Input-Output Stability of Networked Control Systems with Stochastic Protocols and Channels

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***Abstract*— This paper introduces a new definition of stochastic protocols for networked control systems (NCS) and the stochastic analogue of the notion of uniform persistency of excitation of protocols first presented in [1]. Our framework applies directly to common wireless and wireline NCS, including those built on collision-sense multiple access (CSMA) style protocols, with Ethernet and 802.11a/b/g as prime examples of this class. We present conditions for a general class of nonlinear NCS with exogenous disturbances using stochastic protocols in the presence of packet dropouts, random packet transmission times and collisions that are sufficient for L_p stability from exogenous disturbance to NCS state with a linear finite expected gain. Within the same framework, we extend the results of [2] to provide an analysis of deterministic protocols, including try-once-discard (TOD), in the presence of random packet dropouts and inter-transmission times and provide a stochastic analogue of the Lyapunov-theoretic stability properties for network protocols introduced therein.**

I. INTRODUCTION

The premise of networked control systems (NCS) is to spatially distribute a "traditional" control system across a number of *nodes* that exchange data subject to the constraints of a shared data channel. These nodes include sensors, actuators and units that compute various control laws and the data channel is typically a wireless or wireline computer network, many examples of which can be found in [3]. For the vast majority of computer networks described in [3], the primary constraint on the exchange of data between nodes is that the respective channels are exclusive in that the attempt of more than one node to transmit data at a given time will result in data loss, i.e., a collision.

Collisions can be prevented through the use of (contentionless) *scheduling protocols* that decide which node(s) can transmit and at what times. For example, labeling the NCS nodes $\{a_1, a_2, ..., a_N\}$, round-robin scheduling would entail apportioning the channel's time $[0, \infty)$ into slots $\{s_1 := [t_0, t_1), s_2 := [t_1, t_2), \ldots, \}$ such that node a_i is permitted to transmit during slot s_{i+kN}, $k = \{0, 1, \ldots\}$. Depending on the context, this scheduling protocol is also known as time-division multiplexing or Token Ring. Stability properties of NCS employing round-robin scheduling and various other contentionless protocols have been discussed in [1], [4], [2], [5], [6], [7], [8] and [9].

A contentionless scheduling protocol can be thought of as a (time-varying) map $h : [0, \infty) \times X \to \{1, .., N\}$ that selects the node currently being allowed to transmit and an associated dynamical system that evolves the state variable $x \in X$. For spatially separated nodes, this generally means that each node must maintain a copy of the state x that is evolved identically by each node. For networks with a large number of nodes, mobile nodes that are spatially separated across varying distances or networks with a varying number of nodes, it may be impractical or impossible to keep the state information synchronized across all nodes.

The alternative is to accept that collisions may occur, detect and recover from them when they do occur and attempt to to reduce the number that occur by employing various heuristics using data available to each node locally. Concrete and familiar examples of this approach include the family of collision-sense multiple access protocols (CSMA) exemplified by Ethernet, p-persistent CSMA (Bluetooth, 802.11a/b/g) and variants of ALOHA. See [3] for an overview of these protocols and their operational characteristics.

Thus far, the discussion holds true for both computer and control networks. Where computer networks and control networks differ radically is in access patterns – ideally, a continuous-time control system would have nodes constantly transmitting sensors values and constantly receiving control values, in complete contrast to the usual assumption of access in short and irregular bursts for nodes in a computer network. Stated explicitly, *we assume continuous-time controllers and plant outputs are such that there will always be data to transmit when the network channel becomes idle.*

This assumption applies to contentionless and contention protocols in NCS, the key difference being that the latter does not enforce a particular choice of which link to transmit when the channel becomes idle whereas the former does. Despite the lack of collisions in in contentionless protocols, we present a unified approach for the analysis of NCS employing contentionless and contention protocols in the presence of random packet dropouts and random inter-transmission times – effects that are essentially attributes of non-ideal or *stochastic* network channels.

Motivated by the need to design and analyze NCS with stochastic channels in order, for example, to deploy NCS nodes wirelessly, we propose a model of NCS and network protocols analogous to the models presented in [2] and [9].

The NCS design approach adopted in [2], [10], [5], [7], [8], and this paper consists of the following steps:

1) design of a stabilizing controller ignoring the network;
2) and analysis of robustness of stability with respect to effects that the network introduces.

We assume that every link in the NCS contests access to the

This work was supported by the Australian Research Council under the Australian Professional Fellowship & Discovery Grants Scheme.

network at either predetermined time-slots or at times at which the network is sensed to be idle. This results in two potential sources of randomness:

1) At any idle time or transmission slot, either some node j transmits successfully or a collision results or the transmitted packet is dropped. Denoting the probability that a packet is dropped or a collision occurs by p_0, we will always assume that the probabilities of successful transmission of links is identically equal to $(1-p_0)/V$ for a V-link NCS using a contention protocol. While this is not strictly necessary in our analyses, there is no reason to statically (off-line) favor any one link over another during contention by adjusting transmission-success probabilities. Contentionless protocols do, however, enforce a particular choice of which link to transmit in a given slot eliminating the possibility of a collision.
2) Sensing the network as being idle, synchronizing to transmissions time-slots or else randomly waiting for a period of time after any of these events to reduce the likelihood of transmission are common features of network protocols. These uncertainties can be faithfully modeled with a stochastic (renewal) process. For the set of protocols we discuss, it is sufficient to restrict our attention to Poisson processes with some intensity λ or a class of renewal processes where inter-transmission times are uniformly bounded.

In lieu of the notion of a scheduling protocol described in [2] and [9] and the notion of maximum allowable transmission interval (MATI), we now have a stochastic process that determines when transmissions occur and which link, if any, is transmitted at these times.

Within this setup, our analysis framework analyzes the input-output L_p stability (IOS) of NCS (in expectation), the essence of which is that outputs (or state) of an NCS verify a robustness property with respect to exogenous disturbances. We stress that it is only the network protocol and channel delay that induces randomness in our models and that the exogenous disturbances are L_p signals as in [2] and [9].

We show that both contention and contentionless protocols verify stochastic analogues of the protocol stability properties introduced in [1] and [2], respectively. The contention protocols in the sense of our definition, satisfy a property that is similar to the property of uniform persistency of excitation introduced in [1] – that is, links are almost surely (a.s.) transmitted within a finite number of transmissions T. For a V-link NCS in the stochastic setting, the random variable T is closely related to the cover time of an $(V+1)$-vertex undirected graph and the running time of the Coupon Collector problem.[1] For the contentionless protocols we discuss, the stability property we examine is that of a.s. Lyapunov uniform global exponential stability (UGES) with obvious parallels to the analysis approach pursued in [2].

Although link cover times and inter-transmission are now random and, hence, not uniform, we show that if the network-free system is L_p stable, the NCS remains so with any contention protocol, in the sense of our definition, whenever attempted transmissions occur "fast enough". With mild additional technical assumptions, we show that a similar conclusion holds for a.s. Lyapunov UGES protocols and, in particular, holds for the try-once-discard (TOD) protocol, introduced in [7], in the presence of random packet dropouts and inter-transmission times. By "fast enough" we mean that there exists a choice of intensity λ of the transmission process or a choice of uniform bound on inter-transmission times parameterized by properties of the protocol and the NCS dynamics such that the NCS is L_p stable-in-expectation from disturbance to NCS state with a finite expected gain.

Our work builds on the NCS analysis approach and the protocol description methodology described in [2] and further developed in [9] for deterministic systems. Notions of stochastic inter-transmission and delay processes for linear NCS are discussed [12] and elsewhere subsequently with the analysis framework presented in [13] applying aspects of protocol and stability analysis similar in spirit to [2] within a stochastic setting. The focus in [13] is on NCS that employ *contentionless protocols*, as discussed in [1], [4], [2], [5], [6], [7], [8] and [9] and examines mean-square stability of a class of NCS perturbed by a Wiener process when the inter-transmission process is random (a renewal process) and where random data dropouts may occur. This is contrast to the results within this paper that focus on a robustness property (L_p stability-in-expectation) and that consider *protocols* that satisfy a.s. stability properties.

The primary contributions of this paper are the novel definition of stochastic protocols that model typical contention protocols as well as contentionless protocols in the presence of packet dropouts in the NCS setting together with several protocol examples that can be modeled in this way; development of an extension of the Lyapunov UGES property and analysis approach pursued in [2] for non-ideal NCS; characterization of the stochastic analogue of uniform persistency of excitation – the a.s. finite cover time property; and development of several consequences of these definitions including L_p stability-in-expectation of the error dynamics of the NCS that decreases (to zero) as the expected transmission rate increases (to infinity) and development of sufficient conditions for L_p stability-in-expectation of the NCS as a whole.

The paper is divided into five additional sections: Section II introduces notation and technical devices; Section III introduces our model of NCS with stochastic impulses and we discuss the key classes of a.s. covering protocols and a.s Lyapunov UGES protocols in Section IV and Section V, respectively. We also present two typical contention protocol examples that can be faithfully represented with this model and discuss key differences between the use of contention protocols in computer networks and control networks as well as present an example of a contentionless protocol operating in the presence of packet dropouts. We present our main stability results for contention and contentionless protocols in Section VI and Section VII, respectively, where we outline sufficient criteria for the protocol and nominal control system such that the resultant NCS is L_p stable with linear finite expected gain from exogenous disturbance to state. A case study is

[1] See [11, Section 2.4.1], for instance, for a description of the Coupon Collector problem.

presented in Section VIII with proofs of the main results are presented in Section IX before we conclude with some general remarks on analysis framework pursued as well as possible future extensions of this work in Section X.

II. Preliminaries

Let $\mathcal{M}_n$ denote the set of $n \times n$ matrices with zero off-diagonal entries and diagonal entries in the set $\{0, 1\}$. Let $a \vee b$ and $a \wedge b$ denote the maximum and minimum of two real numbers a and b, respectively. Let $Df(t)$ denote the left-handed derivative of $f : \mathbb{R} \to \mathbb{R}^n$:

$$Df(t) = \lim_{h\to 0, h<0} \frac{f(t+h) - f(t)}{h}$$

whenever the above limit exists. Let $\mathbf{E}[\cdot], \mathbf{P}\{\cdot\}$ denote the expectation and probability (measure) operators, respectively. For any random vector $\xi \in \mathbb{R}^d$ with distribution μ, the associated moment generating function (mgf) $\hat{\mu}$ is given by

$$\hat{\mu}(t) = \int \exp(tx)\mu(dx) = \mathbf{E}[\exp(t\xi)].$$

We use the abbreviation *iid* for "independently identically distributed" and the notation $X \sim \text{Exp}(\lambda)$ to indicate that X is an exponentially-distributed random variable with $\mathbf{E}[X] = 1/\lambda$. For distributions on $\mathbb{Z}_+$, we use the probability generating function (pgf) ψ given by

$$\psi(s) = \sum_{n\geq 0} s^n \mathbf{P}\{\chi = n\} = \mathbf{E}[s^\chi], s > 0.$$

We will be considering systems with stochastic impulses of the form:

$$\dot{z}(t) = h(t, z, w) \quad t \in [t_{i-1}, t_i] \tag{1}$$
$$z(t_i^+) = Q_i(z(t_i)), \tag{2}$$

where $Q_i(\cdot) \in \mathcal{M}_{n_z}$ is a sequence of random maps, $t_{i+1} - t_i \sim \text{Exp}(\lambda)$, iid and by $z(t_i^+)$ we mean "evaluated just after the jump": $z(t_i^+) = \lim_{s\to t_i, s>t_i} z(s)$.

Fix $p \in [1, \infty]$. For clarity of the presentation, we assume enough regularity on h for the existence of an absolutely continuous function $z(t, t_0, z_0, w)$ such that $\frac{d}{dt} z(t, t_0, z_0, w) = h(t, z, w), t \in [t_0, a], a > 0$ for every initial condition (t_0, z_0) and any $w \in L_p$. It is then clear how to generate the trajectory process of (1)-(2) with the initialization (t_0, z_0):

$$z(t) = z(t_0) + \int_{t_0}^t h(s, z(s), w(s))ds, t \in (t_0, t_1),$$

where $z(s) := z(s, t_0, z_0, w(s))$ and inductively,

$$z(t) = Q_i(z(t_i)) + \int_{t_i}^t h(s, z(s), w(s))ds, t \in (t_i, t_{i+1}),$$

where $z(s) := z(s, t_i, Q_i(z(t_i)), w(s))$. Note that Zeno solutions are a.s. not possible since $\mathbf{P}\{t_{i+1} - t_i = 0\} = 0$.

III. Hybrid System Model for NCS

We assume that a stabilizing (continuous-time) controller has been designed ignoring the network and consider general nonlinear NCS with disturbances where x_P and x_C are, respectively, states of the plant and controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of the most recently transmitted plant and controller output values via the network and e is the network-induced error defined as

$$e(t) := \begin{pmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{pmatrix}. \tag{3}$$

We model the NCS as a so-called jump-continuous (hybrid) system, where jump times and the associated jump or reset maps are both random. Node data (controller and sensor values) are transmitted at (possibly) random transmission instants $\{t_0, t_1, \ldots, t_i\}, i \in \mathbb{N}$ and our NCS model is prescribed by the following dynamical and jump equations. In particular, for all $t \in [t_{i-1}, t_i]$:

$$\dot{x}_P = f_P(t, x_P, \hat{u}, w) \tag{4}$$
$$\dot{x}_C = f_C(t, x_C, \hat{y}, w) \tag{5}$$
$$u = g_C(t, x_C) \quad y = g_P(t, x_P) \tag{6}$$
$$\dot{\hat{y}} = 0 \quad \dot{\hat{u}} = 0, \tag{7}$$

and at each transmission instant t_i,

$$e(t_i^+) = Q_i(e(t_i)), \tag{8}$$

where $Q_i(\cdot)$ is a *random* jump map. In particular, Q_i may be the identity in the case where nothing was transmitted or a collision or dropout occurred.

We consider two main classes of protocols: *contention* and *contentionless* protocols in the presence of random packet dropouts and random inter-transmission times that we model through appropriate definition of the random error jump maps in (8) and the sequence on transmission instants. Within our modeling framework, we shall see that it is enough to restrict our attention to jump maps of the form

$$e(t_i^+) = Q_i e(t_i),$$

for contention protocols, where Q_i is an iid sequence of diagonal matrices with entries drawn from the set $\{0, 1\}$ and, for contentionless protocols in the presence of dropouts, jump maps of the form:

$$e(t_i^+) = q_i h(i, e(t_i)) + (1 - q_i)e(t_i),$$

where h is a deterministic jump map (e.g., as in [2]) and q_i is an iid sequence of Bernoulli random variables. These classes of protocols together with the sequence of transmission instants are collectively referred to as a *stochastic protocol*.

The effect of the stochastic protocol on the error is such that if the mth to nth nodes are successfully transmitted at transmission instant t_i the corresponding components of error, $e_n, \ldots, e_m$, experience a "jump". It may be the case that a single logical node (a "link") consists of several sensors or several actuators or both with the transmission of that link having the effect of setting multiple components of e to zero. It may also be the case that the network allows the transmission of more than one node at each transmission and our model allows for this extra degree of freedom. For transmission of nodes mth to nth nodes, we will always assume that $e_n(t_i^+), \dots, e_m(t_i^+) = 0$ and, hence, $Q_i(e) = [a_{kj}]e$, where $a_{kj} = 0$ for $k = j \in [n, m] \cup \{k \neq j\}$ and 1 elsewhere. We group the nodes that are transmitted together into logical links, associating a partition of size s_i, denoted by $\mathbf{e}_i = (e_{i1}, e_{i2}, \dots, e_{is_i})$, of the error vector e such that we can write $e = (\mathbf{e}_1, \dots, \mathbf{e}_V)$. We say that the NCS has V links and $\sum_{i=1}^{V} v_i$ nodes. Note that this is purely a notational convenience and simplifies the description of scheduling protocols and the NCS itself. We combine the controller and plant states into a vector $x = (x_P, x_C)$ and similarly to [2, pp. 1653], assuming g_P, g_C are a.e. C^1, for example, we can rewrite (4)-(8):

$$\dot{x} = f(t, x, e, w) \quad t \in [t_{i-1}, t_i] \tag{9}$$

$$\dot{e} = g(t, x, e, w) \quad t \in [t_{i-1}, t_i] \tag{10}$$

$$e(t_i^+) = Q_i(e(t_i)) \tag{11}$$

where $x \in \mathbb{R}^{n_x}, e \in \mathbb{R}^{n_e}, w \in \mathbb{R}^{n_w}$. Implicit in this definition is that there are no (pure) propagation delays. Transmission at time t_i results in the instant reset of the relevant error component to zero. We appeal to the robustness properties verified by the class of systems considered to assert that the results in this paper remain true for sufficiently small delays.

With respect to (4)-(8) and (9)-(11), we further assume that the sequence of (attempted) transmission times $\{t_i\}_{i\in\mathbb{N}}$ is such that $t_{i+1} - t_i$ is exponentially distributed for all i and analyze two classes of jump maps in (8) and (11) which we explore in the proceeding sections.

IV. Contention Protocols

By a contention protocol, we mean the sequence of random transmission times together with iid random jump maps Q_i that are e-independent with reference to (11). That is, Q_i are iid random matrices taking values in the finite set $\mathcal{M}_{n_e} = \{M_0, M_1, \dots, M_V\}$, where $M_0 = I_{n_e}$ and M_j is such that

$$\begin{aligned} M_j e &= M_j(\mathbf{e}_1, \dots, \mathbf{e}_j, \dots, \mathbf{e}_V) \\ &= (\mathbf{e}_1, \dots, \mathbf{e}_{j-1}, \mathbf{0}, \mathbf{e}_{j+1}, \dots, \mathbf{e}_V). \end{aligned}$$

We make this definition more precise shortly. The intuition behind this model is that at a transmission time t_i, either some link j will acquire the channel and have its component of e set to zero, that is,

$$\mathbf{e}_j(t_i^+) = 0, \mathbf{e}_i(t_i^+) = \mathbf{e}(t_i), i \neq j,$$

hence $Q_i = M_j$ or else more than one node attempted to transmit resulting in a collision with e remaining unchanged ($Q_i = M_0$). Due to random "back-off" times, and wait-times inserted into medium access protocols, transmission times are potentially random. Collectively, these issues are the same issues presented in multi-user access in computer and mobile voice networks though the network access patterns are somewhat different. See [3] for an overview.

Remark 1: Note that the definition of NCS error given in (3) and the description of the NCS in (9)-(11) is similar to that presented in [2] with two key differences:

1) the inter-transmission continuous-time dynamics in (9) and (10) are prescribed on a sequence of intervals $[t_{i-1}, t_i]$ of random lengths not necessarily uniformly bounded by a constant, i.e., the notion of MATI does not always make sense for the inter-transmission processes we consider; and,
2) the scheduling protocol (error jump map) (11) is a particular random linear map, where we admit the possibility of $Q_i = I$ with non-zero probability equal to the probability of packet dropout and collision. We believe that this a new and novel approach to modeling contention protocols on non-ideal network channels.

Definition 4.1: For a V-link NCS, abstractly, we define a contention protocol as a discrete Markov chain Q_i subordinated by a renewal process[2] $N(t)$ such that

1) $Q_i \in \mathcal{M}_{n_e}$ are iid random $n_e \times n_e$ with associated link and collision probabilities given by
$$\mathbf{P}\{Q_i = M_i\} = p_i.$$
2) The sequence of arrival times $\{t_i\}_{i\in\mathbb{N}}$ is defined inductively by:
$$t_0 = \tau_0,$$
where $\tau_0 \sim \text{Exp}(\lambda)$ and for each $i > 0$,
$$t_i = t_{i-1} + \tau_i,$$
$\tau_i \sim \text{Exp}(\lambda)$, where the sequence $\{\tau_i\}$ is iid. We set
$$N(t) = \begin{cases} 0 & t \in [0, t_0) \\ k & t \in [t_{k-1}, t_k), \end{cases}$$
hence, $N(t)$ is a Poisson process with intensity λ.

Essentially, the τ_i denotes the wait time after the arrival of a packet (before a new transmission begins). Where not otherwise stated, we will henceforth assume that $\mathbf{P}\{Q_i = M_k\} = \mathbf{P}\{Q_i = M_j\} = (1 - p_0)/V, k, j \neq 0$ i.e., each link is equally likely to be transmitted successfully. As alluded to in the introduction, this assumption is not strictly necessary for our analyses, however, any other distribution of probabilities results in a *static* choice of priorities amongst links where one link may be favored over another during contention. There may be examples of NCS that would benefit from such an adjustment of relative link priorities offline in terms of required transmission rates or greater robustness of stability but as these choices are made offline and not in response to the evolution of the NCS state online, we believe that the scope of exploiting this degree of freedom is limited.

In [9], the analysis framework defined the notion of uniform persistency of excitation of a protocol. To say that a protocol was PE_T was to guarantee that every link is visited after T transmissions. We pursue a stochastic analogue of that here:

Definition 4.2 (Cover Time): Consider a contention protocol in the sense of Definition 4.1 for a V-link protocol and define

$$T_0 = \min\{j \geq 1 : \{M_1, \dots, M_V\} \subset \{Q_0, \dots, Q_{j-1}\}\}$$

[2]More precisely, the process of interest is in fact a marked point-process. See [14] for an exposition.

and, inductively for $i > 0$,

$$T_i = \min\{j \geq 0 : \{M_1, \ldots, M_V\} \subset \{Q_{T_{i-1}}, \ldots, Q_{T_{i-1}+j-1}\}\}.$$

We refer to T_i as the ith cover time and, collectively the cover time process. It is clear from our definition of Q_i that T_i is a stationary process.

Definition 4.3 (Covering sequence): Let $\tau_i = t_{i+1} - t_i$, as in Definition 4.1, that is, τ_i are inter-arrival times. We say that

$$C(j, k) = \{(Q_j, \tau_j), \ldots, (Q_k, \tau_k)\}, k \geq j$$

is a covering sequence iff $\{M_1, \ldots, M_V\} \subset C_{(1)}(j, k)$.[3] It is easy to see that cover times are simply the lengths of consecutive disjoint covering sequences.

Remark 2: From our definition of contention protocols, the distribution of T_n is given by the solution to the (weighted) coupon collectors problem. When $p_i = p_j$, $i, j \neq 0$, we have the closed form expression for the expectation:

$$\mathbf{E}[T] = VH_V/(1 - p_0), \tag{12}$$

where H_V is the Vth harmonic number and we have dropped the time index n since T_n is stationary. We also have the bound for the distribution:

$$\mathbf{P}\{T_n \geq \beta V \ln V/(1 - p_0)\} \leq V^{-(\beta-1)}/(1 - p_0), \tag{13}$$

for any $\beta > 1$. Intuitively, $T_n = \mathbf{E}[T]$ "most of the time" and $\mathbf{P}\{T_n < \infty\} = 1$. $\triangleleft$

Our abstract definition of a contention protocol is a model for the contention protocols discussed in the introduction and to that end we present two natural examples in this setting.

Definition 4.4 (Almost Surely Finite Cover Time): We say that a protocol is *a.s. covering* or has an *a.s. finite cover time* if in Definition 4.2

$$(\forall i \in \mathbb{N}) \quad \mathbf{P}\{T_i < \infty\} = 1.$$

Note that from the preceding discussion, this property is verified by all contention protocols in the sense of Definition 4.1.

Remark 3: The property of persistency of excitation within the context of scheduling protocols discussed in [9] is essentially a protocol stability property closely related to the Lyapunov UGES and UGAS stability properties for scheduling protocols introduced in [2] and [8], respectively. Just as the a.s. covering property introduced in this paper is a stochastic analogue of persistency of excitation of protocols, the Lyapunov UGES and UGAS properties may be recast within our framework to assert L_p stability results in the presence of random data dropouts quite distinct from the unwieldy deterministic characterization of dropouts presented [2]. This generalization is pursued in subsequent sections of the paper within essentially the same analysis framework.

The motivation for studying these stochastic analogues of the stability properties is to naturally extend the results of [2] and [9] to non-ideal networks, that is, networks with random inter-transmission times and random packet dropouts as a result of collisions and transmission errors. Other approaches to analyzing the effects of dropouts have been studied including the asynchronous dynamical systems approach (ADR) presented [15] with applications to NCS pursued in [16]. Dropouts events are characterized by a *rate* which is essentially an ensemble time-average of the dropout indicator function. In principle, assuming that the dropout process is ergodic, for example, this is akin to taking an expectation of the dropout process which we use to characterize the probability of dropouts from transmission errors. In practice, admitting the possibility of collisions introduces a dependence of the probability of dropout on the number of links which is essentially ignored in [15] and [16] as neither work examines scheduling amongst links. This paper characterizes the behavior of contention and contentionless protocols through appropriate definitions of (stochastic) scheduling protocols that would be difficult or impossible to represent within an ADR framework.

[3]The notation $C_{(1)}(j, k)$ refers to the covering sequence of matrices Q_i with no reference to inter-transmission times τ_i i.e., $\{Q_j, \ldots, Q_k\}$.

A. Slotted p-Persistent CSMA

What has been referred to as "scheduling" and the associated scheduling protocols by [9] is generally known as medium access in the communications literature. Collision sense multiple access with collision detection (CSMA/CD) is by far the most widely used medium access protocol by virtue of the sheer volume of Ethernet and Ethernet-like networking devices shipped and manufactured each year.

CSMA/CD is a simple protocol: Links listen for transmissions on the the channel. A link wanting to transmit acquires the channel when it senses that the channel is idle. When more than one link senses that the channel is idle and begins transmission, a collision occurs. At this point, all transmissions are immediately aborted. There are several variants of CSMA/CD that prescribe how transmissions are rescheduled and how links initially acquire the channel.

With slotted p-persistent CSMA, rather than have links transmit whenever the channel is idle, links are only permitted to transmit at prescribed transmission slots that occur every $t_s > 0$ seconds in slotted protocols. At the start of slot s_k, links $S = \{i, .., j\}$ intending to transmit acquire the channel with a probability of p. If a collision occurs, links S^c are permitted to transmit in the next slot and links S^c reschedule their transmissions at slots $\{s_{k+d_i}, \ldots, s_{k+d_j}\}$. The delays $\{d_i, \ldots, d_j\}$ may be deterministic or iid random variables.

As alluded to in the introduction, the primary reason that CSMA protocols and, indeed, all contention protocols work in practice is that the access patterns of computer and voice networks are "bursty" in nature. The assumption is that a link will occasionally transmit a burst of information and remain otherwise idle. Transmissions are expected to eventually succeed as links are "infrequently" contending for the channel.

The situation is quite different for control networks with the implication that medium access patterns are constant rather than bursty and for slotted p-persistent CSMA, we assume that every slot will be in contention. Another key difference between computer networks and NCS is in the treatment of collisions and dropouts. NCS should not buffer failed transmissions of controller or sensor values but, rather, attempt to transmit the latest values when a slot is free. As the maximum number of links contenting slots is constant for every slot, there is no reason for a link to delay transmission for any more than one slot after a collision.

With these assumptions, consider a V-link NCS with the p-persistent CSMA protocol. The probability $\mathbf{P}\{Q_i = M_j\}$ that a *particular* link j transmits successfully during the ith slot is given by

$$\mathbf{P}\{Q_i = M_j\} = p(1-p)^{V-1}.$$

It is clear that $\mathbf{P}\{Q_i = M_j\}$ is maximized when $p = 1/V$. Will henceforth set $p = 1/V$ and have that

$$\mathbf{P}\{Q_i = M_j\} = \frac{1}{V}\left(1 - \frac{1}{V}\right)^{V-1} = \frac{(V-1)^{V-1}}{V^V}.$$

Notice that in this "optimal" case, $\mathbf{P}\{Q_i = M_j\} = \mathbf{P}\{Q_i = M_k\} = (V-1)^{V-1}/V^V$ for $i, k \neq 0$ and the probability of a collision is given by $\mathbf{P}\{Q_i = M_0\} = 1 - (V-1)^{V-1}/V^{V-1}$. Finally, we assume that slots occur every $t_s > 0$ seconds and, hence, p-persistent CSMA is a contention protocol in the sense of Definition 4.1 where inter-arrival times τ_i and, hence, δ_i and η_i, are deterministic.

B. CSMA with Random Waits

Whereas the use of fixed slots tends to improve throughput and reduce collisions with computer networks e.g., slotted versus pure ALOHA, the contention by every link at every slot forces transmissions to happen in lock-step with NCS network access patterns with the potential for a collision at every slot.

Suppose that instead of immediately acquiring the channel with probability p after sensing the channel to be idle or after a new slot arrives, links instead wait a random amount of time before transmitting. In particular, if a particular link j waits for a random time $\eta'_j \sim \text{Exp}(\lambda/V)$ then, $\mathbf{P}\{Q_i = M_j\} = (1-p_0)/V, j \neq 0$. The actual wait time before any particular transmission will be given

$$\tau = \min\{\eta'_1, \ldots, \eta'_V\}$$

that is, the link that waits the least gets to transmit first, hence, $\tau \sim \text{Exp}(\lambda)$. Assuming the wait times are iid for each link, this is the the prototypical example of what mean by a stochastic (contention) protocol and stochastic channel.

In the presence of transmission errors, p_0 is generally nonzero and, conceptually, p-persistent CSMA and CSMA with random waits are essentially the same save for the fact that the transmission process is truly random with the latter. While CSMA with random waits can be thought of as a protocol in its own right when the random waits are enforced explicitly in the implementation, it can also be thought of as a model of medium access with NCS access patterns while using a class of CSMA wireless protocols. Delays in signal detection, multi-path effects and varying processor loads mean that links are only prepared to transmit after some delay upon sensing the channel being idle and although the cumulative effects of these delays may not be exponentially distributed, the principle remains the same.

V. Contentionless Protocols with Dropouts

The premise of a contentionless protocol is that the network channel is a resource shared amongst links and that the simultaneous transmission of data by more than one link will result in data loss. By careful coordination amongst links through the use of a particular scheduling protocol, contention can be eliminated completely and the property that only one link can attempt to transmit at any given instant can be enforced. As alluded to in the introduction, simple round-robin scheduling amongst links is an example of a contentionless protocol as are the protocols discussed in [2], including the so-called try-once-discard (TOD) scheduling protocols.

Despite the elimination of contention, NCS employing contentionless protocols on non-ideal network channels are still subject to packet losses and varying inter-transmission times. With reference to (11), a jump map of the form

$$e(t_i^+) = h(i, e(t_i)) \tag{14}$$

was used to capture the behavior of the protocol in [2] on an ideal network and by assigning a probability, p_0, to the event that the channel drops a packet, we model the behavior of the protocol on non-ideal channels in this paper with jump maps of the form

$$e(t_i^+) = q_i h(i, e(t_i)) + (1 - q_i) e(t_i), \tag{15}$$

where q_i is an iid sequence of Bernoulli random variables that model the dropout process of channel with $\mathbf{P}\{q_i = 1\} = 1 - p_0$. Implicit in this discussion is that, as in Section IV, the sequence of arrival times $\{t_i\}_{i\in\mathbb{N}}$ is defined inductively by:

$$t_0 = \tau_0,$$

where $\tau_0 \sim \text{Exp}(\lambda)$ and for each $i > 0$,

$$t_i = t_{i-1} + \tau_i,$$

$\tau_i \sim \text{Exp}(\lambda)$, where the sequence $\{\tau_i\}$ is iid.

As in [2], it becomes natural to define the associated auxiliary discrete-time system for (15):

$$e(i+1) = q_i h(i, e(i)) + (1 - q_i) e(i) \quad i \in \mathbb{N}, \tag{16}$$

where the sequence $\{q_i\}$ is defined as in (15).

As alluded to in Remark 3, the crux of our NCS analysis framework rests on verifying appropriate stability properties of the protocol in question and inferring a set of sufficient conditions from which robust stability of the NCS can be concluded. For contention protocols, the protocol stability property is that of a protocol being a.s covering. For contentionless protocols, we introduce the following definition with respect to system (16):

Definition 5.1 (Almost surely Lyapunov UGES protocols): Let $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ be given and suppose that κ_i is a sequence of nonegative iid random variables such that, $a_1, a_2 > 0$ such that the following conditions hold for the discrete-time system (16) for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{n_e}$:

$$a_1|e| \leq W(i, e) \leq a_2|e| \tag{17}$$

$$W(i+1, h(i, e)) \leq \kappa_i W(i, e) \tag{18}$$

$$\mathbf{E}[\kappa_i] < 1 \tag{19}$$

then we say that (16) (equivalently, the contentionless protocol) is *almost surely uniformly globally exponentially stable (a.s UGES) with Lyapunov function W*.
Before discussing implications of this definition, we present a motivating example:

Example 5.2 (Try-Once-Discard): The TOD protocol was introduced in [7] and can be expressed with a model of the form (16) where

$$h(e) = (I - \Psi(e))e$$

and $\Psi(e) = \operatorname{diag}\{\psi_1(e)I_{v_1}, \dots, \psi_V(e)I_{v_V}\}$, with I_{v_j} identity matrices of dimension v_j and

$$\psi_j(e) = \begin{cases} 1, & \text{if } j = \min(\arg\max_j |\mathbf{e}_j|) \\ 0, & \text{otherwise.} \end{cases}$$

As in [2][Proposition 5], we set $W(i,e) = |e|$ and claim that TOD is a.s Lyapunov UGES whenever the probability of a dropout, p_0 is such that

$$p_0 + (1-p_0)\sqrt{\frac{V-1}{V}} < 1. \tag{20}$$

The inequality (20) is a particular example of a more general condition that ensures that any Lyapunov UGES protocol in the sense of [2] is an a.s Lyapunov UGES for sufficiently low probability of dropout and admits the following proposition:

Proposition 5.3: Suppose that the protocol (16) on an ideal channel ($p_0 = 0 \Rightarrow q_i = 1$) is Lyapunov UGES in the sense of [2]. That is, there exists $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, $a_1, a_2 > 0$, and $0 \leq \theta < 1$ such that for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{n_e}$:

$$a_1|e| \leq W(i,e) \leq a_2|e| \tag{21}$$

$$W(i+1, h(i,e)) \leq \theta W(i,e). \tag{22}$$

Then (16) is a.s Lyapunov UGES on a non-ideal channel ($p_0 \geq 0$) if

$$p_0 + (1-p_0)\theta < 1. \tag{23}$$

Proof: It is clear that we only need verify (19) to conclude that (16) is a.s Lyapunov UGES with Lyapunov function W. We have $\kappa_i = q_i\theta + (1-q_i)$ and, hence, $\mathbf{E}[\kappa_i] = (1-p_0)\theta + p_0$. The result follows immediately as $\{q_i\}$ are iid. ∎

Remark 4: The rationale of the introduction of the class of a.s Lyapunov UGES protocols is to provide an analysis framework for Lyapunov UGES protocols capable of handling random packet dropouts – any Lyapunov UGES protocol is automatically an a.s Lyapunov UGES protocol for sufficiently low p_0.

VI. L_p Stability of NCS with Contention Protocols

The notion of *robustness* of various stability properties plays a fundamental role in practical design and implementation of control systems as evidenced by the extensive literature discussing e.g., input-to-state stability (ISS), H_2, H_∞ design and variants of robust stability. To that end, [2] and [8] have examined L_p and input-to-state stability of NCS, respectively and it was shown in [9] that persistently exciting scheduling protocols lead to L_p stable NCS when appropriate conditions are imposed on transmission rates and the nominal system.

Intuitively, and despite the presence of collisions, random packet dropouts and random inter-arrival times, it seems natural to expect that the stability of the NCS (4)-(8) for high enough "average" transmission rates and in light of the a.s. cover times of contention protocols and in analogy with persistently exciting scheduling protocols, this stability ought to be robust in an L_p sense. In fact, if we relax our notion of "L_p stability" to "L_p stability-in-expectation", we can prove a positive result in that direction.

Recall that $\|y[t_0,t]\|_p := \left(\int_{t_0}^t |y(s)|^p ds\right)^{1/p}$ for $p \in [1,\infty)$ and $\|y[t_0,t]\|_\infty = \text{ess.}\sup\{|y(s)| : s \in [t_0,t]\}$ and consider the NCS (1)-(2) initialized at (t_0, z_0) with input w and a prescribed output $y = g(t,z)$. We say that (1)-(2) is L_p stable-in-expectation from w to y with expected gain γ if

$$\exists\, K \geq 0 : \mathbf{E}\|y[t_0,t]\|_p \leq K|z_0| + \gamma\mathbf{E}\|w[t_0,t]\|_p.$$

The state z of (1)-(2) is said to be L_p to L_q detectable-in-expectation from output y with expected gain γ if

$$\begin{aligned}\exists\, K \geq 0 : \mathbf{E}\|z[t_0,t]\|_q &\leq K|z_0| \\ &+ \gamma\mathbf{E}\|y[t_0,t]\|_p + \gamma\mathbf{E}\|w[t_0,t]\|_p.\end{aligned}$$

Note that these are essentially the same notions of stability and detectability employed in [2] and [9]. We stress that, as developed in this paper, these notions only apply to hybrid systems of the form (1)-(2), i.e., we insist that w is "essentially" an L_p signal and not a Lèvy process (c.f. [13]) specifically because we are concerned with robustness of stability in the sense of e.g., [17], whereas a Lèvy process characterization of disturbances may be more appropriate in modeling sensor noise and quantization phenomena.

While the following results are stated for the delay and inter-arrival processes presented in Definition 4.1, it is straightforward to extend them to a more general class of processes.

Theorem 6.1: Consider a V-link NCS (9)-(11) and suppose that:

1) the NCS employs a contention scheduling protocol with iid cover times T_i and the inter-arrival process is Poisson with intensity λ and also suppose that the NCS error dynamics satisfy[4]
$$\overline{g}(t,x,e,w) \preceq A\overline{e} + \tilde{y}(x,w) \tag{24}$$
for all $(x,e,w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_w}$ and almost all t, where A is a nonnegative symmetric $n_e \times n_e$ matrix with nonnegative entries and $\tilde{y} = G(x) + H(w)$;
2) system (9) is L_p stable-in-expectation from (e,w) to $G(x)$ with expected gain γ for some $p \in [1,\infty]$; (10) is L_p to L_p detectable-in-expectation from $\tilde{y}$;

Then, there exists $\lambda < \infty$ depending on $(V, |A|, \gamma, \mathbf{E}[T], p_0)$ such that the NCS is L_p stable-in-expectation from w to (x,e)

[4]Let $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{R}^n$. The *vector* partial order $\preceq$ is given by $x \preceq y \iff (x_1 \leq y_1) \wedge \cdots \wedge (x_n \leq y_n)$ and $\overline{e}$ and $\overline{g}$ are given by $\overline{e} := (|e_1|,\dots,|e_{n_e}|)^T$ and $t \overset{\overline{g}}{\mapsto} \overline{g(t)}$, respectively. That is, $\overline{e}$ is the vector that results from taking the absolute value of each scalar component of e and $\overline{g}$ does operates analogously on the image of g.

with a finite linear expected gain $1/(1-\gamma\gamma^*)$. Specifically, λ solves $\gamma^*\gamma < 1$ with

$$\gamma^* = \frac{\mathbf{E}[T](1+\rho)}{(\lambda - |A|)(1-\rho)},$$

where,

$$\rho = (\alpha(1-p_0))^V \prod_{k=1}^{V} \frac{V-(k-1)}{V(1-p_0\alpha)-(k-1)(1-p_0)\alpha} - 1,$$

and $\alpha = \left(\frac{\lambda}{\lambda - |A|}\right)$ and $\lambda > \frac{|A|}{1-p_0}$.

Proof of the results follows from a straightforward extension of classical small-gain theorems, Theorem 3.1, and subsequent results that are developed in the paper in Section IX. The usual detectability assumptions are automatically satisfied when $\tilde{y}$ is defined as above. While no bounds for λ are given, the requisite intensity can be found numerically.

VII. L_p Stability of NCS with Deterministic Protocols in the Presence of Dropouts

In this section, we present the second main result of this paper which shows that under mild conditions a.s Lyapunov protocols induce L_p stability in expectation of NCS for sufficiently high transmission rates. The result is intended to be a stochastic analogue of [2][Theorem 4] where the dependence of the gain and intensity formulae on the dropout probability made explicit. While [2] present sufficient conditions for L_p stability in the presence of (deterministically-characterized) packet dropouts, we believe the following result is a more natural treatment of dropouts and the conditions are directly verifiable.

Theorem 7.1: Consider a V-link NCS (9)-(11) operating on a channel with dropout probability p_0 employs a contentionless scheduling protocol that is a.s. Lyapunov UGES with Lyapunov function W that is locally Lipschitz in e, uniformly in i where (18) is satisfied with an iid sequence $\{\kappa_i\}$ and there exists $L \geq 0$ such that for every $i \in \mathbb{N}$, all t, x, w and almost all e we have that the following holds:

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq LW(i,e) + |\tilde{y}|, \qquad (25)$$

where $\tilde{y} : \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \to \mathbb{R}$ is a continuous function of (x, w); and the intensity of the inter-transmission process λ satisfies

$$\lambda > \frac{\gamma + L}{1 - \mathbf{E}[\kappa]}. \qquad (26)$$

Further suppose that system (9) is L_p stable from (W, w) to $\tilde{y}$ with finite expected gain γ for some $p \in [1, \infty]$; (x, w) is L_p detectable from $\tilde{y}$ with finite expected gain and e is L_p detectable from W with finite expected gain. Then the NCS (9)-(11) is L_p stable from w to (x, e) with finite expected linear gain:

$$\frac{\lambda(1-E[\kappa]) - L}{\lambda(1-E[\kappa]) - L - \gamma}. \qquad (27)$$

Theorem 7.2: We only sketch a proof as the details are similar to the proof of [2][Theorem 4]. In view of 9.5, and condition (26) the error subsystem (10)-(11) is L_p stable with finite expected gain from $\tilde{y}$ to W. In particular, the intensity lower bound (26) yields an expected gain of

$$\frac{1}{\gamma(\lambda(1-E[\kappa]) - L)}.$$

The result follows from the adapted small-gain theorem presented in the Appendix under the detectability assumptions and finite expected gain of the x-subsystem (9).

Remark 5: As the motivation for studying a.s Lyapunov UGES comes from the use of Lyapunov UGES protocols on non-ideal channels, we can restate several of the conditions of Theorem 7.1 in light of Proposition 5.3. Let θ be as in (22) and let the probability of packet dropout p_0 satisfy (23). The requisite intensity in (26) becomes

$$\lambda > \frac{\gamma + L}{(1-p_0)(1-\theta)} \qquad (28)$$

and the resultant gain (27) can be re-expressed in a similar manner.

Remark 6: As in [2] and [9], in both this and the preceding section, several generalizations and specializations of the stability results are possible. With additional technical assumptions on the NCS dynamics, one can conclude uniform global exponential stability (in expectation) and the assumptions on the various reset maps can be relaxed so as to infer ISS-like properties in lieu of L_p stability as discussed [8]. If we forgo the detectability assumptions in the hypotheses of Theorem 6.1 and Theorem 7.1 we can only infer input-to-output stability in expectation.

VIII. Case Study: Batch Reactor

As given in [2], the linearized model of an unstable batch reactor is a two-input-two-output NCS that can be written as:

$$\dot{x}_P = A_P x_P + B_P u \quad y = C_P x_P$$

where $C_P = \left[\begin{smallmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right]$

$$A_P = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} B_P = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}.$$

The system is controlled by a PI controller with a state-space realization prescribed by

$$\dot{x}_C = A_C x_C + B_C y \quad u = C_C x_C + D_C y$$

and

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$-C_C = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \qquad -D_C = \begin{bmatrix} 0 & 2 \\ -5 & 0 \end{bmatrix}.$$

Assuming that only the outputs are transmitted via the network, we have a two link NCS ($V = 2, v_1 = v_2 = 1$) with error and state equations

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad (29)$$

where

$$A_{11} = \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix} \quad A_{12} = \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix}$$
$$A_{21} = -\begin{bmatrix} C_P & 0 \end{bmatrix} A_{11} \quad A_{22} = -\begin{bmatrix} C_P & 0 \end{bmatrix} A_{12}.$$

The error equation is given by

$$\dot{e} = A_{22}e + A_{21}x \tag{30}$$

and we have

$$\bar{\dot{e}} \preceq A\bar{e} + \tilde{y}, \tag{31}$$

where $\tilde{y} = \overline{A_{21}x}$ and $A = A_{22}$, as A_{22} is diagonal and has all nonnegative entries. We assume the NCS uses the CSMA protocol described in Section IV-B and, hence,

$$E[T] = 2 \cdot H_2/(1-p_0) = 3/(1-p_0). \tag{32}$$

By the small-gain theorem described in Proposition II, and Theorem 9.4, the batch reactor system will be L_p stable in expectation from w to x if

$$\frac{\mathbf{E}[T](1+\rho)}{(\lambda - |A|)(1-\rho)}\gamma < 1, \tag{33}$$

where γ is the L_p gain of x subsystem from the input e to an "auxiliary" output $\tilde{y} = \overline{A_{21}x}$.

We compute the L_2 gain for the x subsystem from the input e to an auxiliary output $A_{21}x$ which is $\gamma \approx 15.9222$ however we note that the "gain" from $A_{21}x$ to $\tilde{y}$ is unity, hence, γ is also the gain from input e to output $\tilde{y}$ and we note that $|A| = 15.73$. By solving for λ numerically in (33), subject to the constraint

$$\lambda > \frac{|A|}{1-p_0}$$

from Lemma 9.2, we are able to establish expected transmission rate bounds as a function of p_0 that ensure L_p stability of the batch reactor system The batch reactor system with the CSMA protocol was also simulated using expected transmission rates of $[1, \infty)$ transmissions per second for $p_0 \in [0.1, 0.8]$. The following simulation method was used:

1) For each fixed p_0 and two transmission intensities $\lambda_{u,0} = 10^9, \lambda_{l,0} = 1$, the NCS was simulated with a (pseudo)-random realization of the inter-transmission and protocol processes with a fixed initial state. The simulation was terminated and the NCS deemed unstable if the norm of the NCS exceeded a time-dependent threshold of the form $K_1 + K_2 \exp(-K_3 t)$, otherwise it was deemed to be stable. With the above choices, it is expected that NCS with intensities $\lambda_{u,0}$, $\lambda_{l,0}$ would be stable and unstable, respectively.
2) By bisection on the values of $\lambda_{u,i}$, $\lambda_{l,i}$ depending on the outcome of subsequent simulations, the smallest intensity λ^* resulting in stability can be determined for the given realizations of the (pseudo)-random processes involved. Specifically, if $\lambda_{l,i}$ resulted in instability, $\lambda_{l,i+1} \leftarrow (\lambda_{l,i} + \lambda_{u,i})/2$, $\lambda_{u,i+1} \leftarrow \lambda_{u,i}$, or in the case the NCS was stable, $\lambda_{u,i+1} \leftarrow \lambda_{l,i}$, $\lambda_{l,i+1} \leftarrow (\lambda_{l,i} + \lambda_{l,i-1})/2$. This process was terminated when $\lambda_{u,i} - \lambda_{l,i} < \epsilon^5$ and we set $\lambda^* \leftarrow \lambda_{u,i}$.

With the same p_0 and identical initial conditions, the above procedure was repeated 1000 times and the ensemble average of λ^* to yield the simulation-derived intensity bound.

The expected transmission rate bounds and expected inter-transmission times are shown in Table I as a function of dropout/collision probability p_0 and plotted in Figure 1. Simulation-derived bounds are also listed in Table I.

For the initial condition used, the bounds obtained via Theorem 6.1 are within a factor of 4 of simulation-based bounds and, for example, demonstrate that with a 50% probability of dropout/collision, the network must deliver approximately 922 kbps (116×8 bits) of network throughput to maintain L_p stability. This is well within the realm of ordinary Ethernet and 802.11 wireless technology.

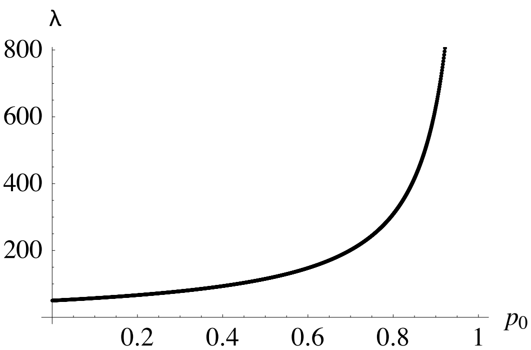

Fig. 1. Batch Reactor expected transmission rate bounds for contention protocols as a function of dropout/collision probability p_0 with identical initial conditions.

p_0	λ	$E[\tau] = 1/\lambda$ (s)	λ^*	$E[\tau^*] = 1/\lambda^*$ (s)
0	50.19	0.02	14.77	0.0677
0.1	57.46	0.017	16.05	0.0623
0.2	66.52	0.015	18.38	0.0544
0.3	78.15	0.013	21.37	0.0468
0.4	93.63	0.011	25.00	0.0400
0.5	115.27	0.0087	31.65	0.0316
0.6	147.71	0.0068	37.74	0.0265
0.7	201.74	0.0049	61.35	0.0163
0.8	309.74	0.0032	145.77	0.00686

TABLE I

TRANSMISSION RATE AND INTER-TRANSMISSION TIME BOUNDS λ AND $E[\tau] = 1/\lambda$ ARE DERIVED VIA THEOREM 6.1; λ AND $E[\tau] = 1/\lambda$ ARE DERIVED VIA SIMULATION.

We can also consider the example within the context of contentionless in protocols. Suppose that the TOD scheduling is employed. From [2] we select $W(i,e) = |e|$ and with respect to Remark 5 we have $\theta = \sqrt{1/2}, L = 15.73$ and $\gamma = 15.9222$ and, hence, the requisite intensity for the conditions of Theorem 7.1 to be verified is

$$\lambda > \frac{108.07}{1-p_0}.$$

[5]The tolerance ϵ was chosen such that intensities were equal within five significant figures.

For an ideal channel (p_0), this corresponds to a transmission at least once every 9.25 msecs compared to a maximum allowable transmission interval (MATI) of 0.01 secs for the deterministic results presented in [2]– a factor of 1.08 improvement in favor of the deterministic results. The notion of MATI implies that every inter-transmission time is *uniformly bounded* whereas the intensity (or reciprocal) is an "average MATI" – individual inter-transmission times can individually exceed or fall short of the average. Notably, both values fall short of the contention protocol figure of 0.02 secs. As the characterization of dropouts in [2] is markedly different from that of this paper, we do not pursue a comparison for $p_0 > 0$. We can, however, compare contention protocols and TOD in the presence of dropouts as presented in this paper and we see that the trend is continued for $p_0 > 0$ e.g., the requisite intensity for $p_0 = 0.5$ is over 216 for TOD and less than 116 for the contention protocol. We cannot immediately conclude that TOD is inferior to essentially to a protocol that transmits links at random when the channel is idle. The disparity in intensity bounds may simply be an artifact of the different stability properties used to characterize each protocol but a similar relative disparity between TOD and the simpler round-robin scheduling protocol is evident in the results presented in [13] and seem to provide some support that PE-like properties lead to sharper results.

IX. Proof of Main Results

The following results imply the stability result presented in Section VI but are of interest in their own right and constitute the substantial technical differences between this paper and [9] despite the superficially similar proof technique.

Lemma 9.1: Let T be the cover time for the sequence $\{(Q_0, \tau_0), \ldots, (Q_{T-1}, \tau_{T-1})\}$. Then the following inequality holds:

$$\left|\prod_{i=0}^{T-1} Q_i \exp(A\tau_i)\right| \leq \exp(|A| \sum_{j=0}^{T-1} \tau_i) - 1.$$

Proof: The proof is a straightforward generalization of [9, Lemma 7.1]. ■

Remark 7: Assuming a V-link NCS, and let p_0 denote the probability of a dropout or collision. Let W_i denote the number of additional transmissions needed to go from having covered $i-1$ links to i links. Then W_i is geometrically distributed with parameter $p_{g,i}$ given by

$$p_{g,i} = \frac{(V-i+1)(1-p_0)}{V}.$$

It is clear that the cover time T can be expressed as $T = \sum_{i=1}^{V} W_i$ and the pgf is given by:

$$\psi_T(s) = (s(1-p_0))^V \prod_{k=1}^{V} \frac{V-(k-1)}{V(1-p_0 s)+(k-1)(1-p_0)s},$$

for $|s| < 1/p_0$.

Lemma 9.2: Suppose that $\tau_i \sim \text{Exp}(\lambda)$. Let T be the cover time for the sequence $\{(Q_0, \tau_0), \ldots, (Q_{T-1}, \tau_{T-1})\}$ and let the random variable Z be given by

$$Z = \exp(|A| \sum_{j=0}^{T-1} \tau_i).$$

Then $\mathbf{E}[Z]$ is given by

$$\mathbf{E}[Z] = (\alpha(1-p_0))^V \prod_{k=1}^{V} \frac{V-(k-1)}{V(1-p_0\alpha)-(k-1)(1-p_0)\alpha},$$

where $\alpha = \left(\frac{\lambda}{\lambda - |A|}\right)$ whenever

$$\lambda > \frac{|A|}{1-p_0}.$$

Proof: Let $W = \sum_{j=0}^{T-1} \tau_i$. The mgf of W is given by

$$\mathbf{E}[\exp(sW)] = \psi_T\left(\frac{\lambda}{\lambda - s}\right),$$

that is, the mgf of W is the pgf of T evaluated at the mgf of an Exp(λ)-distributed random variable.[6] The result follows by setting $s = |A|$. ■

Lemma 9.3: Suppose that $\tau_i \sim \text{Exp}(\lambda)$. Let T be the cover time for the sequence $\{(Q_0, \tau_0), \ldots, (Q_{T-1}, \tau_{T-1})\}$. Then there exists $\lambda < \infty$, depending on $(V, |A|, p_0)$ such that

$$\mathbf{E}\left|\prod_{i=0}^{T-1} Q_i \exp(A\tau_i)\right| < 1.$$

Proof: Letting $\alpha = \left(\frac{\lambda}{\lambda - |A|}\right)$, from Lemma 9.2 we have,

$$\mathbf{E}\left|\prod_{i=0}^{T-1} Q_i \exp(A\tau_i)\right|$$
$$< (\alpha(1-p_0))^V \prod_{k=1}^{V} \frac{V-(k-1)}{V(1-p_0\alpha)-(k-1)(1-p_0)\alpha} - 1.$$

Define $h(V, \lambda, p_0) = \mathbf{E}\left|\prod_{i=0}^{T-1} Q_i \exp(A\tau_i)\right|$. Letting $\alpha \to 1$, and hence, $\lambda \to \infty$, in the above bound yields $h \to 0$. By the implicit function theorem (see e.g., [19, Theorem 2-12]) and since h is a.e. C^1, there exists $\lambda > 0$ such that $0 < h(V, \lambda, p_0) < 1$. It is straightforward to solve for λ numerically. ■

The following theorem asserts L_p stability-in-expectation for the e-subsystem and is they key component of the small-gain-based proof approach that implies Theorem 6.1.

Theorem 9.4: Suppose that a V-link NCS employs a contention scheduling protocol and satisfies hypothesis 1 of Theorem 6.1 with the Poisson intensity λ chosen as in Lemma 9.3. That is, we have $\mathbf{E}\left|\prod_{i=0}^{T-1} Q_i \exp(A\tau_i)\right| < 1$. Then for all $t \geq 0$ we have, for any $p \in [0, \infty]$,

$$\exists : \ K \in [0, \infty) \ : \mathbf{E}\|e[0,t]\|_p \leq K|\bar{e}(0)| + \gamma \mathbf{E}\|\tilde{y}[0,t]\|_p,$$

where,

$$\gamma = \frac{\mathbf{E}[T](1+\rho)}{(\lambda - |A|)(1-\rho)}$$

[6] See [18, Example 1.8.13], for instance.

with $\rho < 1$ a function of $(V, |A|, \lambda, p_0)$. Specifically, $\rho = E[Z] - 1$ where $\mathbf{E}[Z]$ was calculated in Lemma 9.2.

Proof: We write $\tilde{y}(s)$ in place of $\tilde{y}(\overline{x}(s), \overline{w}(s))$. By hypothesis, we have

$$\overline{g}(t, x, e, w) = \dot{\overline{e}} \preceq A\overline{e} + \tilde{y}(t), \tag{34}$$

As in in [9, Section VII-A], we have for all $i \in \mathbb{N}$

$$\begin{aligned}\overline{e}(t_i^+) &\preceq Q_i \exp(A(t_i - t_{i-1}))\overline{e}(t_{i-1}^+) \\ &\quad + Q_i \int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds. \end{aligned}\tag{35}$$

For all $i \in \mathbb{N}$, we can upperbound (35) with

$$\begin{aligned}&\overline{e}(t_i^+) \preceq Q_i \exp(A\tau_i) \times \\ &\left(\overline{e}(t_{i-1}^+) + \exp(-A\tau_i) \int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds\right). \end{aligned}\tag{36}$$

For brevity, define $R_i = Q_i \exp(A\tau_i)$. We can immediately solve the linear recurrence (36) to produce the bound:

$$\begin{aligned}&\overline{e}(t_k^+) \preceq \left(\prod_{i=0}^{k} R_i\right) \overline{e}(0) \\ &+ \exp(-A\tau_k) \sum_{i=0}^{k} \left(\prod_{n=i}^{k} R_n\right) \int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds \end{aligned}\tag{37}$$

for all $k \in \mathbb{N}$.

By hypothesis, we have fixed the intensity of the transmission process such that

$$\mathbf{E}\left|\prod_{i=0}^{T-1} Q_i \exp(A\tau_i)\right| \leq E[Z] - 1 < 1,$$

as in Lemma 9.2 and Lemma 9.1. Let $\rho = E[Z] - 1$. with $\mathbf{E}\left|\prod_{i=0}^{T-1} Q_i \exp(A\tau_i)\right| \leq \rho < 1$. Partition the sequence $\{(Q_0, \tau_0), (Q_1, \tau_1), \ldots\}$ such that each subsequence

$$\begin{aligned}&\{(Q_0, \tau_0), \ldots, (Q_{T_0-1}, \tau_{T_0-1})\}, \\ &\qquad \{(Q_{T_0}, \tau_{T_0}), \ldots, (Q_{T_0+T_1-1}, \tau_{T_0+T_1-1})\}, \ldots\end{aligned}$$

is covering and, hence, T_j are cover times for the respective subsequences. To simplify notation, we use $\tau_{j,i}$ to denote the ith inter-transmission time in the jth covering sequence i.e., $\tau_{j,i} = \tau_{i+G}$, where $G = \sum_{k=0}^{j-1} T_k$, and let ρ_j be given by

$$\rho_j = \exp(|A| \sum_{i=0}^{T_j-1} \tau_{j,i}) - 1 \geq \left|\prod_{i=0}^{T_j-1} R_i\right|.$$

Similarly, let $Q_{j,i}$ denote the ith jump map in the jth covering sequence and set $R_{j,i} = Q_{j,i} \exp(A\tau_{j,i})$. Recall that T_j is stationary. Define the renewal process $N_T(t)$ by

$$N_T(t) = \inf\{M \geq 0 : t \geq \sum_{j=0}^{M-1} \sum_{i=0}^{T_j-1} \tau_{j,i}\}.$$

Let S_M be given by $S_M = \inf\{t \geq 0 : N_T(t) \geq M\}$, that is, S_M is the time it takes to cover V links M times.

We set the disturbance term $\tilde{y} \equiv 0$ and have that

$$\begin{aligned}|\overline{e}(S_M^+)| &\leq \left|\prod_{t_i \leq S_M} R_i\right| |\overline{e}(0)| \\ &\leq \left(\prod_{j=0}^{M-1} \rho_j\right) |\overline{e}(0)| \quad \left(\forall M \in \mathbb{N}^+\right). \end{aligned}\tag{38}$$

With $\tilde{y} = 0$, $D\overline{e} \preceq A\overline{e}$ and for the initial condition $\overline{e}(s_0) = \overline{e}_0$, we have for any $s \geq 0$

$$\overline{e}(s) \preceq \exp(A(s - s_0))\overline{e}_0. \tag{39}$$

Taking the norm of the left and right hand sides of (39) and using the bound in (38) as the initial condition, we have that for all $M \in \mathbb{N}^+$, $\theta \in (S_M, S_{M+1})$, the following bound on $|\overline{e}|$ holds:

$$|\overline{e}(\theta)| \leq \left(\prod_{j=0}^{M-1} \rho_j\right) \exp(|A|(\theta - S_M))|\overline{e}(0)|. \tag{40}$$

Taking the supremum over the interval $[S_M, S_{M+1}]$, we obtain

$$\|\overline{e}[S_M, S_{M+1}]\|_\infty \leq \left(\prod_{j=0}^{M-1} \rho_j\right) \left(\exp(|A| \sum_{i=0}^{T_M-1} \tau_{M,i})\right) |\overline{e}(0)| \tag{41}$$

for all for all $M \in \mathbb{N}^+$. Similarly, we can integrate (40) over the same interval to obtain

$$\begin{aligned}\|\overline{e}[S_M, S_{M+1}]\|_1 &\leq \left(\prod_{j=0}^{M-1} \rho_j\right) \times \\ &\left(\exp(|A| \sum_{i=0}^{T_M-1} \tau_{M,i}) - 1\right) \frac{|\overline{e}(0)|}{|A|}. \end{aligned}\tag{42}$$

We can upperbound both the L_∞ and L_1 bounds (41) and (42) by

$$\begin{aligned}\|\overline{e}[S_M, S_{M+1}]\|_p &\leq \left(\prod_{j=0}^{M-1} \rho_j\right) \times \\ &\left(\exp(|A| \sum_{i=0}^{T_M-1} \tau_{M,i})\right) \frac{|\overline{e}(0)|}{|A| \wedge 1}, \end{aligned}\tag{43}$$

for $p \in \{1, \infty\}$. By our choice of intensity λ and in light of Lemma 9.3 and 9.1, $\mathbf{E}|\exp(|A| \sum_{i=0}^{T_M-1} \tau_{M,i})| < 1 + \rho$ and since ρ_j are iid, $\mathbf{E}[\prod_{j=1}^{M} \rho_j] = \prod_{j=1}^{M} E[\rho_j] = \rho^M$, hence,

$$\mathbf{E}\|\overline{e}[S_M, S_{M+1}]\|_p \leq \rho^M (1 + \rho) \frac{|\overline{e}(0)|}{|A| \wedge 1}. \tag{44}$$

It is also clear that $\mathbf{E}\|\overline{e}[0, S_1]\|_p \leq \frac{1+\rho}{|A| \wedge 1} |\overline{e}(0)|$. Set $S_0 = 0$ and we have by linearity of $\mathbf{E}[\cdot]$,

$$\mathbf{E}\|\overline{e}[0, t]\|_p \leq \sum_{j=0}^{\infty} \rho^j (1 + \rho)|\overline{e}(0)| = \left(\frac{1 + \rho}{1 - \rho}\right) \frac{|\overline{e}(0)|}{|A| \wedge 1} \leq \infty.$$

We now set $\overline{e}(0) = 0$ in (37) and estimate the contribution from the disturbance term to yield:

$$\overline{e}(t_k^+) \preceq \exp(-A\tau_k)\times \sum_{i=0}^{k}\left(\prod_{n=i}^{k} R_n\right)\int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds. \quad (45)$$

Applying the variations of parameters formula to (45), we have

$$\overline{e}(\theta) \preceq \exp(-A\tau_k)\exp(A(\theta - t_k))\times \sum_{i=0}^{k}\left(\prod_{n=i}^{k} R_n\right)\int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds + \int_{t_k}^{\theta} \exp(A(\theta - s))\tilde{y}(s)ds \quad (46)$$

for $\theta \in [t_k, t_{k+1}]$. Consider the term $\prod_{n=i}^{k} R_n = \prod_{n=i}^{k} Q_n \exp(A\tau_n)$ and the associated sequence

$$\{(Q_n, \tau_n), (Q_{n+1}, \tau_{n+1}), \dots\}.$$

Let $W_s = \sum_{j=0}^{s} T_j$ and let $\hat{n}$ be given by

$$\inf\{\hat{n} \geq 0 : W_{\hat{n}} \geq k + i - 1\}.$$

By Proposition 2.3, $\mathbf{E}[W_{\hat{n}}] = \mathbf{E}[\hat{n}] \cdot \mathbf{E}[T]$. Let n^* be given by

$$n^* = \left\lfloor \frac{k+1-i}{\mathbf{E}[T]} \right\rfloor$$

and note that $\mathbf{E}[W_{n^*}] = n^* E[T] \leq k + 1 - i$ and, hence $n^* \leq \mathbf{E}[\hat{n}]$. We now split the product in consideration into $\hat{n}$ products, each of which is associated with a covering sequence and a residual product term

$$\prod_{n=i}^{k} R_n = \left(\prod_{n=0}^{T_0 - 1} R_{0,n}\right)\cdots\left(\prod_{n=0}^{T_{\hat{n}-1}-1} R_{\hat{n}-1,n}\right)\cdot\left(\prod_{n=0}^{r} R_{\hat{n},n}\right)$$

for some random remainder variable $r < T_{\hat{n}}$.

By independence of each product and in view of Lemma 9.3 and the fact that $\hat{n}$ is a stopping time for W_s, we can take expectations as follows:

$$\begin{aligned}\mathbf{E}\left|\prod_{n=i}^{k} R_n\right| &= \mathbf{E}\left|\prod_{n=0}^{T_0-1} R_{0,n}\right| \times \\ &\qquad \mathbf{E}\left|\prod_{n=0}^{T_{\hat{n}-1}-1} R_{\hat{n}-1,n}\right| \cdot \mathbf{E}\left|\prod_{n=0}^{r} R_{\hat{n},n}\right| \\ &= \rho^{\mathbf{E}[\hat{n}]} \cdot \mathbf{E}\left|\prod_{n=0}^{r} R_{\hat{n},n}\right|. \end{aligned} \quad (47)$$

As $\rho < 1$, $r < T_{\hat{n}}$ and $n^* \leq \mathbf{E}[\hat{n}]$, we have the bound

$$\mathbf{E}\left|\prod_{n=i}^{k} R_n\right| \leq \rho^{n^*}(1 + \rho). \quad (48)$$

With this observation, and taking expectation of the supremum of the bound in (46), we have the following:

$$\begin{aligned}\mathbf{E}\|\overline{e}[t_k, t_{k+1}]\|_\infty &\leq \mathbf{E}[\exp(|A|\tau_k)\exp(-|A|\tau_k)]\times \\ &\quad (1+\rho)\sum_{i=0}^{k} \rho^{\left\lfloor \frac{k+1-i}{\mathbf{E}[T]}\right\rfloor} \mathbf{E}\|\varphi[0,\tau_i]\|_1 \mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_\infty \\ &\quad + \mathbf{E}\|\varphi[0,\tau_k]\|_1 \mathbf{E}\|\tilde{y}[t_k, t_{k+1}]\|_\infty \\ &\leq \mathbf{E}\|\varphi[0,\tau]\|_1 (1+\rho)\sum_{i=0}^{k+1} \rho^{\left\lfloor \frac{k+1-i}{\mathbf{E}[T]}\right\rfloor} \mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_\infty, \end{aligned} \quad (49)$$

where $\varphi(s) = \exp(|A|s)$ and we have used independence to split the expectation of products into products of expectation. By upperbounding the term $\exp(|A|(\theta - t_k))$ with $\exp(|A|\tau_k)$ prior to integrating, the L_1 bound can be established in essentially the same way,

$$\begin{aligned}\mathbf{E}\|\overline{e}[t_k, t_{k+1}]\|_1 &\leq \mathbf{E}[\exp(|A|\tau_k)\exp(-|A|\tau_k)]\times \\ &\quad (1+\rho)\sum_{i=0}^{k} \rho^{\left\lfloor \frac{k+1-i}{\mathbf{E}[T]}\right\rfloor} \mathbf{E}\|\varphi[0,\tau_i]\|_1 \mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_1 \\ &\quad + \mathbf{E}\|\varphi[0,\tau_k]\|_1 \mathbf{E}\|\tilde{y}[t_k, t_{k+1}]\|_1 \\ &\leq \mathbf{E}\|\varphi[0,\tau]\|_1 (1+\rho)\sum_{i=0}^{k+1} \rho^{\left\lfloor \frac{k+1-i}{\mathbf{E}[T]}\right\rfloor} \mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_1, \end{aligned} \quad (50)$$

where we have used Hölder's inequality, as in [20, Example 5.2], to split the integrals.

There is an exact expression for $\mathbf{E}\|\varphi[0,\tau]\|_1$ in terms of the mgf of the $\mathrm{Exp}(\lambda)$ random variable τ:

$$\begin{aligned}\mathbf{E}\|\varphi[0,\tau]\|_1 &= \mathbf{E}[\exp(|A|\tau)]/|A| - 1/|A| \quad (51) \\ &= \frac{1}{\lambda - |A|} \quad (52)\end{aligned}$$

and, hence for $p \in \{1, \infty\}$,

$$\mathbf{E}\|\overline{e}[t_k, t_{k+1}]\|_p \leq \frac{1+\rho}{\lambda - |A|}\sum_{i=0}^{k+1} \rho^{\left\lfloor \frac{k+1-i}{\mathbf{E}[T]}\right\rfloor} \mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_p. \quad (53)$$

By linearity of $\mathbf{E}[\cdot]$, we sum (53) to obtain an upperbound on $\mathbf{E}\|\overline{e}[0, t_M]\|_p$:

$$\mathbf{E}\|\overline{e}[0, t_M]\|_p \leq \frac{1+\rho}{\lambda - |A|}\sum_{k=-1}^{M-1}\sum_{i=0}^{k+1} \rho^{\left\lfloor \frac{k+1-i}{\mathbf{E}[T]}\right\rfloor} \mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_p. \quad (54)$$

Applying [9, Appendix, Lemma 1.1] to (54), and taking the limit as $M \to \infty$ in the summation, the L_∞ and L_1 norms can be estimated by

$$\begin{aligned}\mathbf{E}\|\overline{e}[0, t_M]\|_p &\leq \frac{1+\rho}{\lambda - |A|}\mathbf{E}\|\tilde{y}[t_s, t_M]\|_p \sum_{k=0}^{\infty} \rho^{\left\lfloor \frac{k}{\mathbf{E}[T]}\right\rfloor} \quad (55) \\ &= \frac{\mathbf{E}[T](1+\rho)}{(\lambda - |A|)(1-\rho)}\mathbf{E}\|\tilde{y}[t_s, t_M]\|_p \quad (56)\end{aligned}$$

Either $\mathbf{E}\|\tilde{y}[t_s, t_M]\|_p = 0$ or the ratio $\mathbf{E}\|\overline{e}[0, t_M]\|_p / \mathbf{E}\|\tilde{y}[t_s, t_M]\|_p$ is bounded by an expression that is independent of M, hence, (56) remains true with t in lieu of t_M for any $t \geq 0$.

As the L_1 and L_∞ norms are upperbounded by the same expressions, by Theorem 1.6, the error subsystem is L_p stable-in-expectation for any $p \in [1,\infty]$. ■

Analogously, we prove a similar theorem for NCS employing contentionless protocols with dropouts.

Theorem 9.5: Suppose that a V-link NCS with dropout probability p_0 employs a contenionless scheduling protocol that is a.s. Lyapunov UGES with Lyapunov function W that is locally Lipschitz in e, uniformly in i where (18) is satisfied with an iid sequence $\{\kappa_i\}$ and there exists $L \geq 0$ such that for every $i \in \mathbb{N}$, all t, x, w and almost all e we have that the following holds:

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq LW(i,e) + |\tilde{y}|, \tag{57}$$

where $\tilde{y} : \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \to \mathbb{R}$ is a continuous function of (x,w); and the intensity of the inter-transmission process λ satisfies

$$\lambda > \frac{L}{1 - \mathbf{E}[\kappa]}. \tag{58}$$

Then error-subsystem (10)-(11) is L_p stable from $\tilde{y}$ to W with finite expected linear gain:

$$\frac{1}{\lambda(1 - E[\kappa]) - L}. \tag{59}$$

Proof: We write $\tilde{y}(s)$ in place of $\tilde{y}(x(s), w(s))$. Inequality (57) implies[7] that

$$\frac{d}{dt} W(i, e(t)) \leq LW(i, e(t)) + |\tilde{y}| \tag{60}$$

As in in [2, Section X], we have for all $i \in \mathbb{N}$

$$\begin{aligned} W(i+1, e(t_i^+)) \leq \kappa_i \exp(L(t_i - t_{i-1}))W(i, e(t_{i-1}^+)) \\ + \kappa_i \int_{t_{i-1}}^{t_i} \exp(L(t_i - s))|\tilde{y}(s)| ds. \end{aligned} \tag{61}$$

For all $i \in \mathbb{N}^+$, we can upperbound (61) with

$$\begin{aligned} W(i+1, e(t_i^+)) \leq \kappa_i \exp(L\tau_i) \cdot \Bigg(W(i, e(t_{i-1}^+)) + \\ \exp(-L\tau_i) \int_{t_{i-1}}^{t_i} \exp(L(t_i - s))|\tilde{y}(s)| ds \Bigg). \end{aligned} \tag{62}$$

For brevity, define $R_i = \kappa_i \exp(L\tau_i)$. We can immediately solve the linear recurrence (62) to produce the bound:

$$\begin{aligned} W(k+1, e(t_k^+)) \leq \left(\prod_{i=0}^{k} R_i \right) W(0, e(0)) \\ + \exp(-L\tau_k) \sum_{i=0}^{k} \left(\prod_{n=i}^{k} R_n \right) \int_{t_{i-1}}^{t_i} \exp(L(t_i - s))\tilde{y}(s) ds \end{aligned} \tag{63}$$

for all $k \in \mathbb{N}$.

We set the disturbance term $\tilde{y} \equiv 0$ and have that for $M \in \mathbb{N}$,

$$W(M+1, e(t_M^+)) \leq \left(\prod_{j=0}^{M} R_j \right) W(0, e(0)) \tag{64}$$

[7]See [2, Section X] for details.

With the inequality (57) and the initial condition $e(s_0) = e_0$, we have for any $s \geq s_0$

$$W(\cdot, e(s)) \leq \exp(L(s - s_0))W(\cdot, e_0). \tag{65}$$

Taking the norm of the left and right hand sides of (65) and using the bound in (64) as the initial condition, we have that for all $M \in \mathbb{N}$, $\theta \in (t_M, t_{M+1})$, the following bound on $W(\cdot, e)$ holds:

$$W(M+1, e(\theta)) \leq \left(\prod_{j=0}^{M} R_j \right) \exp(L(\theta - t_M))W(0, e(0)). \tag{66}$$

Taking the supremum over the interval $[t_M, t_{M+1}]$, we obtain

$$\|W[t_M, t_{M+1}]\|_\infty \leq \left(\prod_{j=0}^{M} R_j \right) \exp(L\tau_M)W(0, e(0)) \tag{67}$$

for all for all $M \in \mathbb{N}$. Similarly, we can integrate (66) over the same interval to obtain

$$\begin{aligned} \|W[t_M, t_{M+1}]\|_1 \leq \left(\prod_{j=0}^{M} R_j \right) \times \\ (\exp(L\tau_M) - 1) \frac{W(0, e(0))}{L}. \end{aligned} \tag{68}$$

We can upperbound both the L_∞ and L_1 bounds (67) and (68) by

$$\begin{aligned} \|W[t_M, t_{M+1}]\|_p \leq \left(\prod_{j=0}^{M} R_j \right) \times \\ \exp(L\tau_M) \frac{W(0, e(0))}{L \wedge 1}, \end{aligned} \tag{69}$$

for $p \in \{1, \infty\}$. As τ_i is iid sequence and κ_i is iid sequence and they are mutually independent, R_j is an iid sequence and, hence

$$\begin{aligned} \mathbf{E}[\prod_{j=0}^{M} R_j] = \prod_{j=0}^{M} E[R_j] = (\mathbf{E}[\kappa] \cdot \mathbf{E}[\exp(L\tau)])^{M+1} \\ = \left(\frac{\lambda \cdot \mathbf{E}[\kappa]}{\lambda - L} \right)^{M+1}, \end{aligned} \tag{70}$$

where, as in Lemma 9.2, we have used the fact $\mathbf{E}[\exp(L\tau)]$ is given by evaluation the mgf of τ evaluated at L. Hence,

$$\mathbf{E}\|W[t_M, t_{M+1}]\|_p \leq \left(\frac{\lambda \cdot \mathbf{E}[\kappa]}{\lambda - L} \right)^{M+1} \frac{W(0, e(0))}{|A| \wedge 1}. \tag{71}$$

It is also clear that $\mathbf{E}\|W[0, t_0]\|_p \leq \frac{W(0,e(0))}{|A| \wedge 1}$. Set $t_{-1} = 0$ and incrementing the index of summation, we have by linearity of $\mathbf{E}[\cdot]$,

$$\begin{aligned} \mathbf{E}\|W[0, t]\|_p &\leq \frac{W(0, e(0))}{|A| \wedge 1} \sum_{j=0}^{\infty} \left(\frac{\lambda \cdot \mathbf{E}[\kappa]}{\lambda - L} \right)^j \\ &= \frac{W(0, e(0))(\lambda - L)}{(|A| \wedge 1)(\lambda(1 - \mathbf{E}[\kappa]) - L)}, \end{aligned}$$

where condition (58) ensures that the series summand is smaller than unity.

We now set $e(0) = 0$ in (63) and estimate the contribution from the disturbance term to yield:

$$W(k+1, e(t_k^+)) \leq \exp(-L\tau_k)\times \sum_{i=0}^{k}\left(\prod_{n=i}^{k} R_n\right)\int_{t_{i-1}}^{t_i} \exp(L(t_i - s))|\tilde{y}(s)|ds. \quad (72)$$

Applying the variations of parameters formula to (72), we have

$$W(k+1, e(\theta)) \leq \exp(-L\tau_k)\exp(L(\theta - t_k))\times \sum_{i=0}^{k}\left(\prod_{n=i}^{k} R_n\right)\int_{t_{i-1}}^{t_i} \exp(L(t_i - s))|\tilde{y}(s)|ds + \int_{t_k}^{\theta} \exp(L(\theta - s))|\tilde{y}(s)|ds \quad (73)$$

for $\theta \in [t_k, t_{k+1}]$. Tking expectation of the supremum of the bound in (73) yields the following:

$$\begin{aligned}\mathbf{E}\|W[t_k, t_{k+1}]\|_\infty &\leq \mathbf{E}[\exp(L\tau_k)\exp(-L\tau_k)]\times \sum_{i=0}^{k}\left(\frac{\lambda\cdot\mathbf{E}[\kappa]}{\lambda - L}\right)^{k+1-i}\mathbf{E}\|\varphi[0,\tau_i]\|_1\mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_\infty \\ &\quad + \mathbf{E}\|\varphi[0,\tau_k]\|_1\mathbf{E}\|\tilde{y}[t_k, t_{k+1}]\|_\infty \\ &\leq \mathbf{E}\|\varphi[0,\tau]\|_1 \sum_{i=0}^{k+1}\left(\frac{\lambda\cdot\mathbf{E}[\kappa]}{\lambda - L}\right)^{k+1-i}\mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_\infty, \end{aligned} \quad (74)$$

where $\varphi(s) = \exp(Ls)$ and we have used independence to split the expectation of products into products of expectation. By upperbounding the term $\exp(L(\theta - t_k))$ with $\exp(L\tau_k)$ prior to integrating, the L_1 bound can be established in essentially the same way,

$$\begin{aligned}\mathbf{E}\|W[t_k, t_{k+1}]\|_1 &\leq \mathbf{E}[\exp(L\tau_k)\exp(-L\tau_k)]\times \sum_{i=0}^{k}\left(\frac{\lambda\cdot\mathbf{E}[\kappa]}{\lambda - L}\right)^{k+1-i}\mathbf{E}\|\varphi[0,\tau_i]\|_1\mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_1 \\ &\quad + \mathbf{E}\|\varphi[0,\tau_k]\|_1\mathbf{E}\|\tilde{y}[t_k, t_{k+1}]\|_1 \\ &\leq \mathbf{E}\|\varphi[0,\tau]\|_1 \sum_{i=0}^{k+1}\left(\frac{\lambda\cdot\mathbf{E}[\kappa]}{\lambda - L}\right)^{k+1-i}\mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_1, \end{aligned} \quad (75)$$

where we have used Hölder's inequality, as in [20, Example 5.2], to split the integrals.

As in the proof of Theorem 9.4 we have

$$\mathbf{E}\|\varphi[0,\tau]\|_1 = \frac{1}{\lambda - L} \quad (76)$$

and, hence,

$$\mathbf{E}\|W[t_k, t_{k+1}]\|_p \leq \sum_{i=0}^{k+1}\left(\frac{\lambda\cdot\mathbf{E}[\kappa]}{\lambda - L}\right)^{k+1-i}\frac{\mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_p}{\lambda - L}. \quad (77)$$

By linearity of $\mathbf{E}[\cdot]$, we sum (77) to obtain an upperbound on $\mathbf{E}\|W[0, t_M]\|_p$:

$$\mathbf{E}\|W[0, t_M]\|_p \leq \sum_{k=-1}^{M-1}\sum_{i=0}^{k+1}\left(\frac{\lambda\cdot\mathbf{E}[\kappa]}{\lambda - L}\right)^{k+1-i}\frac{\mathbf{E}\|\tilde{y}[t_{i-1}, t_i]\|_p}{\lambda - L}. \quad (78)$$

Applying [9, Appendix, Lemma 1.1] to (54), and taking the limit as $M \to \infty$ in the summation, the L_∞ and L_1 norms can be estimated by

$$\mathbf{E}\|W[0, t_M]\|_p \leq \frac{\mathbf{E}\|\tilde{y}[t_s, t_M]\|_p}{\lambda - L}\sum_{k=0}^{\infty}\left(\frac{\lambda\cdot\mathbf{E}[\kappa]}{\lambda - L}\right)^{k} \quad (79)$$

$$= \frac{\mathbf{E}\|\tilde{y}[t_s, t_M]\|_p}{\lambda(1 - E[\kappa]) - L}, \quad (80)$$

where we have again used the fact that the series summand is smaller than unity in view of condition (58). Either $\mathbf{E}\|\tilde{y}[t_s, t_M]\|_p = 0$ or the ratio $\mathbf{E}\|\bar{e}[0, t_M]\|_p / \mathbf{E}\|\tilde{y}[t_s, t_M]\|_p$ is bounded by an expression that is independent of M, hence, (56) remains true with t in lieu of t_M for any $t \geq 0$.

As the L_1 and L_∞ norms are upperbounded by the same expressions, by Theorem 1.6, the error subsystem is L_p stable-in-expectation for any $p \in [1, \infty]$. ■

X. Conclusions and Future Works

This paper generalized the notion of persistency of excitation of scheduling protocols and developed an L_p stability result suitable for analysis of NCS employing Ethernet and Ethernet-like wireless and wireline contention protocols. We introduced the notion of protocol cover times and an abstract definition of stochastic protocols and demonstrated several consequences that led to the development of the stability result. We also presented an extension of the Lyapunov UGES protocol stability property introduced in [2] that allowed the effects of packet dropouts on NCS employing contentionless protocols to be characterized.

The analysis tools and derived bounds compare favorably with simulations and demonstrate that Ethernet-like protocols and contentionless protocols are capable of ensuring robust stability of systems even in the presence of packet dropouts and collisions.

Several important extensions of these results seem natural including: extending the results to treat arbitrary random time-varying delays; consideration of stochastic exogenous perturbations as well the treatment of a more general class of renewal processes modeling contention protocols and we believe that these extensions are important directions for future research.

Appendix I
Riesz-Thorin Interpolation System for Random Linear Operators

Definition 1.1: Fix a measurable space $(S, \mathcal{S})$, an index set T and a subset $U \subset S^T$. Then a function $X : \Omega \to U$ is $U \cap \mathcal{S}^T$-measurable iff $X_t : \Omega \to S$ is $\mathcal{S}$-measurable for every $t \in T$. The mapping X is called an S-valued (random) process on T with paths in U. In an analogous way, we say that X is a random L_p process if $\|X_t(\omega)\|_p$ is $\mathcal{S}$-measurable for every $t \in T$ and $\mathbf{E}|X_t| = m$, for all $t \in T$. We denote the space of of random L_p process defined on the index set T that are jointly S-measurable by $L_p^r(\mathcal{S}, T)$. The processes we consider will always be defined on $\mathbb{R}_{\geq 0}$ and a common σ-algebra $\mathcal{S}$. Henceforth, we write L_p^r and drop the dependence of X on ω.

Definition 1.2: We say that $T(\lambda)$ is a random linear operator on L_p^r if

$$T(\lambda)(\alpha f+\beta g)=T(\lambda)\alpha f+T(\lambda)\beta g,$$

for all $\lambda\in\Lambda$, all $\alpha,\beta\in\mathbb{R}$ and all $f,g\in L_p^r$. We henceforth drop the dependence λ with the tacit understanding that the operators we consider are random.

Lemma 1.3: Let $f:\Omega\times U\to\mathbb{C}$ (where $U\subseteq\mathbb{C}$, open) be a holomorphic non-constant random function. That is, for each fixed ω, $f(\omega;\cdot)$ is holomorphic and non-consant. Then $\mathbf{E}|f|$ attains its maximal value on any compact $K\subseteq U$ on the boundary ∂K of K.

Proof: Fix $\omega\in\Omega$, hence, $f(\omega,\cdot):U\to\mathbb{C}$ is holomorphic and therefore continuous, so $|f(\omega,\cdot)|$ will also be continuous on U. The subset $K\subset U$ is compact and since $|f(\omega,\cdot)|$ is continuous on K it must attain a maximum and a minimum value there. Suppose the maximum of $|f(\omega,\cdot)|$ is attained at z_0 in the interior of K. By definition there will exist $r>0$ such that the set $S_r=\left\{z\in\mathbb{C}:|z-z_0|^2\le r^2\right\}\subset K$.

Consider C_r the border of the previous set parametrized counter-clockwise. Since $f(\omega,\cdot)$ is holomorphic by hypothesis, the Cauchy integral formula implies that

$$f(\omega,z_0)=\frac{1}{2\pi i}\oint_C\frac{f(\omega,z)}{z-z_0}dz \tag{81}$$

a canonical parametrization of C_r is $z=z_0+re^{i\frac{\theta}{r}}$, for $\theta\in[0,2\pi r]$ and, hence,

$$f(\omega,z_0)=\frac{1}{2\pi r}\int_0^{2\pi r}f(\omega,z_0+re^{i\frac{\theta}{r}})d\theta. \tag{82}$$

Taking modulus on both sides and estimating the contour integral yields

$$|f(\omega,z_0)|\le\max_{z\in C_r}|f(\omega,z)|$$

but since $|f(\omega,z_0)|$ is a maximum, the we must have that

$$|f(\omega,z_0)|=\max_{z\in C_r}|f(\omega,z)|.$$

In particular, this holds for any $r'\le r$ and, hence, $|f(\omega,\cdot)|$ is constant in the interior of S_r. By the Identity Theorem, $f(\omega,\cdot)$ is constant throughout U. Thus if the maximum of $|f(\omega,\cdot)|$ is attained in the interior of K, then $f(\omega,\cdot)$ is constant but this is a contradiction and we must have that the maximum is attained at ∂K. Since the maximum of $|f(\omega,\cdot)|$ is attained at ∂K for each $\omega\in\Omega$, we have that the maximum of $\mathbf{E}|f|$ is attained at ∂K. ■

Lemma 1.4 (Three lines lemma): Suppose that $f:\Omega\times\mathbb{C}$ is holomorphic and non-consant in the strip $S=\{z:a\le\Re\{z\}\le b\}$ and bounded for each ω and

$$M_a=\mathbf{E}\sup|f(a+it)|\text{ and }M_b=\mathbf{E}\sup|f(a+it)|$$

then

$$\mathbf{E}|f(x+iy)|\le M_a^{\frac{b-x}{b-a}}M_b^{\frac{x-a}{b-a}}.$$

Proof: We consider

$$f_\epsilon(x+iy)=\exp(\epsilon(x+iy)^2)f(x+iy)M_a^{\frac{x+iy-b}{b-a}}M_b^{\frac{a-(x+iy)}{b-a}}$$

for $\epsilon>0$. This function satisfies

$$\mathbf{E}|f_\epsilon(a+iy)|\le\exp(\epsilon a^2)\text{ and }\mathbf{E}|f_\epsilon(b+iy)|\le\exp(\epsilon b^2)$$

and

$$\lim_{y\to\pm\infty}\mathbf{E}\sup_{a\le x\le b}|f_\epsilon(x+iy)|=0.$$

By application of Lemma 1.3 on sufficiently large rectangles, we can conclude that for each $z\in S$

$$\mathbf{E}|f_\epsilon(z)|\le\exp(\epsilon a^2)\vee\exp(\epsilon b^2).$$

Letting $\epsilon\to0^+$ competes the proof. ■

Lemma 1.5: Let p_0, p_1 and p, $p_0<p<p_1$ be given and consider the simple random function $s=\sum_k\sum_j\alpha_{k,j}a_{k,j}\chi_{E_{k,j}}\chi_{S_k}$, with $\alpha_{k,j}\in\mathbb{C}$, $|\alpha_{k,j}|=1$, $a_{k,j}>0$, for each k, $\{E_{k,j}\}$ is a pairwise disjoint collection of measurable sets, each of finite measure and $\{S_k\}\in\Omega$ pairwise disjoint with $\sum\mathbf{P}\{S_k\}=1$. Suppose that $\mathbf{E}\|s\|_p=1$. Let

$$\frac{1}{p_z}=\frac{1-z}{p_0}+\frac{z}{p_1}$$

and define

$$s_z=\sum_k\sum_j\alpha_{k,j}a_{k,j}^{p/p_z}\chi_{E_{k,j}}\chi_{S_k}.$$

This family satisfies

$$\mathbf{E}\|s_z\|_{p_{\Re\{z\}}}=1,\quad 0<\Re\{z\}<1.$$

The proof is trivial since

$$\mathbf{E}\int|s_z|^{p_{\Re\{z\}}}dz=\sum_k\sum_j a_{k,j}^p\mu(E_{k,j})\mathbf{P}\{S_k\}=\mathbf{E}\|s\|_p=1.$$

Theorem 1.6: Let p_j,q_j, $j=0,1$ be exponents in the range $[1,\infty]$ and suppose that $p_0<p_1$. If T is a random linear operator defined (at least) on and *independent of* a simple random process $X:\Omega\times t\to\mathbb{R}^n$ in L_1^r that satisfies

$$\mathbf{E}\|TX\|_{q_j}\le M_j\mathbf{E}\|X\|_{p_j}.$$

If we define p_t and q_t by

$$\frac{1}{p_t}=\frac{1-t}{p_0}+\frac{t}{p_1}$$

and

$$\frac{1}{q_t}=\frac{1-t}{q_0}+\frac{t}{q_1}$$

we will have that T extends to a random bounded-in-expectation linear operator from L_{p_t} to L_{q_t}:

$$\mathbf{E}\|TX\|_{q_t}\le M_t\mathbf{E}\|X\|_{p_t}.$$

The operator norm, M_t, satisfies $M_t\le M_0^{1-t}M_1^t$.

Proof: Fix $p=p_{t_0}$, $0<t_0<1$, fix $\omega\in\Omega$ and consider simple functions $s(\omega,\cdot),s'$ on $\mathbb{R}^n$ which satisfy $\mathbf{E}\|s\|_{p_{t_0}}=1$ and $\|s'\|_{q'_{t_0}}=1$. Let $s_z(\omega,\cdot)$ and s'_z be families of simple functions constructed as in Lemma 1.5, where $s_z(\omega,\cdot)$ is constructed using p_j, $j=0,1$ and s'_z is constructed using the exponents q'_j, $j=0,1$. By hypothesis,

$$\phi(\omega,z)=\int_{\mathbb{R}^n}s'_z(x)Ts_z(\omega,x)d\mu(x)$$

is a non-constant analytic function of z for each fixed $\omega \in \Omega$. By Lemma 1.5 and the assumption on T,

$$\mathbf{E} \sup_{y\in\mathbb{R}} |\phi(j+iy)| \le M_j, \quad j = 0, 1.$$

By Lemma 1.4, we can conclude that

$$\mathbf{E} \left| \int s' T s(\omega, \cdot) d\mu \right| \le M_0^{1-t_0} M_1^{t_0}.$$

Since, s' is an arbitrary simple function with unit norm in $L_{q'}$, we can conclude that

$$\mathbf{E} \|Ts\|_{q_{t_0}} \le M_0^{1-t_0} M_1^{t_0}.$$

As simple functions (on the product measure space) are dense in L_{p_t}, T can be extended to all of L_p^r and is bounded in norm. ■

Appendix II
Probability and Stochastic Processes

Proposition 2.1: Suppose that $\{X_i\}_{i=1}^N$ is a set of N iid random variables where N is assumed to be a random variable independent of each X. Let S_N be given by

$$S_N = \sum_{i=1}^N X_i, \tag{83}$$

that is, S_N is the random sum of iid random variables. Then the mgf of S_N is given by evaluating the pgf of N at the mgf of X:

$$\mathbf{E}[\exp(sS_N)] = \psi_N(\phi(s)), \tag{84}$$

where ψ_N and ϕ denote the pgf of N and mgf of X, respectively.

Proof: The mgf of S_n can be evaluated using conditional expectation as follows:

$$\mathbf{E}[\exp(sS_n)|N=n] = \mathbf{E}[\exp(s(X_1 + \cdots + X_n)) = \phi(s)^n,$$

hence, $\mathbf{E}[\exp(sS_N)|N] = \phi(s)^N$. Finally,

$$\begin{aligned}\mathbf{E}[\exp(sS_N)] &= \mathbf{E}[\mathbf{E}[\exp(sS_N)|N]] \\ &= \mathbf{E}[\phi(s)^N] = \mathbf{E}[z^N]_{z=\phi(s)} = \psi_N(\phi(s)).\end{aligned}$$

■

Theorem 2.2 (Optional Sampling): Let M be a martingale on some countable index set T with filtration $\mathcal{F}$, and consider two optional times σ and τ, where τ is bounded. Then M_τ is integrable, and

$$M_{\sigma\wedge\tau} = \mathbf{E}[M_\tau|\mathcal{F}_\sigma] \quad \text{a.s.}$$

See e.g., [21, Theorem 6.12].

Proposition 2.3: Suppose that $\{T_i\}_{i=1}^s$ is a set of s iid random variables. Let S_s be given by

$$S_s = \sum_{i=1}^s T_i. \tag{85}$$

Let $m = \mathbf{E}[T]$ and define M_s by $M_s = S_s - sm$. It is clear that M is a martingale since

$$\begin{aligned}\mathbf{E}[M_{n+1}|\mathcal{F}_n] &= \mathbf{E}[S_{n+1} - (n+1)m|\mathcal{F}_n] \\ &= \mathbf{E}[S_{n+1}|\mathcal{F}_n] - (n+1)m \\ &= m + \mathbf{E}[S_n|\mathcal{F}_n] - (n+1)m \\ &= S_n - nm = M_n,\end{aligned}$$

where $\mathcal{F}_n$ is the obvious filtration.

Suppose that h is defined by

$$h = \inf\{t \ge 0 : S_t \ge S'\}.$$

It is clear that h is a stopping time for M. Then

$$\mathbf{E}[h]\mathbf{E}[T] = \mathbf{E}[S_h].$$

Proof: Fix $N > 1$ and by the the optional sampling theorem, Theorem 2.2, we have

$$\begin{aligned}&\mathbf{E}[S_{h\wedge N} - (h\wedge N)m] = 0 \\ &\Rightarrow \mathbf{E}[S_{h\wedge N}] = m\mathbf{E}[(h\wedge N)] \\ &\Rightarrow S'/m \ge \mathbf{E}[(h\wedge N)] \\ &\Rightarrow S' \ge m\mathbf{E}[(h\wedge N)].\end{aligned}$$

Since $\mathbf{E}[(h\wedge N)]$ is uniformly bounded by S'/m, we let $N \to \infty$ to yield $\lim_{N\to\infty} \mathbf{E}[(h\wedge N)] = \mathbf{E}[h] = S'/m$ and, hence $\mathbf{E}[h]m = \mathbf{E}[h]\mathbf{E}[T] = \mathbf{E}[S_h]$. ■

Appendix III
Small-gain analysis

The following result is a stochastic analogue of [20, Theorem 5.6] and is proved in much the same way.

Theorem 3.1: Suppose that $H_1 : L_p^r([0,a],\mathbb{R}^n) \to L_p^r([0,a],\mathbb{R}^m)$ and $H_2 : L_p^r([0,a],\mathbb{R}^m) \to L_p^r([0,a],\mathbb{R}^n)$ are random operators that satisfy

$$\mathbf{E}\|y_1\|_p \le \gamma_1 \mathbf{E}\|e_1\|_p + \beta_1, \quad e_1 \in L_p^r([0,a],\mathbb{R}^n) \tag{86}$$

$$\mathbf{E}\|y_2\|_p \le \gamma_2 \mathbf{E}\|e_2\|_p + \beta_2, \quad e_2 \in L_p^r([0,a],\mathbb{R}^m) \tag{87}$$

for each $a \in [0,\infty)$. Suppose further that the system is well defined in the sense that for each pair of inputs $u_1 \in L_p^r([0,a],\mathbb{R}^n)$, $u_2 \in L_p^r([0,a],\mathbb{R}^m)$, there exist unique outputs $e_1, y_2 \in L_p^r([0,a],\mathbb{R}^n)$ and $e_2, y_1 \in L_p^r([0,a],\mathbb{R}^m)$. Define $u = (u_1,u_2)$, $y = (y_1,y_2)$ and $e = (e_1,e_2)$. Under the preceding assumptions, the feedback connection (when viewed as a mapping from u to y) is finite expected-gain L_p stable if $\gamma_1\gamma_2 < 1$.

Proof: Assuming existence of solutions on the interval $[0,a)$, we write

$$e_1 = u_1 - (H_2 e_2), \quad e_2 = u_2 + (H_1 e_1).$$

Then,

$$\begin{aligned}\mathbf{E}\|e_1\|_p &\le \mathbf{E}\|u_1\|_p + \mathbf{E}\|(H_2e_2)\|_p \le \mathbf{E}\|u_1\|_p + \gamma_2\mathbf{E}\|e_2\|_p + \beta_2 \\ &\le \mathbf{E}\|u_1\|_p + \gamma_2(\mathbf{E}\|u_2\|_p + \gamma_1\mathbf{E}\|e_1\|_p + \beta_1) + \beta_2 \\ &= \gamma_1\gamma_2\mathbf{E}\|e_1\|_p + (\mathbf{E}\|u_1\|_p + \gamma_2\mathbf{E}\|u_2\|_p + \beta_2 + \gamma_2\beta_1).\end{aligned}$$

Since $\gamma_1\gamma_2 < 1$,

$$\mathbf{E}\|e_1\|_p \le \frac{1}{1-\gamma_1\gamma_2}(\mathbf{E}\|u_1\|_p + \gamma_2\mathbf{E}\|u_2\|_p + \beta_2 + \gamma_2\beta_1)$$

for each $a \in [0, \infty)$. Similarly,

$$\mathbf{E}\|e_2\|_p \leq \frac{1}{1-\gamma_1\gamma_2}(\mathbf{E}\|u_2\|_p + \gamma_1\mathbf{E}\|u_1\|_p + \beta_1 + \gamma_1\beta_2)$$

for each $a \in [0, \infty)$. The proof is complete since $\mathbf{E}\|e\|_p \leq \mathbf{E}\|e_1\|_p + \mathbf{E}\|e_2\|_p$. ■

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