Averaging with disturbances and closeness of solutions

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March 16, 2000

Abstract

We establish that, under appropriate conditions, the solutions of a time-varying system with disturbances converge uniformly on compact time intervals to the solutions of the system's average as the rate of change of time increases to infinity. The notions of "average" used for systems with disturbances are the "strong" and "weak" averages introduced in [5].

Keywords: (weak and strong) averaging, continuity of solutions.

1 Introduction

Averaging is an important approximate method for analysis of time-varying systems. In its classical form it applies to ordinary differential equations of the form

$$\dot{x} = f\left(\frac{t}{\epsilon}, x, \epsilon\right) \tag{1}$$

where $\epsilon > 0$ and where f has an average f_{av} satisfying a condition like

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(\tau, x, 0)d\tau \qquad \forall t \geq 0 \ .$$

Classical averaging results (see, for example, [1, 2, 3, 6, 8] and references therein) state that, under appropriate smoothness assumptions on f, the solutions of (1) converge uniformly on compact time intervals to the solutions of

$$\dot{x} = f_{av}(x) \tag{2}$$

as ϵ tends to zero.[1] Moreover, if the system (2) has an exponentially stable equilibrium point p that is an equilibrium point of (1) for small $\epsilon > 0$ then p is an exponentially stable equilibrium point of (1) for small $\epsilon > 0$.

Except for [5], we are not aware of any results on averaging that consider systems with exogenous disturbances. However, systems with disturbances occur frequently in control theory. Recently, in

*Research supported in part by NSF under grant ECS-9896140.

[1]Sometimes averaging is applied to systems of the form $\frac{dx}{d\tau} = \epsilon f(\tau, x, \epsilon)$ instead of the form (1). In this case, the convergence of solutions is established on intervals of length proportional to $1/\epsilon$. This is seen to be an equivalent result by scaling time as $\tau = t/\epsilon$.

[5], two different definitions of average for systems with disturbances ("strong" and "weak" average) were introduced and results were presented on deducing input-to-state stability (ISS) for a system from input-to-state stability for the system's strong or weak average. These results generalize, in a sense, the exponential stability result mentioned above, as well as more recent stability results based on averaging, like in [7].

In this note we also study systems with disturbances that possess a weak or strong average (see Definitions 1 and 2, respectively). In particular, we study systems of the form

$$\dot{x} = f\left(\frac{t}{\epsilon}, x, w, \nu\right) \tag{3}$$

where w is the exogenous disturbance and ν is a parameter vector with small norm. We use $x_{\epsilon,\nu}(t, t_\circ, x_\circ, w)$ to denote the solution of (3) (for given values of ϵ, ν) at time t, emanating from $x_\circ$ at the initial time $t_\circ$, under the action of the disturbance w. We are interested in conditions that guarantee that the solutions of (3) are close, on compact time intervals, to the solutions of (3)'s average which is taken when $\nu = 0$. The average system has the form

$$\dot{y} = f_{av}(y, w) \tag{4}$$

where f_{av} is assumed to be locally Lipschitz. We will not impose any stability assumptions on the average system. Moreover, for f we will only assume that it is continuous in (x, w, ν) uniformly in t. In particular, we do not assume uniqueness of solutions for (3). We assume that w belongs to a set of functions that is equi-bounded (see Definition 4) for the case of strong averages, or equi-bounded and equi-uniformly continuous (see Definition 5) for the case of weak averages. We will show, among other things, that when a trajectory for the averaged system is defined on a given compact time interval, the trajectories of the actual system converge to that average trajectory uniformly on the compact time interval.

The paper is organized as follows: In Section 2 we present some preliminary definitions including the definitions of weak and strong average taken from [5]. Our main results are stated formally in Theorems 1 and 2 of Section 3. In Section 4 we provide the proofs of our theorems.

2 Preliminaries

For our purposes, a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is nondecreasing in its first argument and converging to zero in its second argument. Given a measurable function w, we define its infinity norm $\|w\|_\infty := \operatorname{ess\,sup}_{t\geq 0} |w(t)|$. If we have $\|w\|_\infty < \infty$, then we write $w \in \mathcal{L}_\infty$. If w is absolutely continuous, its derivative is defined almost everywhere and we can write $w(t) - w(t_\circ) = \int_{t_\circ}^{t} \dot{w}(\tau) d\tau$.

The following two definitions of strong and weak average for a time-varying system with exogenous disturbances were introduced in [5]:

Definition 1 (weak average) *A locally Lipschitz function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the weak average of $t \mapsto f(t, x, w, \nu)$ if there exist $\beta \in \mathcal{KL}$ and $T^* > 0$ such that $\forall T \geq T^*$ and $\forall t \geq 0$ we have*[2]

$$\left| f_{wa}(x, w) - \frac{1}{T} \int_t^{t+T} f(s, x, w, 0) ds \right| \leq \beta\Big(\max\{|x|, |w|, 1\}, T\Big), \quad \forall x \in \mathbb{R}^n, w \in \mathbb{R}^m. \tag{5}$$

The weak average of system (3) is then defined as

$$\dot{y} = f_{wa}(y, w). \tag{6}$$

□

[2]Note that w in the integral is a constant vector.

Definition 2 (strong average) *A locally Lipschitz function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the strong average of $t \mapsto f(t, x, w, \nu)$ if there exist $\beta \in \mathcal{KL}$ and $T^* > 0$ such that $\forall T \geq T^*$ and $\forall t \geq 0$ the following holds:*

$$\left| \frac{1}{T} \int_t^{t+T} \Big[f_{sa}(x, w(s)) - f(s, x, w(s), 0) \Big] ds \right| \leq \beta \Big(\max\{|x|, \|w\|_\infty, 1\}, T \Big), \quad \forall x \in \mathbb{R}^n, w \in \mathcal{L}_\infty. \quad (7)$$

The strong average of system (3) is then defined as

$$\dot{y} = f_{sa}(y, w). \quad (8)$$

□

It has been shown in [5] that functions f that have a strong average are, in essence, functions of the form $f(t, x, w, 0) = \widetilde{f}(t, x) + g(x, w)$ where $\widetilde{f}(t, x)$ has a well-defined (weak) average. More precisely, it was proved in [5]:

Proposition 1 *Suppose that $f(t, x, w)$ is continuous and periodic in t of period $T_1 > 0$. Then, there exists a strong average $f_{sa}(x, w)$ for $t \mapsto f(t, x, w)$ if and only if $f(t, x, w)$ can be written as the sum of a term that is independent of t plus a term that is independent of w and that has a well-defined average.* □

We also need definitions of forward completeness, equi-boundedness and equi-uniform continuity.

Definition 3 *Let $\mathcal{W}$ be a set of locally essentially bounded functions and let $f_{av} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function. The system*

$$\dot{x} = f_{av}(x, w) \qquad x(0) = x_\circ \quad (9)$$

is said to be $\mathcal{W}$-forward complete if for each $r > 0$ and $T > 0$ there exists $R \geq r$ such that, for all $|x_\circ| \leq r$ and $w \in \mathcal{W}$, the solutions of (9) exist and are contained in a closed ball of radius R for all $t \in [0, T]$.

Definition 4 *Let $\mathcal{W}$ be a set of locally essentially bounded functions. The set $\mathcal{W}$ is equi-(essentially) bounded if there exists a strictly positive real number Ω such that, for all $w \in \mathcal{W}$, $||w||_\infty \leq \Omega$;*

Definition 5 *Let $\mathcal{W}$ be a set of locally essentially bounded functions. The set $\mathcal{W}$ is equi-uniformly continuous if for each $\rho > 0$ there exists $\delta > 0$ such that, for all $w \in \mathcal{W}$ and all $t \geq 0$,*

$$\tau \in [0, \delta] \qquad \Longrightarrow \qquad |w(t + \tau) - w(t)| \leq \rho \ .$$

Remark 1 A sufficient condition for $\mathcal{W}$ to be equi-uniformly continuous is that all $w \in \mathcal{W}$ are absolutely continuous (on $[0, \infty)$) and there exists a strictly positive real number Ω_1 such that, for all $w \in \mathcal{W}$, $||\dot{w}||_\infty \leq \Omega_1$. □

3 Main Results

Our main results give conditions under which the solutions of (3) are close to the solutions of (3)'s weak or strong average, when these averages exist.

Theorem 1 (Closeness to weak average) *Suppose*

1. *the function $t \mapsto f(t, x, w, \nu)$ is measurable in t for each (x, w, ν), the function $(x, w, \nu) \mapsto f(t, x, w, \nu)$ is continuous in (x, w, ν) uniformly in t, and the function $t \mapsto f(t, 0, 0, 0)$ is bounded;*
2. *the set $\mathcal{W}$ is equi-(essentially) bounded and equi-uniformly continuous;*

3. the weak average of the system (3) exists and is $\mathcal{W}$-forward complete.

Then, for each triple (T, δ, r) of strictly positive real numbers there exists a triple (ϵ^*, ν^*, μ) of strictly positive real numbers such that, for each $\epsilon \in (0, \epsilon^*)$, $|\nu| < \nu^*$, $t_\circ \geq 0$, $|y_\circ| \leq r$, $w \in \mathcal{W}$ and each $x_\circ$ such that $|x_\circ - y_\circ| \leq \mu$, each solution $x_{\epsilon,\nu}(t, t_\circ, x_\circ, w)$ of (3) and the solution $y(t - t_\circ, y_\circ, w)$ of the weak average satisfy

$$|x_{\epsilon,\nu}(t, t_\circ, x_\circ, w) - y(t - t_\circ, y_\circ, w)| \leq \delta \qquad \forall t \in [t_\circ, t_\circ + T] \ . \tag{10}$$

Without the assumption that $\mathcal{W}$ is equi-uniformly continuous, the conclusion of Theorem 1 is not correct, in general. This is demonstrated by the system

$$\dot{x} = -0.25x^3 + \cos\left(\frac{t}{\epsilon}\right) x^3 w \ , \tag{11}$$

which was discussed in detail in [5]. There is was shown that the weak average of the system (11) is

$$\dot{y} = -0.25y^3 \tag{12}$$

but the system (11) under the input $w_\epsilon(t) = \cos\left(\frac{t}{\epsilon}\right)$ exhibits finite escape time.

Remark 2 Weak averages play an important role in cases when stability of an overall system is to be investigated by viewing the system as the interconnection of subsystems. In this case, the disturbances to one subsystem are generated by the states of the other subsystem. Because of this, the disturbances may be slowly-varying compared to the rate of change of the time-varying signals. An important example of this is the class of cascaded system (see [5]).

The assumption that $\mathcal{W}$ is equi-uniformly continuous can be removed when the strong average exists and is $\mathcal{W}$-forward complete:

Theorem 2 (Closeness to strong average) *Suppose*

1. *the function $t \mapsto f(t, x, w, \nu)$ is measurable in t for each (x, w, ν), the function $(x, w, \nu) \mapsto f(t, x, w, \nu)$ is continuous in (x, w, ν) uniformly in t, and the function $t \mapsto f(t, 0, 0, 0)$ is bounded;*
2. *the set $\mathcal{W}$ is equi-(essentially) bounded;*
3. *the strong average of the system (3) exists and is $\mathcal{W}$-forward complete.*

Then, for each triple (T, δ, r) of strictly positive real numbers there exists a triple (ϵ^*, ν^*, μ) of strictly positive real numbers such that, for each $\epsilon \in (0, \epsilon^*)$, $|\nu| < \nu^*$, $t_\circ \geq 0$, $|y_\circ| \leq r$, $w \in \mathcal{W}$ and each $x_\circ$ such that $|x_\circ - y_\circ| \leq \mu$, each solution $x_{\epsilon,\nu}(t, t_\circ, x_\circ, w)$ of (3) and the solution $y(t - t_\circ, y_\circ, w)$ of the strong average satisfy

$$|x_{\epsilon,\nu}(t, t_\circ, x_\circ, w) - y(t - t_\circ, y_\circ, w)| \leq \delta \qquad \forall t \in [t_\circ, t_\circ + T] \ . \tag{13}$$

The above results can also be applied to average systems that are not $\mathcal{W}$-forward complete. For example, suppose we are given an initial condition $y_\circ$, a disturbance $w(\cdot)$, a time interval $[t_\circ, t_\circ + T]$ and a bounded domain $\mathcal{D}$ such that the solution of the average system $y(t - t_\circ, y_\circ, w)$ is defined and remains in $\mathcal{D}$ for all $t \in [t_\circ, t_\circ + T]$, and we want to know whether solutions of the system (3) are close to this particular solution of the average on the interval $[t_\circ, t_\circ + T]$, at least for initial conditions that are close to $y_\circ$ and under the same disturbance w. In this case the system (3) can be modified to be of the form

$$\dot{x} = \ell(x) f\left(\frac{t}{\epsilon}, x, w, \nu\right) \tag{14}$$

where $\ell : \mathbb{R}^n \to [0,1]$ is a locally Lipschitz function with compact support that is equal to one on a neighborhood of the set

$$\mathcal{Y} := \{z \in \mathbb{R}^n : \ z = y(t, y_\circ, w) \ , \ t \in [0,T]\} \ . \tag{15}$$

It follows from the definition of average that the (weak or strong) average of (14) exists and is given by

$$\dot{y} = \ell(y) f_{av}(y,w) \tag{16}$$

where f_{av} is the average of f. Since $\ell(y)$ has compact support, this average is $\mathcal{W}$-forward complete for all sets $\mathcal{W}$ of locally essentially bounded functions. The results of the theorems can then be applied to conclude closeness of the solutions of (14) to the solutions of (16). Choosing $\delta > 0$ small enough in the theorems so that a δ-neighborhood of the set $\mathcal{Y}$ is contained in the neighborhood of $\mathcal{Y}$ where $\ell(\cdot)$ is equal to one, it follows that solutions of (3) are close to the particular solution of the average of (3) on the interval of consideration, at least for initial conditions that are close.

Similar observations can be used to work with systems that have averages only on certain subsets of (x,w)-space.

4 Proofs

4.1 Proof of Theorem 1

Step 1: Definition of ϵ^*, ν^* and μ

The triple (T, δ, r) is given. Without loss of generality, assume $\delta < 1$. Let $R \geq r$ come from $\mathcal{W}$-forward completeness of the weak average (Definition 1) and let Ω come from equi-(essential) boundedness of $\mathcal{W}$ (Definition 4). From the definition of weak average (Definition 1) it follows that there exists $L > 0$ such that, for all (x,y) satisfying $|x| \leq R+1$, $|y| \leq R$, and for all w satisfying $|w| \leq \Omega$ we have

$$|f_{wa}(x,w) - f_{wa}(y,w)| \leq L|x-y| \ . \tag{17}$$

Then define

$$\mu := \exp\left(-\frac{1}{2}LT\right)\frac{\delta}{2} \tag{18}$$

and let ν^* be such that

$$\left| f\left(\frac{t}{\epsilon}, x, w, \nu\right) - f\left(\frac{t}{\epsilon}, x, w, 0\right) \right| \leq \frac{L\delta^2}{8\left\{\exp(LT) - 1\right\}} \qquad \forall |\nu| \leq \nu^* \ , \ |w| \leq \Omega \ , \ |x| \leq R+1 \ , \ t \geq 0 \ . \tag{19}$$

In preparation for defining ϵ^*, let $\beta \in \mathcal{KL}$ and $T^* > 0$ come from Definition 1 and let $\widetilde{T} \geq T^*$ satisfy

$$\beta\left(\max\left\{R+1, \Omega\right\}, \widetilde{T}\right) \leq \frac{\delta^2}{12T\exp(LT)} \ . \tag{20}$$

According to the first supposition and Definition 1, the quantity

$$B := \sup_{\left\{ \ t \geq 0, \ |x| \leq R+1, \ |y| \leq R, \ |w| \leq \Omega, \ |\nu| \leq \nu^* \ \right\}} \max\Big\{|f(t,x,w,\nu)| \ , \ |f_{wa}(y,w)|\Big\} \tag{21}$$

is finite. Define

$$g\left(\frac{t}{\epsilon}, \widetilde{w}\right) := \widetilde{w}_1^T \left[f\left(\frac{t}{\epsilon}, \widetilde{w}_2, \widetilde{w}_3, 0\right) - f_{wa}\left(\widetilde{w}_2, \widetilde{w}_3\right)\right] \ , \tag{22}$$

the $\widetilde{w}_i$ being components, of appropriate dimension, of a vector $\widetilde{w}$. Define $\widetilde{\mathcal{W}}$ to be the set of functions

$$\widetilde{w}(\cdot) = \begin{bmatrix} \widetilde{w}_1(\cdot) \\ \widetilde{w}_2(\cdot) \\ \widetilde{w}_3(\cdot) \end{bmatrix} \tag{23}$$

such that $\widetilde{w}_3 \in \mathcal{W}$, and $\widetilde{w}_1$, $\widetilde{w}_2$ are absolutely continuous with $||\widetilde{w}_1||_\infty \leq 1$, $||\dot{\widetilde{w}}_1||_\infty \leq 2B$, $||\widetilde{w}_2||_\infty \leq R+1$, and $||\dot{\widetilde{w}}_2||_\infty \leq B$. Let $\rho > 0$ be such that, for all $\widetilde{w} \in \widetilde{\mathcal{W}}$ and all $t_i \geq 0$,

$$s \in [t_i, t_i + \rho] \qquad \Longrightarrow \qquad \left|\exp[L(t_i - s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) - g\left(\frac{s}{\epsilon}, \widetilde{w}(t_i)\right)\right| \leq \frac{\delta^2}{12T\exp(LT)} \; . \tag{24}$$

This ρ exists since g is continuous in $\widetilde{w}$ uniformly in t, $\widetilde{\mathcal{W}}$ is equi-uniformly continuous and for $s = t_i$ the quantity being bounded in (24) is zero. Then define

$$\epsilon^* := \min\left\{\frac{\rho}{\widetilde{T}}, \frac{\delta^2}{12(2B\widetilde{T}\exp(LT))}\right\} . \tag{25}$$

<u>Step 2: Comparison of solutions</u>

Let $\epsilon \in (0, \epsilon^*)$, $|\nu| < \nu^*$, $|y_\circ| \leq r$, $t_\circ \geq 0$, $w \in \mathcal{W}$ and consider any $x_\circ$ such that $|x_\circ - y_\circ| \leq \mu$. Define

$$e_{\epsilon,\nu}(t) := x_{\epsilon,\nu}(t, t_\circ, x_\circ, w) - y(t - t_\circ, y_\circ, w) \tag{26}$$

and note that $|e_{\epsilon,\nu}(t_\circ)| \leq \mu < \frac{\delta}{2} \leq \frac{1}{2} < 1$. If $|e_{\epsilon,\nu}(t)| < 1$ for all $t \in [t_\circ, t_\circ + T]$ then define $\bar{t} = t_\circ + T$. Otherwise, define

$$\bar{t} := \inf\left\{t \in [t_\circ, t_\circ + T] : |e_{\epsilon,\nu}(t)| = 1\right\} \; . \tag{27}$$

Note that $\bar{t} > t_\circ$ and $e_{\epsilon,\nu}(\cdot)$ and $x_{\epsilon,\nu}(\cdot, t_\circ, x_\circ, w)$ are defined and absolutely continuous on $[t_\circ, t_\circ + \bar{t}]$. Let $\widetilde{w} \in \widetilde{\mathcal{W}}$ be such that, for all $t \in [t_\circ, \bar{t}]$,

$$\begin{bmatrix} \widetilde{w}_1(t) \\ \widetilde{w}_2(t) \\ \widetilde{w}_3(t) \end{bmatrix} = \begin{bmatrix} e_{\epsilon,\nu}(t) \\ x_{\epsilon,\nu}(t, t_\circ, x_\circ, w) \\ w(t) \end{bmatrix} \; . \tag{28}$$

Such a $\widetilde{w} \in \widetilde{\mathcal{W}}$ exists since, for all $t \in [t_\circ, \bar{t}]$, $|e_{\epsilon,\nu}(t)| \leq 1$. Indeed, since $|y(t - t_\circ, y_\circ, w)| \leq R$ for all $t \in [t_\circ, t_\circ + T]$, it follows that $|x_{\epsilon,\nu}(t, t_\circ, x_\circ, w)| \leq R + 1$ for all $t \in [t_\circ, \bar{t}]$. In turn, it follows from (21) that, for almost all $t \in [t_\circ, \bar{t}]$, $|\dot{e}_{\epsilon,\nu}(t)| \leq 2B$ and $|\dot{x}_{\epsilon,\nu}(t, t_\circ, x_\circ, w)| \leq B$.

For almost all $t \in [t_\circ, \bar{t}]$ we have (dropping the arguments of signals for notational convenience)

$$\begin{aligned} \dot{e}_\epsilon &= f\left(\frac{t}{\epsilon}, x_{\epsilon,\nu}, w, \nu\right) - f_{wa}(y, w) \\ &= [f_{wa}(x_{\epsilon,\nu}, w) - f_{wa}(y, w)] + \left[f\left(\frac{t}{\epsilon}, x_{\epsilon,\nu}, w, 0\right) - f_{wa}(x_{\epsilon,\nu}, w)\right] \\ &\quad + \left[f\left(\frac{t}{\epsilon}, x_{\epsilon,\nu}, w, \nu\right) - f\left(\frac{t}{\epsilon}, x_{\epsilon,\nu}, w, 0\right)\right] \; . \end{aligned} \tag{29}$$

For the scalar-valued function $V(t) := \frac{1}{2}e^T_{\epsilon,\nu}(t)e_{\epsilon,\nu}(t)$, which is also absolutely continuous on $[t_\circ, \bar{t}]$, we have $V(t_\circ) \leq \frac{1}{2}\mu^2 = \exp(-LT)\dfrac{\delta^2}{8}$ and, for almost all $t \in [t_\circ, \bar{t}]$,

$$\begin{aligned} \dot{V} &\leq LV + e^T_{\epsilon,\nu}\left[f\left(\frac{t}{\epsilon}, x_{\epsilon,\nu}, w, 0\right) - f_{wa}(x_{\epsilon,\nu}, w)\right] + \frac{L\delta^2}{8\left\{\exp(LT) - 1\right\}} \\ &= LV + g\left(\frac{t}{\epsilon}, \widetilde{w}(t)\right) + \frac{L\delta^2}{8\left\{\exp(LT) - 1\right\}} \end{aligned} \tag{30}$$

where we have used the definition of $\bar{t}$, (19), (22) and (28). By standard comparison theorems it follows that for all $t \in [t_\circ, \bar{t}]$,

$$\begin{aligned} V(t) &\le \exp(LT)V(t_\circ) + \frac{\delta^2}{8} + \int_{t_\circ}^{t} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) ds \\ &\le \frac{\delta^2}{4} + \int_{t_\circ}^{t} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) ds \ . \end{aligned} \tag{31}$$

Fix $t \in [t_\circ, \bar{t}]$ and define k to be the largest nonnegative integer such that $k \le \frac{t-t_\circ}{\epsilon\widetilde{T}}$. For $i = 0, \ldots, k$, define $t_i = t_\circ + i\epsilon\widetilde{T}$ and note that, from the definition of k and (25), we have

$$t - t_k \le \epsilon\widetilde{T} \le \frac{\delta^2}{12(2B\exp(LT))} \ , \qquad t_{i+1} - t_i = \epsilon\widetilde{T} \le \rho \ . \tag{32}$$

We split the interval of integration in (31) using the times t_i to obtain

$$V(t) \le \frac{\delta^2}{4} + \int_{t_k}^{t} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) ds + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) ds \ . \tag{33}$$

It follows from the definition of $\widetilde{\mathcal{W}}$, (21), (22) and (32) that the first integral on the right-hand side of (33) is bounded as

$$\int_{t_k}^{t} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) ds \le \epsilon\widetilde{T}\exp(LT)2B \le \frac{\delta^2}{12} \ . \tag{34}$$

To bound the second integral on the right-hand side of (33), we split the integrand into two pieces: one that will be used to exploit the two time-scale behavior of g as a function of s and the other that will be used to exploit the continuity properties of g with respect to s. In the calculation that follows, we will use the following bound, which is a result of the definition of $\widetilde{\mathcal{W}}$, (5), (20), (22) and Hölder's inequality:

$$\begin{aligned} \left| \int_{t_i}^{t_{i+1}} g\left(\frac{s}{\epsilon}, \widetilde{w}(t_i)\right) ds \right| &= \left| \widetilde{w}_1^T(t_i) \int_{t_i}^{t_i+\epsilon\tilde{T}} \left\{ f\left(\frac{s}{\epsilon}, \widetilde{w}_2(t_i), \widetilde{w}_3(t_i), 0\right) - f_{wa}\left(\widetilde{w}_2(t_i), \widetilde{w}_3(t_i)\right) \right\} ds \right| \\ &\le \epsilon\tilde{T}\beta\left(\max\left\{R+1, \Omega\right\}, \tilde{T}\right) \le \epsilon\tilde{T}\frac{\delta^2}{12T\exp(LT)} \ . \end{aligned} \tag{35}$$

We will also use (24) and (32) and the fact that $k\epsilon\widetilde{T} \le t - t_\circ \le T$. We have

$$\begin{aligned} &\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) ds \\ &\quad = \sum_{i=0}^{k-1} \exp[L(t-t_i)] \left\{ \int_{t_i}^{t_{i+1}} g\left(\frac{s}{\epsilon}, \widetilde{w}(t_i)\right) ds + \int_{t_i}^{t_{i+1}} \left\{ \exp[L(t_i-s)]g\left(\frac{s}{\epsilon}, \widetilde{w}(s)\right) - g\left(\frac{s}{\epsilon}, \widetilde{w}(t_i)\right) \right\} ds \right\} \\ &\quad \le k\exp(LT)\left\{ \epsilon\widetilde{T}\frac{\delta^2}{12T\exp(LT)} + \epsilon\widetilde{T}\frac{\delta^2}{12T\exp(LT)} \right\} \le \frac{\delta^2}{6} \ . \end{aligned} \tag{36}$$

Combining (33), (34) and (36), it follows that $V(t) \le \dfrac{\delta^2}{2}$ for all $t \in [t_\circ, \bar{t}]$. Since $V(t) = \frac{1}{2}e_{\epsilon,\nu}^T(t)e_{\epsilon,\nu}(t)$, it follows that $|e_{\epsilon,\nu}(t)| \le \delta < 1$ for all $t \in [t_\circ, \bar{t}]$. From the definition of $\bar{t}$, it follows that $\bar{t} = t_\circ + T$ so that $|e_{\epsilon,\nu}(t)| \le \delta$ for all $t \in [t_\circ, t_\circ + T]$. This establishes the result. □

4.2 Sketch of proof of Theorem 2:

The proof of Theorem 1 follows exactly the same steps as the proof of Theorem 1 with the following changes:

The value $\rho > 0$ in (24) is chosen so that for all $\widetilde{w} \in \widetilde{\mathcal{W}}$ we have that $s \in [t_i, t_i + \rho]$ implies

$$\left|\exp[L(t_i - s)]g\left(\frac{s}{\epsilon}, \widetilde{w}_1(s), \widetilde{w}_2(s), \widetilde{w}_3(s)\right) - g\left(\frac{s}{\epsilon}, \widetilde{w}_1(t_i), \widetilde{w}_2(t_i), \widetilde{w}_3(s)\right)\right| \leq \frac{\delta^2}{12T\exp(LT)} \ . \tag{37}$$

(Note that we are now using

$$g\left(\frac{s}{\epsilon}, \widetilde{w}_1(t_i), \widetilde{w}_2(t_i), \widetilde{w}_3(s)\right)$$

instead of

$$g\left(\frac{s}{\epsilon}, \widetilde{w}_1(t_i), \widetilde{w}_2(t_i), \widetilde{w}_3(t_i)\right)$$

which was used in (24).) This ρ exists since g is continuous in $\widetilde{w}$ uniformly in t, $\widetilde{\mathcal{W}}$ is equi-(essentially) bounded, $\widetilde{w}_1$ and $\widetilde{w}_2$ are absolutely continuous and for $s = t_i$ the left hand side in (37) is zero.

Then we arrive in the same way as in proof of Theorem 1 at the inequality (33). Moreover, the bound (34) still holds. Instead of (35) we use:

$$\begin{aligned}
&\left|\int_{t_i}^{t_{i+1}} g\left(\frac{s}{\epsilon}, e_{\epsilon,\nu}(t_i), x_{\epsilon,\nu}(t_i), w(s)\right) ds\right| \\
&\qquad = \left|e^T_{\epsilon,\nu}(t_i)\int_{t_i}^{t_i+\epsilon\tilde{T}} \left\{f\left(\frac{s}{\epsilon}, x_{\epsilon,\nu}(t_i), w(s), 0\right) - f_{sa}\left(x_{\epsilon,\nu}(t_i), w(s)\right)\right\} ds\right| \\
&\qquad \leq \epsilon\tilde{T}\beta\left(\max\left\{R+1, \Omega\right\}, \tilde{T}\right) \leq \epsilon\tilde{T}\frac{\delta^2}{12T\exp(LT)} \ ,
\end{aligned} \tag{38}$$

which was obtained using the definition of $\mathcal{W}$, (7), (20), (22) and Hölder's inequality. In the same way as in Theorem 1 and using (38) we can write:

$$\begin{aligned}
&\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \exp[L(t-s)]g\left(\frac{s}{\epsilon}, e_{\epsilon,\nu}(s), x_{\epsilon,\nu}(s), w(s)\right) ds \\
&\quad = \sum_{i=0}^{k-1}\exp[L(t-t_i)]\left\{\int_{t_i}^{t_{i+1}} g\left(\frac{s}{\epsilon}, e_{\epsilon,\nu}(t_i), x_{\epsilon,\nu}(t_i), w(s)\right) ds \, + \right. \\
&\quad \left. \int_{t_i}^{t_{i+1}}\left\{\exp[L(t_i-s)]g\left(\frac{s}{\epsilon}, e_{\epsilon,\nu}(s), x_{\epsilon,\nu}(s), w(s)\right) - g\left(\frac{s}{\epsilon}, e_{\epsilon,\nu}(t_i), x_{\epsilon,\nu}(t_i), w(s)\right)\right\} ds\right\} \\
&\quad \leq k\exp(LT)\left\{\epsilon\widetilde{T}\frac{\delta^2}{12T\exp(LT)} + \epsilon\widetilde{T}\frac{\delta^2}{12T\exp(LT)}\right\} \leq \frac{\delta^2}{6} \ .
\end{aligned} \tag{39}$$

Using now (33), (34) and (39) we obtain that $V(t) \leq \frac{\delta^2}{2}, \forall t \in [t_0, \bar{t}]$. This establishes the result in the same way as in the proof of Theorem 1. □

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