}_s \in \widetilde{F}_{av}(x_s) \quad (59)$$

where, for each x_s, $\widetilde{F}_{av}(x_s)$ is nonempty, compact and convex, and the set-valued map $\widetilde{F}_{av}$ is upper semicontinuous.[4] This includes the case where the right-hand side is a continuous function. Suppose further that the differential inclusion (59) has an asymptotically stable attractor $\mathcal{A}_s$ with basin of attraction $\mathcal{H}_s$. According to the main results of [66], $\mathcal{H}_s$ is an open set and, letting $\omega_{s,o}$ be any proper indicator function [5] for $\mathcal{A}_s$ on $\mathcal{H}_s$, there exists $\beta_s \in \mathcal{KL}$ and for each $\delta > 0$ and each compact subset $\mathcal{X}_s$ of $\mathcal{H}_s$ there exists a set-valued map $\widetilde{F}_{av,\ell}$ $\widetilde{L} > 0$, $\sigma > 0$ such that $\widetilde{F}_{av,\ell}(x_s)$ is nonempty, compact and convex for each x_s, $\widetilde{F}_{av,\ell}$ is locally Lipschitz,

$$\widetilde{F}_{av}(x_s) \subseteq \widetilde{F}_{av,\ell}(x_s)\ , \quad (60)$$

$$\widetilde{F}_{av,\ell}(x_s) \subseteq \widetilde{F}_{av,\ell}(y_s) + \widetilde{L}|x_s - y_s|\mathcal{B} \qquad \forall x_s, y_s \in \mathcal{X}_s + \sigma\mathcal{B} \quad (61)$$

and Assumption 5 is satisfied for

$$\dot{x}_s \in \widetilde{F}_{av,\ell}(x_s) \quad (62)$$

[4] A set-valued map F is said to be upper semicontinuous if for each x and each $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi - x| \le \delta$ implies $F(\xi) \subseteq F(x) + \varepsilon\mathcal{B}$.

[5] When $\mathcal{A}$ is compact and $\mathcal{H} \supset \mathcal{A}$ is open, $\omega : \mathcal{H} \to \mathbb{R}_{\ge 0}$ is said to be a proper indicator function for $\mathcal{A}$ on $\mathcal{H}$ if $\omega(x) = 0$ iff $x \in \mathcal{A}$ and $\omega(x) \to \infty$ as $x \to \partial\mathcal{H}$ or $|x| \to \infty$ if $\mathcal{H}$ is unbounded.

with $\omega_{s,o}$, β_s, $\mathcal{H}_s$, and $\omega_{s,i} \equiv \delta$. Now, according to [10, Theorem 9.7.2] or [15, Proposition 2.22], there exists a function $(x_s, e) \mapsto F_{av}(x_s, e)$ such that

$$F_{av}(x_s, \mathcal{B}) = \widetilde{F}_{av,\ell}(x_s) \tag{63}$$

and such that (57) holds with $L = 10 n_s \widetilde{L}$.

So we see that when F_{av} is a continuous function or an upper semicontinuous set-valued map with nonempty, compact, convex values and $\omega_{s,o}$ is a proper indicator function for a compact set on the set's basin of attraction, we can easily maneuver into the situation where (57) holds.

C. Weakly nonlinear oscillators

Our results can easily be applied to what [52, Chapter 5] refers to as "averaging over spatial variables". As an example, we consider systems of the form

$$\begin{aligned} \dot{x}_s &= \varepsilon g\left(x_s \sin(x_f), x_s \cos(x_f)\right) \cos(x_f) \\ \dot{x}_f &= 1 - \varepsilon \frac{1}{x_s} g\left(x_s \sin(x_f), x_s \cos(x_f)\right) \sin(x_f) \end{aligned} \tag{64}$$

where g is continuous. This type of system arises when casting into polar coordinates the weakly nonlinear oscillator equation $\ddot{y} + y = \varepsilon g(y, \dot{y})$. The state x_s is associated with the magnitude of $(y, \dot{y})$ while the state x_f is associated with the phase angle. See, for example, [24, Section V.2], [52, Chapter 5] or [28, Section 8.4].

This system satisfies our Assumptions 2 and 1 since we have no disturbances and $\dot{x}_s$ vanishes when $\varepsilon = 0$. The boundary layer system associated with this system is

$$\begin{aligned} \dot{x}_s &= 0 \\ \dot{x}_f &= 1 \,. \end{aligned} \tag{65}$$

For this system we can take $\mathcal{H}_f = \mathbb{R}$, and $\omega_{f,i}$ and $\omega_{f,o}$ identically zero so that Assumption 3 is automatically satisfied. For Assumption 4 we take $\mathcal{K}_f = \mathbb{R}$, $\mathcal{R}_s$ to be an arbitrarily large, bounded interval, and

$$F_{av}(x_s) = \frac{1}{2\pi} \int_0^{2\pi} g\left(x_s \sin(t), x_s \cos(t)\right) \cos(t) dt \,. \tag{66}$$

With these choices, Assumption 4 is satisfied. We now suppose that the average system has an asymptotically stable equilibrium point x_s^* with basin of attraction $\mathcal{H}_s$. In this case, Assumption 5 holds with $\omega_{s,i}$ equal zero and $\omega_{s,o}$ equal to any function that is zero at x_s^*, positive otherwise, blows up at the boundary of $\mathcal{H}_s$, and is continuous on $\mathcal{H}_s$. We also take $\mathcal{K}_s$ to be any compact subset of $\mathcal{H}_s$. While it is not to difficult to see directly that Assumptions 6 and 7 hold, it also follows easily from our definitions for $\omega_{f,o}$, $\omega_{f,i}$, $\omega_{s,o}$, $\omega_{s,i}$, $\mathcal{K}_s$ and $\mathcal{K}_f$ that Assumption 8 holds, and from the periodicity of g with respect to x_f and the continuity of g that, with the observations of Section VI-B, Assumption 9 can be made to hold. (At this point we fix $\mathcal{R}_s$ to be large enough so that the second item of Assumption 9 holds.) We then recover from our main results the well-known fact that the magnitude variable x_s for the system (64) converges to an arbitrarily small neighborhood of x_s^* from a set arbitrarily close to $\mathcal{H}_s$ as ε becomes arbitrarily small.

D. Two time-scale averaging in adaptive control

In this section we consider systems of the form

$$\begin{aligned} \dot{x}_s &= \varepsilon \widetilde{F}_s(x_s, x_{1_f}, x_{2_f}) \\ \dot{x}_{1_f} &= A(x_s) x_{1_f} + B(x_s, x_{2_f}) + \varepsilon \widetilde{G}(x_s, x_f) \\ \dot{x}_{2_f} &= 1 \end{aligned} \tag{67}$$

which cover the class of systems studied in [50], [49], [53, Chapter 4] and [24, V.3]. Assumptions 2 and 1 are satisfied since we are not considering disturbances and $\dot{x}_s$ vanishes when $\varepsilon = 0$. The boundary layer system is

$$\begin{aligned} \dot{x}_s &= 0 \\ \dot{x}_{1_f} &= A(x_s) x_{1_f} + B(x_s, x_{2_f}) \\ \dot{x}_{2_f} &= 1 \,. \end{aligned} \tag{68}$$

Under suitable assumptions on $A(\cdot)$ and $B(\cdot,\cdot)$, like those in the references mentioned above, there is a uniformly globally asymptotically stable invariant manifold for the boundary layer system given by the set of points

$$\mathcal{A}_f := \left\{ (x_s, x_{1_f}, x_{2_f}) : x_{1_f} = v(x_s, x_{2_f}) \right\} \,. \tag{69}$$

For the purposes of Assumption 3, we take

$$\omega_{f,o}(z_{bl}) = |z_{bl}|_{\mathcal{A}_f} \,. \tag{70}$$

We take $\mathcal{H}_f$ to be $\mathbb{R}^{n_s} \times \mathbb{R}^{n_{1f}} \times \mathbb{R}$. This set can be refined if $A(\cdot)$ and $B(\cdot)$ only have appropriate properties for certain ranges of x_s. We next suppose that $\widetilde{F}_s$ has an average in the sense of Assumption 4, which is the same as asking that the function $\widetilde{F}_s(x_s, v(x_s, t), t)$ have an average in the classical sense (not requiring periodicity). This can be seen by noting that the solution of the last component of the boundary layer has the form $x_{2_f} + t$, where x_{2_f} is an initial condition which can also be thought of as an initial time, and then changing the variable of integration in the integral in Assumption 4 from t to $x_{2_f} + t$. If the average system has an asymptotically stable attractor $\mathcal{A}_s$ having basin of attraction $\mathcal{H}_s$ then Assumption 5 is easily satisfied with an appropriate choice for $\omega_{s,o}$. Finally, under the Lipschitz and boundedness conditions of Assumptions 9 and with the help of Proposition 2 and Theorem 1 we get convergence to an arbitrarily small neighborhood of the set

$$\{(x_s, x_f) : x_s \in \mathcal{A}_s \;,\; (x_s, x_f) \in \mathcal{A}_f\} \tag{71}$$

from a sets of the form $\Omega_1 \times \Omega_2 \times \mathbb{R}$ where Ω_1 is an arbitrarily large compact subset of $\mathcal{H}_s$ and Ω_2 is an arbitrarily large compact subset of $\mathbb{R}^{n_{1f}}$.

E. Systems with a slow equilibrium manifold

E.1 ISS manifold

Consider the situation where the boundary layer system (16) has an equilibrium manifold, given by the set of points

$$\mathcal{A}_f := \{(x_s, x_f, d_s) : x_f = h(x_s, d_s)\} \tag{72}$$

that is "input-to-distance to the manifold" stable. By this we mean that Assumption 3 holds with $\omega_{f,o}(z_{bl}) = |z_{bl}|_{\mathcal{A}_f}$

and $\omega_{f,i}(d_f) = \gamma(|d_f|)$ for some function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is continuous, nondecreasing, and zero at zero. For the special case of $\gamma(s) \equiv 0$, this is the problem considered in [13].

We now clarify the meaning of Assumption 4 for the general problem. We propose as an average the function

$$F_{av}(x_s, d_s, d_f, e) := \lim_{\varepsilon \to 0^+} \frac{F_s(x_s, h(x_s + e_1, d_s + e_2) + e_3, \varepsilon)}{\varepsilon} \tag{73}$$

and restrict our attention to situations where this limit makes sense and the convergence is uniform over the sets of interest. (A simple situation is when $F_s = \varepsilon \widetilde{F}_s$ where $\widetilde{F}_s$ is independent of ε.) We now check Assumption 4 with an appropriate choice for $\omega_{av,i}$. We note that the "steady-state" solutions of the boundary layer are *not* restricted to the equilibrium manifold because of the presence of the disturbances d_f. Nevertheless, we do get to restrict our attention to solutions ϕ_{bl} of the boundary layer that satisfy

$$|\phi_{bl}(0)|_{\mathcal{A}_f} \leq \gamma\left(\|d_f\|_\infty\right) \tag{74}$$

and hence, by Assumption 3,

$$|\phi_{bl}(t)|_{\mathcal{A}_f} \leq \max\left\{\beta_f\left(\gamma\left(\|d_f\|_\infty\right), t\right) \ , \ \gamma\left(\|d_f\|_\infty\right)\right\} =: b(t) \ . \tag{75}$$

By definition this means that for each $t \geq 0$ there exists $e(t) = (e_1(t), e_2(t), e_3(t))$ with

$$|e(t)| \leq b(t) \tag{76}$$

such that

$$\phi_{f_{bl}}(t) = h(x_s + e_1(t), d_s + e_2(t)) + e_3(t) \ . \tag{77}$$

Indeed, for each t let $a(t) = (a_1(t), a_2(t), a_3(t))$ satisfy

$$a_2(t) = h(a_1(t), a_3(t)) \tag{78}$$

and

$$|\phi_{bl}(t) - a(t)| \leq b(t) \ . \tag{79}$$

Define $e(t) := \phi_{bl}(t) - a(t)$. Then, since the first and third components of $\phi_{bl}(t)$ are constant and equal to x_s and d_s, we can write

$$\begin{aligned} \phi_{f_{bl}}(t) &= a_2(t) + e_2(t) \\ &= h(a_1(t), a_3(t)) + e_2(t) \\ &= h(x_s + e_1(t), d_s + e_3(t)) + e_2(t) \ . \end{aligned} \tag{80}$$

It follows that with the choice of F_{av} in (73) that Assumption 4 is satisfied with

$$\omega_{av,i}(d_f) := \max\left\{ \ \beta_f(\gamma(|d_f|), 0) \ , \ \gamma(|d_f|) \ \right\} \ . \tag{81}$$

With a little more work it is possible to reduce $\omega_{av,i}$ to

$$\omega_{av,i}(d_f) = \gamma(|d_f|) \tag{82}$$

when F_{av} is uniformly bounded and uniformly continuous on the sets of interest by splitting the interval of integration into two parts. This goes as follows: Given $\rho > 0$, let $\kappa > 0$ be such that

$$|e - \widetilde{e}| \leq \kappa \implies |F_{av}(x_s, d_s, d_f, e) - F_{av}(x_s, d_s, d_f, \widetilde{e})| \leq \rho/4 \ . \tag{83}$$

Then let $T_1^* > 0$ be such that

$$\beta_f(c_{f,i}, t) \leq \kappa \qquad \forall t \geq T_1^* \tag{84}$$

where $c_{f,i} = \sup_{d_f \in \mathcal{V}_f} \gamma(|d_f|)$. Define $\widetilde{e}(t)$ to be the projection of $e(t)$ onto the closed ball of radius $\gamma(\|d_f\|_\infty)$ so that

$$\|\widetilde{e}\|_\infty \leq \gamma(\|d_f\|_\infty) \ . \tag{85}$$

It follows that for all $t \geq T_1^*$, $|e(t) - \widetilde{e}(t)| \leq \kappa$. Now define $T^* := 4MT_1^*/\rho$ where

$$|F_s(x_s, \phi_{bl}(t), d_s, d_f, \varepsilon) - \varepsilon F_{av}(x_s, d_s, d_f, \widetilde{e})| \leq \varepsilon M \tag{86}$$

over the sets of interest. Using the uniform convergence that produced the average, let $\varepsilon^* > 0$ be such that

$$|F_s(x_s, \phi_{bl}(t), d_s, d_f, \varepsilon) - \varepsilon F_{av}(x_s, d_s, d_f, e(t))| \leq \varepsilon\rho/2 \ . \tag{87}$$

Then, combining (83), (86), (87) and the definition of T^*, we have that, for all $T \geq T^*$,

$$\begin{aligned} &\left| \int_0^T [F_s(x_s, \phi_{bl}(t), d_s, d_f(t), \varepsilon) - \right. \\ &\qquad \left. \varepsilon F_{av}(x_s, d_s, d_f(t), \widetilde{e}(t))] \, dt \right| \\ &\qquad\qquad \leq T\varepsilon\rho/2 + T_1^*\varepsilon M + T\varepsilon\rho/4 \leq \varepsilon T\rho \ . \end{aligned} \tag{88}$$

Now with the average system in place and the measuring function $\omega_{s,av}$ determined, we impose Assumption 5 for the average system and the proposed regularity conditions of Assumptions 8 and 9 to derive the benefits of Theorem 1. This extends the result of [13] in several ways, e.g., the slow manifold of the boundary layer is allowed to be ISS with a nonzero gain, the ISS property for average system is allowed to be with respect to a set, nonglobal basins of attractions are explicitly addressed, and the preservation of ISS gain does not require a Lyapunov formulation.

E.2 Unstable manifold

We now demonstrate, like in [5, Remark 5.1] (see also [9, Section 7]), that our formulation allows the boundary layer system to have an unstable equilibrium manifold.

Consider the system

$$\begin{aligned} \dot{x}_s &= \varepsilon \widetilde{F}_s(x_s, x_f, d_f) \\ \dot{x}_{1_f} &= x_{1_f}^2 (x_{1_f} - x_{2_f}) + x_{2_f}^5 \\ \dot{x}_{2_f} &= x_{2_f}^2 (x_{2_f} - 2x_{1_f}) \end{aligned} \tag{89}$$

where the fast subsystem comes from the classical example by Vinograd [68] of a system with an unstable equilibrium point that is globally attractive. We will show that if $\widetilde{F}(x_s, x_f, d_s)$ is bounded on appropriate sets and $\widetilde{F}_s(x_s, \cdot, d_f)$ is continuous at the origin, uniformly in (x_s, d_f) over the sets of interest, then we can take $F_{av}(x_s, d_f) = \widetilde{F}_s(x_s, 0, d_f)$ in Assumption 4. The analysis here readily extents to the case where the equilibrium point for the fast subsystem is a function of the state of the slow subsystem.

Let $\mathcal{A}_f$ denote the (closure of the) set of points reachable for the fast subsystem in forward time from the set $|x_f| \leq 1$. Since this set is globally attractive and captures all trajectories in a finite time that depends only on the distance from the set, it follows from [66, Proposition 4] this set is globally asymptotically stable, i.e., Assumption 3 holds with $\omega_{f,o}(z) = |x_f|_{\mathcal{A}_f}$ and $\mathcal{H}_f = \mathbb{R}^{n_s} \times \mathbb{R}^2$.

Now to generate the average, we consider trajectories of the boundary layer that start in the set $\mathcal{A}_f$. It is readily apparent from the phase portrait of the boundary layer system (see, for example, [22, Figure 40.3]) that for each $\rho > 0$, the quantity

$$\frac{1}{T}\int_0^T I_\rho(\phi_f(t, x_f))dt \ ,$$

where $I_\rho(x_f) = 1$ if $x_f \in \rho\mathcal{B}$ and is zero otherwise, converges to zero as $T \to \infty$ uniformly in $x_f \in \mathcal{A}_f$. (In the framework of [9], [5], the trajectories of the boundary layer converge in distribution to the origin and the boundary layer has a unique invariant measure, namely the Dirac measure supported at the origin.) It then follows, if $\widetilde{F}_s$ is uniformly bounded over $\mathcal{R}_s \times \mathcal{A}_f \times \mathcal{V}_f$ and continuous in x_f near the origin uniformly in $(x_s, d_f) \in \mathcal{R}_s \times \mathcal{V}_f$, that Assumption 4 holds with $F_{av}(x_s, d_f) = \widetilde{F}_s(x_s, 0, d_f)$.

F. Regular and slowly varying perturbations

F.1 The generic case

Consider the special case where the fast dynamics don't depend on the slow dynamics and we are only interested in the fast dynamics. In this case we can take F_s, F_{av}, $\omega_{s,o}$ and $\omega_{s,i}$ all equal to zero so that Assumptions 4 and 5 automatically hold. We will suppose Assumptions 2 - 3 hold, where our boundary layer system is

$$\begin{array}{rcl} \dot{x}_f &=& F_f(x_f, d_s, d_f(t), 0) \\ \dot{d}_s &=& 0 \ . \end{array} \tag{90}$$

For this system's stability property, given in Assumption 3, we are interested in robustness to regular and slowly varying perturbations. We impose Assumptions 8 and 9 which mainly become conditions on $\mathcal{K}_f$, $\omega_{f,o}$, $\omega_{f,i}$, and F_f. In fact, it turns out that item 4 in Assumption 9 can be dropped for this special case. We extract what remains of Assumptions 8 and 9:

Assumption 8 (b) The following conditions hold:
1. $\mathcal{K}_f \subseteq \mathcal{H}_f$,
2. $$\sup_{z \in \mathcal{K}_f} \omega_{f,o}(z) =: c_{f,o} < \infty \tag{91}$$
3. $$\sup_{d_f \in \mathcal{V}_f} \omega_{f,i}(d_f) =: c_{f,i} < \infty \tag{92}$$
4. there exists $\rho > 0$ such that
$$\{z = (x_f, d_s) : d_s \in \mathcal{V}_s \ , \ \omega_{f,o}(z) \leq c_{f,i} + \rho\} \subseteq \mathcal{K}_f \ . \tag{93}$$

Assumption 9 (b) There exist $L > 0$ and $\sigma > 0$ and for each $\rho > 0$ there exists $\varepsilon^* > 0$ such that, defining

$$\mathcal{Z}_f := \{z : \omega_{f,o}(z) \leq \max\{\ \beta_f(c_{f,o}, 0) \ , \ c_{f,i}\}\ \} \tag{94}$$

and

$$\mathcal{U}_f(\sigma) := \{z = (x_f, d_s) : z \in \mathcal{Z}_f + \sigma\mathcal{B} \ , \ d_s \in \mathcal{V}_s\} \tag{95}$$

we have
1. $\omega_{f,o}$ is uniformly continuous on $\mathcal{U}_f(\sigma)$,
2. for all $d_f \in \mathcal{V}_f$, $(x_f, d_s), (y_f, w_s) \in \mathcal{U}_f(\sigma)$, $|d_s - w_s| \leq \varepsilon^*$, $\varepsilon \in (0, \varepsilon^*]$,

$$\begin{array}{r} |F_f(x_f, d_s, d_f, \varepsilon) - F_f(y_f, w_s, d_f, 0)| \leq \\ L\left(|x_f - y_f| + \rho\right) \ . \end{array} \tag{96}$$

Proposition 3: If Assumptions 2 - 3 hold, F_s, F_{av}, $\omega_{s,o}$ and $\omega_{s,i}$ are zero, and Assumptions 8(b) and 9(b) hold then Assumptions 4-7 hold.

Proof. See Section VII-B. ■

Remark VI.1: Due to space constraints, we have not considered the case where the discrepancy in F_f in (96) is not always small instantaneously but is small on average, like in the results of Vrkoč mentioned in the Introduction. The easiest way to address this is to single out part of the disturbance vector d_{sm_f} as belonging to a set of small in the mean signals and allowing the right-hand side of the bound in (96) to depend on the norm of d_{sm_f}. ■

F.2 Total stability

A simple situation that should be emphasized is when there are no disturbances, at least when $\varepsilon = 0$ and the "boundary layer" has an asymptotically stable compact attractor $\mathcal{A}_f$ having basin of attraction $\mathcal{H}_f$. (Consider, for example, the van der Pol equation with asymptotically stable periodic orbit and basin of attraction $\mathbb{R}^2 \setminus \{0\}$.) A straightforward consequence of Proposition 3 is that, for the case where F_f is locally Lipschitz, the attractor is semiglobally (with respect to $\mathcal{H}_f$) practically asymptotically stable in the parameter ε, where $\varepsilon \neq 0$ can introduce arbitrary bounded disturbances scaled by ε. This is a fairly well-known result but certainly more widely appreciated for an asymptotically stable equilibrium which is the situation covered by classical "total stability" results. The more general version can also be proved directly, even for the case where F_f is only continuous, by appealing to converse Lyapunov functions (e.g., see [66]). For the case of F_f continuous, the approach taken here would need results like those mentioned in Remark V.4.

F.3 Systems with a slow equilibrium manifold

Many of the classical results on robustness to slowly varying parameters require the existence of a continuously differentiable equilibrium manifold (see, for example, [28, Section 5.7]. It is noteworthy that our results, specialized to equilibrium manifolds, impose no differentiability requirements. For example, we address slowly varying results for the system

$$\dot{x}_f = -x_f^3 + d_s \tag{97}$$

which has a slow equilibrium manifold given by the set of points satisfying $x_f = (d_s)^{1/3}$. Differentiability conditions are avoided by working with measuring functions like the distance to the manifold, which is a globally Lipschitz function regardless of the continuity properties of the function defining the manifold. This choice is made rather than to trying to work via a coordinate transformation and evolution of the error between the fast variable x_f and the value $h(d_s)$ where h is the function characterizing the equilibrium manifold. Through the distance to the manifold, we are able to see that slowly varying parameters do not cause drift far from the manifold, without assuming differentiability of the function defining the manifold.

G. Classical averaging

We again consider the case where the fast dynamics don't depend on the slow dynamics, this time by virtue of the assumption that x_f is a scalar and

$$F_f(x_s, x_f, d_s, d_f, \varepsilon) = 1 \ . \tag{98}$$

In this case, the solution to the fast subsystem can be associated with time and different initial conditions correspond to different starting times. In order for the fast subsystem to satisfy Assumptions 3 and 6 we will take $\omega_{f,i}$ and $\omega_{f,o}$ to be identically zero, and $\mathcal{H}_f$ and $\mathcal{K}_f$ to be $\mathbb{R}$. We impose Assumptions 2 and 1. In Assumption 4, $\phi_{f_{bl}}(t) = x_f + t$. In the special case where $\omega_{av,i} \equiv 0$ Assumption 4 in this setting is, for all practical purposes, the weak/strong average introduced in [46]. For the average system (29) we impose the stability condition in Assumption 5 and we are interested in the degree to which the actual system inherits the properties of the average system. We impose Assumptions 8 and 9 which mainly become conditions on $\omega_{s,o}$, $\omega_{s,i}$, $\mathcal{K}_s$, F_s and F_{av}. In fact, it turns out that the continuity of F_s with respect to x_f in Assumption 9 can be dropped. We extract what remains of Assumptions 8 and 9:

Assumption 8 (c) The following conditions hold:

1. $\mathcal{K}_s \subseteq \mathcal{H}_s$,
2. $$\sup_{x_s \in \mathcal{K}_s} \omega_{s,o}(x_s) =: c_{s,o} < \infty \tag{99}$$
3. $$\sup_{d_s, d_f \in \mathcal{V}_f} \omega_{s,i}(d_s, d_f) =: c_{s,i} < \infty \tag{100}$$
4. there exists $\rho > 0$ such that
$$\{x_s : \omega_{s,o}(x_s) \le c_{s,i} + \rho\} \subseteq \mathcal{K}_s \ . \tag{101}$$

Assumption 9 (c) There exist $L > 0$, $M > 0$ and $\sigma > 0$ and for each $\rho > 0$ there exists $\varepsilon^* > 0$ such that, defining

$$\mathcal{X}_s := \{x_s : \omega_{s,i}(x_s) \le \max\{\beta_s(c_{s,o}, 0) \ , \ c_{s,i}\} \ \} \ , \tag{102}$$

we have

1. $\omega_{s,o}$ is uniformly continuous on $\mathcal{X}_s + \sigma\mathcal{B}$,
2. $\mathcal{X}_s + \sigma\mathcal{B} \subseteq \mathcal{R}_s$,
3. for all $d_f \in \mathcal{V}_f$, $x_s, y_s \in \mathcal{X}_s + \sigma\mathcal{B}$, $d_s, w_s \in \mathcal{V}_s$, $|d_s - w_s| \le \varepsilon^*$, $\varepsilon \in (0, \varepsilon^*]$, $t \in \mathbb{R}$, $|e| \le \sup_{d_f \in \mathcal{V}_f} \omega_{av,i}(d_f)$,

$$\begin{aligned}
|F_s(x_s, t, d_s, d_f, \varepsilon)| &\le \varepsilon M \\
|F_{av}(x_s, d_s, d_f, e)| &\le M \\
|F_{av}(x_s, d_s, d_f, e) - F_{av}(y_s, d_s, d_f, e)| &\le L|x_s - y_s| \\
|x_s - y_s| \le \varepsilon^* &\Longrightarrow \\
|F_s(x_s, t, d_s, d_f, \varepsilon) - F_s(y_s, t, w_s, d_f, \varepsilon)| &\le \varepsilon\rho \ .
\end{aligned}$$

■

Proposition 4: If Assumptions 2 and 1 hold, $F_f = 1$, $\omega_{f,o} = 0$, $\omega_{f,i} = 0$, $\mathcal{K}_f = \mathcal{H}_f = \mathbb{R}$, and Assumption 4, 8(c) and 9(c) hold then Assumptions 3,5-7 hold.

Proof. See Section VII-C. ■

Remark VI.2: The result of this proposition is similar to the results presented in [64] and [65]. ■

H. Partial averaging

Partial averaging is the label given to the case where the differential equation depends on a slow time parameter in addition to a fast time parameter. Early partial averaging results are obtained in [42] (for finite time intervals), [54], and [50] (in the context of two time-scale averaging).

Our results address partial averaging by including as state variables a slow time state and a fast time state, i.e.,

$$\dot{p} = \varepsilon \tag{103}$$

and

$$\dot{q} = 1 \tag{104}$$

where p is a part of x_s and q is a part of x_f.

We note that the regularity of F_s and F_{av} with respect to p that is needed is less than what is indicated in Assumption 9 since the evolution for p, in the transformed time-scale, is not a function of ε. Due to space limitations, we don't pursue this relaxation here.

I. Pulse-width modulated control systems

In this section we study nonlinear systems controlled by pulse-width modulation. We consider the model

$$\begin{aligned}
\dot{x}_s &= \varepsilon\left[f(x_s) + g(x_s)\left[h_\circ(x_s) + u(h_1(x_s) - p(x_f))\right]\right] \\
\dot{x}_f &= 1
\end{aligned} \tag{105}$$

where $u : \mathbb{R} \to [0,1]$, $u(s) = 1$ for $s \ge 0$, $u(s) = 0$ for $s < 0$, and $p : \mathbb{R} \to \mathbb{R}$ is measurable, bounded, and periodic, with period one. (Periods different from one are accommodated by rescaling time.) The analysis of the system (105) hinges on the nondecreasing, possibly discontinuous function

$$v \mapsto \sigma(v) := \text{measure}\{x_f \in [0,1] : p(x_f) \le v\} \ , \tag{106}$$

which takes values in $[0,1]$, and it's corresponding upper semicontinuous set-valued map

$$S(v) := \left[\lim_{q \to v-} \sigma(q), \lim_{q \to v^+} \sigma(q)\right] \ . \tag{107}$$

The limits used in the definition of $S(v)$ are well-defined, due to the monotonicity of $\sigma(\cdot)$. Where $\sigma(\cdot)$ is continuous, $S(v) = \sigma(v)$. A common example of a function $p(\cdot)$, analyzed in [34] for instance, is a periodic ramp with period one taking values in $[0,1]$ with slope one. In this case σ is continuous and $\sigma(v) = v$ for all $v \in [0,1]$. Another special case corresponds to sliding mode control and arises when $p(x_f) \equiv 0$ and $h_\circ(x_s) = -1/2$ so that $h_\circ(x_s) + \sigma(h(x_s)) = h_\circ(x_s) + u(h(x_s) - p(x_f)) = 1/2\mathrm{sgn}(h(x_s))$.

We assume that the functions f, g, $h_\circ$ and h_1 are continuous and that the system

$$\dot{x}_s \in f(x_s) + g(x_s)\left[h_\circ(x_s) + S(h(x_s))\right] \tag{108}$$

has a compact set $\mathcal{A}_s$ that is asymptotically stable with basin of attraction $\mathcal{H}_s$. We will apply our main results to show that the system (105) has the set $\mathcal{A}_s$ semiglobally (with respect to $\mathcal{H}_s$) practically asymptotically stable. The key is to establish Assumption 4 for a family of functions $F_{s,\delta}$ and $F_{av,\delta}$ that satisfy Assumption 9, where each element $F_{s,\delta}$ covers (105) and there are elements of $F_{av,\delta}$ arbitrarily close to the right-hand side of (108).

We first note that the x_f component of the solution to the boundary layer is given by $x_f + t$ and, by definition, for all x_f,

$$\int_0^1 [u(h(x_s) - p(x_f + t))]dt = \sigma(h(x_s)) \; . \tag{109}$$

This fact is the basis for forming an average for the right-hand side of the system (105), however the problem is that the right-hand side of (105) and possibly also the average are discontinuous in x_s. The initial step in remedying this is to convert the function $u(\cdot)$ into its corresponding upper semicontinuous set-valued map with nonempty compact convex values containing $u(\cdot)$:

$$U(s) := [\lim_{q \to s^-} u(q), \lim_{q \to s^+} u(q)] \; . \tag{110}$$

Again by definition

$$\int_0^1 [U(h(x_s) - p(x_f + t))]dt = S(h(x_s)) \; . \tag{111}$$

However, if we try to parameterize the set-valued maps $U(\cdot)$ and $S(\cdot)$ with exogenous disturbances, the resulting parameterizations will not be continuous. Instead we define, for each $\delta > 0$, a δ-inflation of U given by

$$U_\delta(s) := U(s + \delta\mathcal{B}) + \delta\mathcal{B} \tag{112}$$

which is upper semicontinuous and has nonempty, compact convex values. Then we use [14, §8, Theorem 4] together with the argument used to prove [14, §8, Corollary 2] and the periodicity of $p(\cdot)$ to assert that for each $K > 0$ and $\rho > 0$ there exists $\delta^* > 0$ such that, for all $|x_s| \le K$, all x_f, and all $\delta \in [0, \delta^*]$,

$$\int_0^1 [U_\delta(h(x_s) - p(x_f + t))]dt \subseteq S(h(x_s)) + \rho\mathcal{B} \; . \tag{113}$$

A parameterization of $U_\delta(\cdot)$ would again not be continuous, but we can easily construct a globally Lipschitz set-valued map $U_{\delta,\ell}$ having nonempty, compact, convex values satisfying

$$U(s) \subseteq U_{\delta,\ell}(s) \subseteq U_\delta(s) \tag{114}$$

and, using [10, Theorem 9.7.2] or [15, Proposition 2.22], we can parameterize $U_{\delta,\ell}$ with a Lipschitz function, i.e., there exist a function $u_{\delta,\ell}$ and a positive number L_δ such that

$$u_{\delta,\ell}(s, \mathcal{B}) = U_{\delta,\ell}(s) \tag{115}$$

and, for all $d_f \in [-1,1]$, $s_1, s_2 \in \mathbb{R}$,

$$|u_{\delta,\ell}(s_1, d_f) - u_{\delta,\ell}(s_2, d_f)| \le L_\delta |s_1 - s_2| \; . \tag{116}$$

We define $\mathcal{D}_f$ to be the set of measurable functions taking values in $[-1,1] =: \mathcal{V}_f$. With the definitions

$$\begin{aligned} &F_{s,\delta}(x_s, x_f, d_f, \varepsilon) = \\ &\qquad \varepsilon[f(x_s) + g(x_s)[h_\circ(x_s) + u_{\delta,\ell}(h(x_s) - p(x_f), d_f)]] \end{aligned} \tag{117}$$

and

$$F_{av}(x_s, e) = f(x_s) + g(x_s)[h_\circ(x_s) + S(h(x_s)) + e] \tag{118}$$

it follows from the above calculations that $F_{s,\delta}$ covers (105) for each $\delta > 0$ that for each $\rho > 0$ and each compact set $\mathcal{R}_s$ there exists $\delta > 0$ such that Assumption 4 is satisfied with $m = 1$, $F_{av,e}$, and $\omega_{av,i} \equiv \rho$. If f, g, h and $\sigma = S$ are locally Lipschitz, the result follows by applying our total stability results to the system $\dot{x}_s = F_{av}(x_s, e)$ having boundary layer $\dot{x}_s = F_{av}(x_s, 0)$. If any of f, g, h and σ are only continuous, we first follow the outline of Section VI-B to turn F_{av} into a parameterized locally Lipschitz function.

VII. Technical Proofs

A. Proof of Proposition 2

Claim 1: Let $\sigma > 0$ come from Assumption 9. For each $\delta > 0$ and $T > 0$ there exists $\varepsilon^* > 0$ such that

$$\left.\begin{aligned} x_s(t) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B} \quad \forall t \in [0,T] \\ |z(0) - z_{bl}(0)| \le \varepsilon^* \\ z(0), z_{bl}(0) \in \mathcal{K}_f \\ \varepsilon \in (0, \varepsilon^*] \end{aligned}\right\} \tag{119}$$

imply

$$z(t), z_{bl}(t) \in \mathcal{U}_f(\sigma) \qquad \forall t \in [0,T] \tag{120}$$

and

$$|z(t) - z_{bl}(t)| \le \delta \qquad \forall t \in [0,T] \; . \tag{121}$$

Proof. Throughout this proof we will use, without loss of generality,

$$|(x_s, x_f, d_s)| := \max\{|x_s|, |x_f|, |d_s|\} \; . \tag{122}$$

Let $T > 0$ and $\delta > 0$ be given and without loss of generality assume that $\delta < \sigma$ where σ comes from Assumption 9. Also let $L > 0$ and $M > 0$ come from Assumption 9. Define

$$\varepsilon_1^* := \frac{\delta}{2\exp(LT)} \tag{123}$$

and

$$\rho := \frac{\delta}{2\left[\exp(LT) - 1\right]} \ . \tag{124}$$

For this ρ let Assumption 9 generate $\varepsilon_2^* > 0$. Then define $\widetilde{\rho} = \min\{\delta, \varepsilon_2^*\}$ and define

$$\varepsilon_3^* := \frac{\widetilde{\rho}}{2MT} \ . \tag{125}$$

Also let ε_4^* come from the second part of Assumption 2 for $\widetilde{\rho}$ and T. Then define

$$\varepsilon^* := \min\{\varepsilon_1^*, \varepsilon_2^*/2, \varepsilon_3^*, \varepsilon_4^*\} \ . \tag{126}$$

Assume the conditions in (119).

Using the definition of ε^*, the definition of ε_4^* and Assumption 2, we have for all $t \in [0,T]$,

$$|d_s(t) - d_s(0)| \leq \widetilde{\rho} \ . \tag{127}$$

Next, since

$$x_s(0) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B} \tag{128}$$

and

$$|x_s(0) - x_{s_{bl}}| \leq \varepsilon^* \leq \varepsilon_1^* < \delta/2 < \sigma/2 \ , \tag{129}$$

it follows that $x_{s_{bl}} \in \mathcal{X}_s + \sigma\mathcal{B}$.

Then, from the definition of $c_{f,o}$ and $c_{f,i}$ in Assumption 8, the definition of $\mathcal{Z}_f$ in Assumption 9, Assumption 3, and $z_{bl}(0) \in \mathcal{K}_f \subseteq \mathcal{H}_f$, it follows that $z_{bl}(t) \in \mathcal{U}_f(\sigma)$ for all $t \in [0,T]$.

We define

$$\begin{aligned} \bar{t} := \sup\{\ t \in [0,T] : |x_s(\tau) - x_{s_{bl}}| \leq \varepsilon_2^* \ , \\ z(\tau) \in \mathcal{U}_f(\sigma) \quad \forall \tau \in [0,t]\ \} \ . \end{aligned} \tag{130}$$

It follows from $\varepsilon^* \leq \varepsilon_2^*/2$, $z(0) \in \mathcal{K}_f$ and $x_s(0) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B}$ that $\bar{t}$ is well-defined, $\bar{t} > 0$, $z_{bl}(t) \in \mathcal{U}_f(\sigma)$ for all $t \in [0,\bar{t})$, and if $\bar{t} < T$ then either $|z(\bar{t}) - z_{bl}(t)| = \sigma$ or $|x_s(\bar{t}) - x_{s_{bl}}| = \varepsilon_2^*$.

Suppose $\bar{t} < T$. Due to the computations above, and the last point of Assumption 9, we have for almost all $t \in [0,\bar{t}]$,

$$|\dot{x}_s(t)| \leq \varepsilon M \tag{131}$$

and

$$|\dot{x}_f(t) - \dot{x}_{f_{bl}}(t)| \leq L\left(|x_f(t) - x_{f_{bl}}(t)| + \rho\right) \ . \tag{132}$$

From these two conditions it follows that

$$\begin{aligned} |x_s(\bar{t}) - x_{s_{bl}}| &\leq |x_s(\bar{t}) - x_s(0)| + |x_s(0) - x_{s_{bl}}| \\ &\leq \varepsilon M\bar{t} + \varepsilon^* \\ &< \tfrac{1}{2}\min\{\delta, \varepsilon_2^*\} + \varepsilon^* \\ &\leq \min\{\delta, \varepsilon_2^*\} \end{aligned} \tag{133}$$

and

$$\begin{aligned} |x_f(\bar{t}) - x_{f_{bl}}(\bar{t})| &\leq \varepsilon^* \exp(L\bar{t}) + \rho\left[\exp(L\bar{t}) - 1\right] \\ &< \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta < \sigma \ . \end{aligned} \tag{134}$$

From these computations it follows that $\bar{t} = T$ and then that (120) and (121) hold. ■

Claim 2: There exists $T^* > 0$ and for each $T \geq T^*$ and $\delta > 0$ there exists $\varepsilon^* > 0$ such that

$$\left.\begin{array}{r} x_s(t) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B} \quad \forall t \in [0,T] \\ z_{bl}(0) = z(0) \in \mathcal{K}_f \\ \varepsilon \in (0,\varepsilon^*] \end{array}\right\} \tag{135}$$

imply that

$$z(t) \in \mathcal{U}_f(\sigma) \qquad \forall t \in [0,T] \tag{136}$$

and

$$|\omega_{f,o}(z(t)) - \omega_{f,o}(z_{bl}(t))| \leq \delta \qquad \forall t \in [0,T] \tag{137}$$

and

$$z(t) \in \mathcal{K}_f \qquad \forall t \in [T^*,T] \ . \tag{138}$$

Proof. Let $\rho > 0$ come from the last item of Assumption 8. Define $\widetilde{\rho} := \min\{\delta, \rho\}$. Let $\sigma > 0$ from from Assumption 9. Relying on the uniform continuity of $\omega_{s,o}$ and $\omega_{f,o}$ that is provided in Assumption 9, let $\widetilde{\delta} > 0$ be such that

$$\left.\begin{array}{r} x_s, y_s \in \mathcal{X}_s + \sigma\mathcal{B} \\ |x_s - y_s| \leq \widetilde{\delta} \end{array}\right\} \implies |\omega_{s,o}(x_s) - \omega_{s,o}(y_s)| \leq \widetilde{\rho} \tag{139}$$

and

$$\left.\begin{array}{r} z, z_{bl} \in \mathcal{U}_f(\sigma) \\ |z - z_{bl}| \leq \widetilde{\delta} \end{array}\right\} \implies |\omega_{f,o}(z) - \omega_{f,o}(z_{bl})| \leq \frac{\widetilde{\rho}}{2} \ . \tag{140}$$

Now let $T^* > 0$ be such that

$$\beta_f(c_{f,o}, t) \leq \frac{\widetilde{\rho}}{2} \qquad \forall t \geq T^* \ . \tag{141}$$

Let $T \geq T^*$ and for the pair $(\widetilde{\delta}, T)$, let Claim 1 generate ε^*. With $z_{bl}(0) = z(0) \in \mathcal{K}_f$, we use the result of Claim 1, (140) and $\widetilde{\rho} \leq \delta$, we get (136) and (137).

Next, using that $\mathcal{K}_f \subseteq \mathcal{H}_f$ (which is given by the second item of Assumption 8), the conclusion of Claim 1, (139), (140), (141), the last item of Assumption 8, Assumption 3 and Assumption 5, we get (138). ■

Claim 3: For each $\delta > 0$ there exists $\varepsilon^* > 0$ such that

$$\left.\begin{array}{r} x_s(t) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B} \quad \forall t \in [0,T] \\ z(0) \in \mathcal{K}_f \\ \varepsilon \in (0,\varepsilon^*] \end{array}\right\} \tag{142}$$

imply

$$z(t) \in \mathcal{U}_f(\sigma) \qquad \forall t \in [0,T] \tag{143}$$

and

$$\omega_{f,o}(z(t)) \leq \max\{\beta(\omega_{f,o}(z(0)), t), ||\omega_{f,i}(d_f)||_\infty\} + \delta \tag{144}$$

for all $t \in [0,T]$.

Proof. With the result of Claim 2, the proof follows the same calculations as those in the proof of Theorem 1. ■

Claim 4: For each $\rho > 0$ there exist $T^* > 0$ and $\varepsilon^* > 0$ such that for each

$$\left.\begin{array}{r} T \geq T^* \\ \varepsilon \in (0,\varepsilon^*] \\ (x_s(0), d_s(0), d_f) \in \mathcal{X}_s \times \mathcal{V}_s \times \mathcal{D}_f \\ z(0) = (x_s(0), x_f(0), d_s(0)) \in \mathcal{K}_f \end{array}\right\} \tag{145}$$

if $x_s(t) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B}$ for all $t \in [0,T]$ then there exists a measurable function $e : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ satisfying

$$||e||_\infty \leq ||\omega_{av,i}(d_f)||_\infty \tag{146}$$

such that

$$\left| \int_0^T [F_s(x_s(t), x_f(t), d_s(t), d_f(t), \varepsilon) - \right. \tag{147}$$

$$\left. \varepsilon F_{av}(x_s(t), d_s(t), d_f(t), e(t))]\, dt \right| \leq T\rho\varepsilon \ .$$

Proof. Let
- $M > 0$, $L > 0$, $\sigma > 0$, $\mathcal{X}_s$, $\mathcal{Z}_f$ come from Assumption 9;
- $\rho > 0$ be given and, without loss of generality, assume that $\rho < \sigma$, perhaps by decreasing ρ;
- $\varepsilon_0^* > 0$ come from Assumption 9 for $\rho/8$;
- $T_1^* > 0$ and $\varepsilon_1^* > 0$ come from Assumption 4 for $\rho/4$;
- $\varepsilon_2^* > 0$ come from Claim 1 for T_1^* and $\min\left\{\varepsilon_0^*, \frac{\rho}{8L}\right\}$;
- $\widetilde{\rho} \in (0, \rho]$ be such that, for each $c \in [0, c_{f,i}]$, if $\omega_{f,o}(z) \leq c + \widetilde{\rho}$ then there exists z_c such that $\omega_{f,o}(z_c) \leq c$ and $|z_c - z| \leq \varepsilon_2^*$;
- $\varepsilon_3^* > 0$ come from Claim 3 for $\widetilde{\rho}/2$;
- $T_2^* > 0$ be such that

$$\beta_f(c_{f,o}, t) \leq \frac{\widetilde{\rho}}{2} \qquad \forall t \geq T_2^* \ ; \tag{148}$$

- k be an integer such that

$$\frac{2M \max\{T_1^*, T_2^*\}}{T_2^* + kT_1^*} \leq \rho/4 \ ; \tag{149}$$

- $T^* := T_2^* + kT_1^*$.
- $\varepsilon^* := \min\{\varepsilon_0^*, \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*\}$;
- $T \geq T^*$;
- m be the largest integer so that $T_2^* + mT_1^* \leq T$. (Note that $m \geq k$.)

Suppose $z(0) \in \mathcal{K}_f$ and $x_s(t) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B}$ for all $t \in [0,T]$. Then, using Claim 3,

$$z(t) \in \mathcal{U}_f(\sigma) \qquad \forall t \in [0,T] \tag{150}$$

and

$$\omega_{f,o}(z(t)) \leq ||\omega_{f,i}(d_f)||_\infty + \widetilde{\rho} \qquad \forall t \in [T_2^*, T] \ . \tag{151}$$

We consider the integral in (147) by breaking the interval of integration into the subintervals
1. $[0, T_2^*]$,
2. $[T_2^* + (j-2)T_1^*, T_2^* + (j-1)T_1^*]$, $j = 2, \ldots, m+1$,
3. $[T_2^* + mT_1^*, T]$.

The length of the first and last interval is bounded by $\max\{T_1^*, T_2^*\}$ and thus, using Assumption 9, (149), the definition of T^* and the relation $T \geq T^*$, the first and last subinterval are each bounded by

$$2M\varepsilon \max\{T_1^*, T_2^*\} \leq \varepsilon(T_2^* + kT_1^*)\rho/4 \leq \varepsilon T\rho/4 \tag{152}$$

Now we show that each of the $m-1$ remaining intervals can be bounded in norm by $\varepsilon T\rho/(2m)$. This will establish our desired result. For the jth interval, $j = 2, \ldots, m+1$, we add and subtract F_s and εF_{av} evaluated along the solutions $z_{bl_j}(\cdot)$ of the boundary layer system starting at the point z_c where $\omega_{f,o}(z_c) \leq ||\omega(d_f)||_\infty$ and $|z_c - z(t_j)| \leq \varepsilon_2^*$ where $t_j = T_2^* + (j-2)T_1^*$. It follows from our construction that $|z(t) - z_{bl_j}(t)| \leq \min\left\{\varepsilon_0^*, \frac{\rho}{8L}\right\}$ for all t in the interval. In turn, by construction of ε_0^*, we have that the difference between the F_s evaluated along $z_{bl_j}(t)$ and F_s evaluation along $z(t)$ can be bounded by $\varepsilon\rho/8$, and similarly for εF_{av}. Multiplying by the length of the time interval T_1^* gives the bound

$$T_1^*\varepsilon\rho/4 \leq \varepsilon\frac{T}{m}\rho/4 \tag{153}$$

for these terms. The remaining terms in the jth interval are exactly like in Assumption 4 and can be bounded as

$$T_1^*\varepsilon\rho/4 \leq \varepsilon\frac{T}{m}\rho/4 \ . \tag{154}$$

Adding these bounds together proves the result. ■

Claim 5: For each $\delta > 0$ and $T > 0$ there exists $\varepsilon^* > 0$ such that

$$\left.\begin{array}{r} x_s(0), x_{s,av}(0) \in \mathcal{K}_s \\ z(0) \in \mathcal{K}_f \\ |x_s(0) - x_{s,av}(0)| \leq \varepsilon^* \\ \varepsilon \in (0, \varepsilon^*] \end{array}\right\} \tag{155}$$

imply

$$|x_s(t) - x_{s,av}(t)| \leq \delta \qquad \forall t \in [0, T/\varepsilon] \ . \tag{156}$$

Proof. Let $T > 0$ and $\delta > 0$ be given and, without loss of generality, assume that $\delta < \sigma/2$ where σ comes from Assumption 9. Also let $L > 0$ and $M > 0$ come from Assumption 9. Define

$$\varepsilon_1^* := \frac{\delta}{2\exp(LT)} \tag{157}$$

and

$$\rho := \frac{\delta}{2[\exp(LT) - 1]}\frac{L}{L+1} \ . \tag{158}$$

For this $\rho > 0$ let Assumption 9 generate $\varepsilon_2^* > 0$. Also for this ρ let Claim 4 generate T^* and ε_3^*. Next define

$$\varepsilon_4^* := \frac{\delta}{4MT^*} \ . \tag{159}$$

Also define

$$\varepsilon_5^* := \frac{T}{T^*} \ . \tag{160}$$

Also let ε_6^* come from Claim 2 for any $\delta > 0$ (the only thing to be used from Claim 2 is (136).) Then define

$$\varepsilon^* := \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*, \varepsilon_5^*, \varepsilon_6^*\} \ . \tag{161}$$

Assume the conditions of (155) hold.

From the definition of $c_{s,o}$ and $c_{s,i}$ in Assumption 8, the definition of $\mathcal{X}_s$ in Assumption 9, Assumption 5, and $x_{s,av}(0) \in \mathcal{K}_s \subseteq \mathcal{H}_s$, it follows that $x_{s,av}(t) \in \mathcal{X}_s$ for all $t \in [0,T]$.

We define

$$\bar{t} := \sup\left\{t \in [0, T/\varepsilon] : x_s(\tau) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B} \qquad \forall \tau \in [0,t]\right\} . \tag{162}$$

It follows from that fact that $x_s(0) \in \mathcal{K}_s$ and the definition of $\mathcal{X}_s$ that $\bar{t}$ is well-defined, $\bar{t} > 0$, $x_s(t) \in \mathcal{X}_s + \frac{\sigma}{2}\mathcal{B}$ for all $t \in [0, \bar{t})$ and if $\bar{t} < T/\varepsilon$ then $|x_s(\bar{t}) - x_{s,av}(\bar{t})| \geq \sigma/2$. It also follows from Claim 2 that $z(t) \in \mathcal{U}_f(\sigma)$ for all $t \in [0, \bar{t})$.

Suppose $\bar{t} < \min\{T^*, T/\varepsilon\}$. Due to the computations above and the last point of Assumption 9, we have for almost all $t \in [0, \bar{t}]$,

$$|\dot{x}_s(t) - \dot{x}_{s,av}(t)| \leq \varepsilon 2M \leq \frac{\delta}{2T^*} . \tag{163}$$

From this it follows that for all $t \in [0, \bar{t}]$,

$$|x_s(t) - x_{s,av}(t)| \leq \varepsilon^* + \frac{\delta}{2} < \frac{\sigma}{2} . \tag{164}$$

It follows that $\bar{t} \geq \min\{T/\varepsilon, T^*\}$ and that the result has been proved for the case where $T/\varepsilon \leq T^*$.

Now suppose $T^* < \bar{t} < T/\varepsilon$. Due to the computations above, the last point of Assumption 9, and the results of Claims 3 and 4, we have, for all $t \in [T^*, \bar{t}]$,

$$\begin{aligned} &|x_s(t) - x_{s,av}(t)| \leq \\ &\quad \varepsilon^* + \int_0^t \varepsilon L\left(|x_s(\tau) - x_{s,av}(\tau)| + \frac{L+1}{L}\rho\right) d\tau . \end{aligned} \tag{165}$$

It follows from this condition and the definitions (157), (158) and (161) that, for all $t \in [T^*, \bar{t}] \subseteq [T^*, T/\varepsilon]$,

$$\begin{aligned} |x_s(t) - x_{s,av}(t)| &\leq \varepsilon^* \exp(\varepsilon L t) + \rho \tfrac{L+1}{L}\left[\exp(\varepsilon L t) - 1\right] \\ &\leq \tfrac{\delta}{2} + \tfrac{\delta}{2} < \sigma/2 . \end{aligned} \tag{166}$$

It follows that $\bar{t} = T/\varepsilon$ and the result is established. ■

Claim 6: Assumption 6 holds.

Proof. This result follows from the combination of Claim 2 and Claim 5. ■

Claim 7: Assumption 7 holds.

Proof. This proof of this result follows the same lines as the proof of Claim 2. ■

B. Proof of Proposition 3

The proof of Proposition 3 follows the same lines as the proofs of Claims 1 and 2 in Section VII-A. Indeed, notice that the conclusion of Claim 2 when there is no x_s component to the solution is exactly the form of Assumption 6. Moreover, Claims 1 and 2 were proved using only the assumptions that are made in Proposition 3. ■

C. Proof of Proposition 4

The proof of Proposition 4 follows the same lines as the proofs of Claims 4 and 5 in Section VII-A. Indeed, since Claims 1-3 of Section VII-A hold by assumption, we can enter the proof of Proposition 2 at Claim 4 and continue to establish the result. We note that, in the proof of Claim 4 continuity of F_s with respect to x_f is not needed because of the fact that the x_f component of the solution to the boundary layer system is the same as the x_f component of the actual solution. ■

VIII. Conclusion

We have developed a unified framework for studying robustness of the input-to-state stability (ISS) property and have presented new results on robustness of ISS to slowly-varying parameters, to highly oscillatory signals, and to generalized singular perturbations. The framework assumes a time-scale separation between slow and fast variables which permits the definition of a boundary layer system like in classical singular perturbation theory. To address various robustness problems simultaneously, we have allowed the asymptotic behavior of the boundary layer to be complex and have required it to generate an average for the derivative of the slow state variables. Our main result have shown that if the boundary layer and averaged systems are ISS then the ISS bounds also hold for the actual system with an offset that converges to zero with the parameter that characterizatizes the separation of time-scales. The ISS notion that we have used permits general attractors and general measures on disturbances. We have shown how our general framework connects to many different types of robustness established in the literature.

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