Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems

D. Nešić, A.R. Teel and D. Carnevale

***Abstract*— The purpose of this note is to apply recent results on stabilization of networked control systems to obtain an explicit formula for the maximum allowable sampling period (MASP) that guarantees stability of a nonlinear sampled-data system with an emulated controller. Such formulas are of great value to control practitioners.**

***Index Terms*— Sampled-data, nonlinear systems, emulation, stability.**

I. Introduction

Design of controllers for sampled-data systems is often carried out by using the emulation approach in which we first design a continuous-time controller for a continuous-time plant ignoring sampling and then we discretize the controller and implement it digitally[1]. It is obvious that this approach can be successful only if the sampling period T is sufficiently small. This approach has been investigated for linear systems (see [3] and references cited therein) and nonlinear systems (see [9] and references cited therein).

The central issue in the emulation design is the choice of the sampling period T that guarantees stability of the sampled-data system with the emulated controller. It was shown for linear systems in [3] and nonlinear systems in [7], [9] that for commonly used emulation schemes there exists a *maximum allowable sampling period*[2] *(MASP)*, denoted as $T^* > 0$, such that for any fixed $T \in (0, T^*)$ the sampled-data system is stable in an appropriate sense. Obviously, it is quite useful to have an a priori estimate of MASP as the sampling period T is a design parameter that the control engineer needs to choose before implementing the controller digitally. However, analytic computation of MASP is typically not carried out in the literature (rare exceptions are [6], [7], [16]). We note that a somewhat similar (but more general) problem is relevant in scheduling control tasks on embedded processors [14] where event-triggered sampling is considered instead of time-triggered sampling that we concentrate on.

The purpose of this note is to provide an explicit formula for MASP that guarantees asymptotic or exponential stability of sampled-data nonlinear systems with emulated controllers. We provide results for regional and global stability. Our results follow from the recent results on stabilization of networked control systems [2] (similar results were also reported in [10], [13], [15]). The main result of this paper follows directly from [2] by showing that general sampled-data systems can be modelled using the hybrid systems framework that was proposed in [10], [2] to model networked control systems. We believe that reporting this result separately in the specific context of sampled-data systems is important since such formulas are quite useful to practitioners implementing controllers using the emulation method. We have just become aware of related unpublished results in [6] that deal with a computation of MASP for global stabilization with sampled feedback. Our modelling framework, approach and proofs are different from [6]. We compare our bounds for MASP on an example taken from [6] where our results give a less conservative bound on MASP than the ones obtained in [6]. We note however that both our approach and the approach in [6] are quite flexible and one can not expect that our bounds would always be better than the ones given in [6].

The paper is organized as follows. In Sections 2 and 3 we present respectively the preliminaries and the class of models that we consider. Section 4 contains the main result and the discussion that links it with other relevant literature. Conclusions are given in the last section and the sketch of the proof of our main result is given in the appendix.

II. Notation and definitions

We denote by $\mathbb{R}$ and $\mathbb{Z}$ the sets of real and integer numbers, respectively. Also $\mathbb{R}_{\geq 0} = [0, +\infty)$, and $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. The Euclidean norm is denoted $|\cdot|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t \to \infty} \beta(s, t) = 0$ for each $s \geq 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said

D. Nešić is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia, d.nesic@ee.unimelb.edu.au. His work was partially supported by the Australian Research Council under the Australian Professorial Fellow and Discovery Grant schemes. A.R. Teel is with the Electrical and Computer Engineering Department, The University of California, Santa Barbara, CA 93106-9560, USA, teel@ece.ucsb.edu. His work supported in part by AFOSR grant F49620-03-1-0203, ARO grant DAAD19-03-1-0144 and NSF grant CCR-0311084 and ECS-0324679. D. Carnevale is with the Dipartimento di Informatica, Sistemi e Produzione The University of Rome "Tor Vergata", Italy, carnevaledaniele@gmail.com.

[1] In this paper, computational delays and related scheduling issues arising from implementation of controllers on embedded computing systems are ignored. For more details on how to deal with these and related issues see [1].

[2] A similar notion of a *maximum allowable transfer interval (MATI)* was introduced in [15] in the context of networked control systems. However, we adopt the term "MASP" to be more consistent with the sampled-data literature.

to be of class $\mathcal{KLL}$ if, for each $r \geq 0$, $\beta(\cdot, r, \cdot)$ and $\beta(\cdot, \cdot, r)$ belong to class $\mathcal{KL}$. To simplify notation, we sometimes write $(x, e) := [x^T \ e^T]^T$ for two vectors x and e.

We recall definitions given in [5] that we will use to develop a hybrid model of a NCS. The reader should refer to [5] for the motivation and more details on these definitions.

Definition 1: A *compact hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ given by $\mathcal{D} = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$, where $J \in \mathbb{Z}_{\geq 0}$ and $0 = t_0 \leq t_1 \cdots \leq t_J$. A *hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that, for each $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain.

Definition 2: A *hybrid trajectory* is a pair (dom ξ, ξ) consisting of hybrid time domain dom ξ and a function ξ defined on dom ξ that is continuously differentiable in t on (dom ξ) $\cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{Z}_{\geq 0}$.

Definition 3: For the hybrid system $\mathcal{H}$ given by the open state space $O \subset \mathbb{R}^n$ and the data (F, G, C, D) where $F : O \to \mathbb{R}^n$ is continuous, $G : O \to O$ is locally bounded, and C and D are subsets of O, a hybrid trajectory ξ : dom $\xi \to O$ is a *solution to* $\mathcal{H}$ if

1) For all $j \in \mathbb{Z}_{\geq 0}$ and for almost all $t \in I_j$ where I_j is such that $I_j \times \{j\} :=$ dom $\xi \cap (\mathbb{R}_{\geq 0} \times \{j\})$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j))$.
2) For all $(t, j) \in$ dom ξ such that $(t, j+1) \in$ dom ξ, we have $\xi(t, j) \in D$ and $\xi(t, j+1) = G(\xi(t, j))$.

Hence, the hybrid system models that we consider are of the form:

$$\begin{array}{rcll} \dot{\xi}(t,j) & = & F(\xi(t,j)) & \xi(t,j) \in C \\ \xi(t_{j+1}, j+1) & = & G(\xi(t_{j+1}, j)) & \xi(t_{j+1}, j) \in D \ . \end{array}$$

We sometimes omit the time arguments and write:

$$\begin{array}{rcll} \dot{\xi} & = & F(\xi) & \xi \in C \\ \xi^+ & = & G(\xi) & \xi \in D \ , \end{array} \tag{1}$$

where we denoted $\xi(t_{j+1}, j+1)$ as ξ^+ in the last equation. We also note that typically $C \cap D \neq \emptyset$ and, in this case, if $\xi(0,0) \in C \cap D$ we have that either a jump or flow is possible, the latter only if flowing keeps the state in C. Hence, the hybrid model we consider may have non-unique solutions.

III. A HYBRID MODEL OF SAMPLED-DATA SYSTEMS

We consider general nonlinear sampled-data systems and we find it convenient to write the model of the system in the following form[3]:

$$\begin{array}{rcll} \dot{x}_P & = & f_P(x_P, \hat{u}) & t \in [t_{i-1}, t_i] \\ y & = & g_P(x_P) & \\ \dot{x}_C & = & f_C(x_C, \hat{y}) & t \in [t_{i-1}, t_i] \\ u & = & g_C(x_C) & \\ \dot{\hat{y}} & = & 0 & t \in [t_{i-1}, t_i] \\ \dot{\hat{u}} & = & 0 & t \in [t_{i-1}, t_i] \\ \hat{y}(t_i^+) & = & y(t_i) & \\ \hat{u}(t_i^+) & = & u(t_i) & , \end{array} \tag{2}$$

where $t_i = iT, i \in \mathbb{N}$ and $T > 0$ is the sampling period[4]; x_P and x_C are respectively states of the plant and the controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of most recently transmitted plant and controller output values. Note that the last two formulas in (2) model the sampling process and the two formulas before that model the zero order hold mechanism. We introduce the sampling induced error e defined as

$$e(t) := \begin{pmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{pmatrix} = \begin{pmatrix} e_y \\ e_u \end{pmatrix} ,$$

and $x := (x_P^T \ x_C^T)^T$ and we can rewrite the equations (2) in the following manner:

$$\dot{x} = f(x, e) \qquad \forall t \in [t_{i-1}, t_i] \tag{3}$$

$$\dot{e} = g(x, e) \qquad \forall t \in [t_{i-1}, t_i] \tag{4}$$

$$e(t_i^+) = 0 \ , \tag{5}$$

where $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$ and[5]

$$f(x, e, w) := \begin{pmatrix} f_P(x_P, g_C + e_u) \\ f_C(x_C, g_P + e_y) \end{pmatrix};$$

$$g(x, e) := \begin{pmatrix} -\frac{\partial g_P}{\partial x_P} f_P(x_P, g_C + e_u) \\ -\frac{\partial g_C}{\partial x_C} f_C(x_C, g_P + e_y) \end{pmatrix} ,$$

and we omitted the arguments of $g_P := g_P(x_P)$ and $g_C := g_C(x_C)$ for space reasons. In order to apply results from [2], we map the model (3), (4), (5) into a hybrid system of the type (1) discussed in the preliminaries section. In particular, we consider systems of the form

$$\left.\begin{array}{rcl} \dot{x} & = & f(x, e) \\ \dot{e} & = & g(x, e) \\ \dot{\tau} & = & 1 \end{array}\right\} \quad \tau \in [0, T^*] \qquad \left.\begin{array}{rcl} x^+ & = & x \\ e^+ & = & 0 =: h(e) \\ \tau^+ & = & 0 \end{array}\right\} \quad \tau \in [\varepsilon, \infty) \tag{6}$$

where $\varepsilon > 0$ can be arbitrarily small, $T^* \geq \varepsilon$ and $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$ and $\tau \in \mathbb{R}_{\geq 0}$. Note that the hybrid model above allows for non-equidistant sampling in case $\varepsilon < T^*$ and in this case sampling times satisfy $\varepsilon \leq t_{i+1} - t_i \leq T^*$ for all i. On the other hand, if $\epsilon = T^*$ we recover the case of equidistant sampling where $t_i = iT^*$.

In what follows we will consider the behavior of all possible solutions to the hybrid system (6) subject to $\tau(0,0) \geq 0$. Since the derivative of τ is positive (equal to one) and when τ jumps it is reset to zero, it follows that τ will never take on negative values. According to the definition of solution for a hybrid system, the error vector e can jump, after ϵ seconds have elapsed from the previous jump. This is because at the previous jump τ was reset to zero, when the system is not jumping we have $\dot{\tau} = 1$, and the D set, which enables jumps, is the set $\{(x, e, \tau, \kappa) : \tau \in [\varepsilon, \infty)\}$. On the other hand, if T^* seconds have elapsed from the previous jump then the error vector e must jump. This is because the C set is $\{(x, e, \tau, \kappa) : \tau \in [0, T^*]\}$, and thus flows are not allowed after

[3]We note that it is possible to consider the problem in more generality, such as nonequdistant sampling times satisfying $t_i - t_{i-1} \leq \tau$ and systems with exogenous disturbances (see [2], [10]).

[4]Our results actually apply to non-equidistant sampling as it will become clear in the next section.

[5]We assume that g_P and g_C are differentiable in all their arguments.

τ reaches T^*. In this way, the time-invariant hybrid system (6) covers all of the possible behaviors described by (3), (4), (5).

Standing Assumption 1: f and g are continuous and h is locally bounded. ■

We will give an upper bound on T^* (MASP) to guarantee asymptotic or exponential stability.

Definition 4: For the hybrid system (6), the set $\{(x, e, \tau, \kappa) : x = 0\ ,\ e = 0\}$ is *uniformly asymptotically stable or UAS* if there exist $\Delta > 0$ and $\beta \in \mathcal{KLL}$ such that, for each initial condition $\tau(0,0) \in \mathbb{R}_{\geq 0}$, $|(x(0,0), e(0,0))| \leq \Delta$, and each corresponding solution,

$$\left| \left[\begin{array}{c} x(t,j) \\ e(t,j) \end{array} \right] \right| \leq \beta \left(\left| \left[\begin{array}{c} x(0,0) \\ e(0,0) \end{array} \right] \right|, t, \varepsilon j \right) \tag{7}$$

for all (t, j) in the solution's domain. The set is *uniformly exponentially stable or UES* if β can be taken to have the form $\beta(s, t, k) = Ms \exp(-\lambda(t + k))$ for some $M > 0$ and $\lambda > 0$. The set is uniformly globally asymptotically stable or UGAS (respectively, uniformly globally exponentially stable or UGES) if the system is UAS (respectively UES) and the above bound holds for all $x(0,0) \in \mathbb{R}^{n_x}$ and $e(0,0) \in \mathbb{R}^{n_e}$. ■

Remark 1: It is worth noting that when $\varepsilon = 0$ there are (instantaneous Zeno) solutions to (6) satisfying $x(0,j) = x(0,0)$ and $\tau(0,j) = \tau(0,0)$ for all $j \in \mathbb{Z}_{\geq 0}$. This motivates the factor ε multiplying j on the right-hand side of (7). ■

In order to guarantee asymptotic or exponential stability, we make the following assumption:

Assumption 1: There exist $\widetilde{\Delta}_x, \widetilde{\Delta}_e > 0$, a function $W : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz, a locally Lipschitz, positive definite, radially unbounded function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, real numbers $L \geq 0$, $\gamma > 0$, $\underline{\alpha}_W, \overline{\alpha}_W \in \mathcal{K}_\infty$ and a continuous, positive definite function ϱ such that, $\forall e \in \mathbb{R}^{n_e}$

$$\underline{\alpha}_W(|e|) \leq W(e) \leq \overline{\alpha}_W(|e|) \tag{8}$$

and for almost all $|x| \leq \widetilde{\Delta}_x$ and $|e| \leq \widetilde{\Delta}_e$,

$$\left\langle \frac{\partial W(e)}{\partial e}, g(x,e) \right\rangle \leq LW(e) + H(x)\ ; \tag{9}$$

moreover, for almost all $|x| \leq \widetilde{\Delta}_x$ and $|e| \leq \widetilde{\Delta}_e$,

$$\langle \nabla V(x), f(x,e) \rangle \leq -\varrho(|x|) - \varrho(W(e)) - H^2(x) + \gamma^2 W^2(e)\ . \tag{10}$$

We say that Assumption 1 holds globally if (8), (9) and (10) hold for almost all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$. ■

Assumption 2: Suppose all conditions of Assumption 1 hold and, in addition, there exist strictly positive real numbers $\underline{\alpha}_W, \overline{\alpha}_W$, a_1, a_2, and a_3 such that we have

$$\begin{aligned} \underline{\alpha}_W|e| \leq W(e) \leq \overline{\alpha}_W|e| \quad \forall e \in \mathbb{R}^{n_e} \\ a_1|x|^2 \leq V(x) \leq a_2|x|^2 \quad \forall x \in \mathbb{R}^{n_x} \\ \varrho(s) \geq a_3 s^2 \quad \forall s \geq 0. \end{aligned} \tag{11}$$

We say that Assumption 2 holds globally if (9) and (10) hold for almost all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$ and (11) holds. ■

Remark 2: Assumption 1 is essentially the same as the main assumption of [2]. An extra requirement in [2] was that there also exists $\lambda \in (0,1)$ such that the function W from Assumption 1 satisfies:

$$W(h(e)) \leq \lambda W(e) \qquad \forall e \in \mathbb{R}^{n_e}\ . \tag{12}$$

Since in [2] we could have in general that $h(e) \neq 0$, where $h(\cdot)$ defines the jump equation for the error e in (6), then it was necessary to explicitly assume (12). However, in our case we have that $h(e) = 0$ (see (6)) and, hence, for any W that satisfies Assumption 1, (12) holds for arbitrary $\lambda \in [0,1)$. An important consequence of this fact is that the formulas for MASP that we provide in our Theorems 1 and 2 for sampled-data systems are much simpler than the corresponding formulas for MATI that are given in [2] for a more general class of networked control systems. ■

Remark 3: Assumption 1 is very related to the main assumptions in [10]. The condition on $\dot{x} = f(x,e)$ is expressed here in terms of a Lyapunov function that establishes an $\mathcal{L}_2$ gain γ from W to H whereas in [10, Theorem 4] it is stated directly in terms of the $\mathcal{L}_2$ gain γ. However, in practice the $\mathcal{L}_2$ gain is typically verified with a Lyapunov function V that satisfies (10). We note that finding these functions may be hard for general nonlinear systems. ■

IV. Main result

In this section we present our main results, which contain an explicit formula for MASP that guarantees stability of a sampled-data system with an emulated controller. In particular, we assume that the controller is designed so that the following closed loop system:

$$\dot{x} = f(x,e) \tag{13}$$

is stable in an appropriate sense (more precisely, Assumption 1 holds). The system (13) represents the continuous-time closed loop system in which the error e accounts for the mismatch between the sampled-data and continuous-time values of controls due to emulation. The sampled-data system (2) consists of a zero-order hold equivalent implementation of the continuous-time controller designed so that the continuous-time system $\dot{x} = f(x,0)$ is stable. The goal is to determine MASP so that the system (2) is exponentially/asymptotically stable. More precisely, we consider the following question:

> *Suppose that the controller is designed so that Assumption 1 holds (respectively Assumption 2 holds). Find a value T^* of MASP that guarantees UAS or UGAS (respectively UES or UGES) of the sampled-data system (2) with the emulated controller for all $T \in (0, T^*)$.*

We emphasize that in this paper we provide an explicit computation of MASP, which is typically not done in the literature (a rare exception is the unpublished result in [6]). To state our main results we introduce the following function[6]:

$$\mathcal{T}(\gamma, L) := \begin{cases} \dfrac{1}{Lr} \arctan(r) & \gamma > L \\ \dfrac{1}{L} & \gamma = L \\ \dfrac{1}{Lr} \operatorname{arctanh}(r) & \gamma < L\ , \end{cases} \tag{14}$$

[6]Note that in the first and last expressions in (14) we use respectively the trigonometric and hyperbolic functions.

where

$$r := \sqrt{\left|\left(\frac{\gamma}{L}\right)^2 - 1\right|} \ . \tag{15}$$

In particular, we obtain the following result:

Theorem 1: Under Assumption 1, if MASP in (6) satisfies $T^* < \mathcal{T}(\gamma, L)$ and $0 < \varepsilon \leq T^*$ then, for the system (6), the set $\{(x, e, \tau) : x = 0, e = 0\}$ is uniformly asymptotically stable. Moreover, if Assumption 1 holds globally, then the set $\{(x, e, \tau) : x = 0, e = 0\}$ is UGAS.

The proof of Theorem 1 is given in the Appendix. The proof of the following result follows almost the same steps.

Theorem 2: Suppose Assumption 2 holds, MASP satisfies $T^* < \mathcal{T}(\gamma, L)$ and $0 < \varepsilon \leq T^*$. Then, for the system (6), the set $\{(x, e, \tau) : x = 0, e = 0\}$ is uniformly exponentially stable. Moreover, if Assumption 2 holds globally, then the set $\{(x, e, \tau) : x = 0, e = 0\}$ is UGES.

Remark 4: The proof of Theorem 1 will show that there exists an appropriate function ϕ (see Claim 1) such that $V(x)+\gamma\phi(\tau)W^2(e)$ is a strict Lyapunov function for the discrete-time system that is generated as the composition of flows and jumps in the system (6). In other words, for each solution and each (t_j, j) and $(t_{j+1}, j+1)$ belonging to the domain of the solution,

$$\begin{aligned} V(x(t_{j+1}, j+1)) + \gamma\phi(0)W^2(e(t_{j+1}, j+1)) \\ < V(x(t_j, j)) + \gamma W^2(e(t_j, j)) \ . \end{aligned}$$

■

Remark 5: The formula (14) is similar to formulas for MASP that were obtained in [6]. However, the two approached are notably different as well as the obtained formulas. Below we revisit an example taken from [6] and show that in this case our formula gives MASP that is less conservative (larger) than the one obtained in [6]. However, this can not be expected in general as both our and the approach in [6] are quite flexible.

Remark 6: The formula for T^* depends on the growth properties of g (i.e. constants L) and the robustness of the x subsystem (i.e. the gain γ) to the errors that come from implementing the continuous time controller using a zero order hold equivalent.

Remark 7: The model that we consider in (2) is a very special case of the model of general networked control systems (NCS) considered in [10]. The last four formulas in (2) are more general in [10] and they take the following form:

$$\begin{array}{rcll} \dot{\hat{y}} & = & \hat{f}_P(t, x_P, x_C, \hat{y}, \hat{u}, w) & t \in [t_{i-1}, t_i] \\ \dot{\hat{u}} & = & \hat{f}_C(t, x_P, x_C, \hat{y}, \hat{u}, w) & t \in [t_{i-1}, t_i] \\ \hat{y}(t_i^+) & = & y(t_i) + h_y(i, e(t_i)) & \\ \hat{u}(t_i^+) & = & u(t_i) + h_u(i, e(t_i)) & . \end{array} \tag{16}$$

The first two formulas in (16) allow for more general implementations than a simple zero order hold case that we consider in (2) when $\dot{\hat{y}} = 0$ and $\dot{\hat{u}} = 0$. A more significant difference is the last two formulas in (16) that allow much more general sampling/transmissions to occur. Indeed, it was shown in [10] that by choosing h_y and h_u in (16) one can model a range of commonly used network protocols that schedule access of different nodes to the network (see [10] for more details). The situation that we consider in this paper corresponds to the case of a single node which is a very special case of (16) with $h_y = 0$ and $h_u = 0$. By specifying appropriately h_y and h_u, our results in [10] can be used for the cases of multi-rate or event driven sampling. Hence, NCS considered in [10] can be viewed as an appropriate generalization of the classical sampled-data nonlinear systems. More importantly, many results in the area of NCS are directly relevant to the classical sampled-data nonlinear systems considered here, as this note clearly illustrates.

Remark 8: It is possible to state various other versions and generalizations of Theorem 1 but we do not present all the details here for reasons of brevity. Instead, we only discuss them briefly and point to the relevant literature. It is possible to state a result on $\mathcal{L}_p$ stability with respect to exogenous disturbances (see [10]). Moreover, instead of $\mathcal{L}_p$ stability one can use variants of input-to-state stability in order to find prove ISS from w to x, e (see [10], [11]). However, in some of these cases we do not obtain an explicit formulas for MASP.

Remark 9: The formula for MASP in Theorem 1 may be conservative in general. However, examples in [2], [10], [13] illustrate in a much more general context of NCS that the formula is often not overly conservative.

Remark 10: Our approach is flexible and the formula for MASP may be further improved if one uses the structure of the system, such as in the example below. In this case, the x subsystem takes a very special form:

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2) \\ \dot{x}_2 & = & f_2(x_2, e) \end{array}$$

where the first system is ISS when x_2 is regarded as its input. Hence, we can regard the overall system as a cascade of the (x_2, e) subsystem and the x_1 subsystem. We note that stability results for cascades of continuous-time [8] and discrete-time [12] systems are well known. In our case, we consider a cascade consisting of a hybrid (x_2, e) subsystem and a continuous-time x_1 subsystem. Since the hybrid subsystem does not exhibit Zeno behavior, this hybrid cascade can be treated in a manner that is almost identical to the continuous-time case (see e.g. [8]). In particular, asymptotic stability for the hybrid subsystem (x_2, e) plus ISS for the x_1 subsystem with respect to the input x_2 implies asymptotic stability for the overall hybrid system. As a result, we apply our formula for MASP by using the function f_2 instead of the full function $f(x, e) := (f_1(x_1, x_2), f_2(x_2, e))$. Moreover, we can apply our results in analysis of stability of families of system (i.e. robust stability) as it was done in [6].

Example 1: Consider a family of nonlinear systems as in [6]

$$\dot{x}_1 = -2x_1 - d_1 x_1^3 + x_2, \tag{17}$$

$$\dot{x}_2 = d_2 x_2^2 - x_2^3 + \hat{u}, \tag{18}$$

$$y = x_2, \tag{19}$$

with unknown and possibly time-varying $d_1 \geq 0$, $|d_2| \leq 1$. Note that our results also apply in this case although we did not write our main result in such generality. The emulated controller is given by

$$u = -2\hat{y}. \tag{20}$$

Note the cascade structure of (17)–(18) and note that the subsystem (17) is ISS from x_2 uniformly in $d_1 \geq 0$. We define the sampling error

$$e(t) = \big(\hat{y}(t) - y(t)\big) = \big(e_y\big) \ . \tag{21}$$

Since the controller is not dynamic, we can write

$$\hat{u} = -2\hat{y} = -2(x_2 + e_y). \tag{22}$$

We consider the system

$$\left.\begin{array}{rcl} \dot{x}_2 &=& f_2(x_2, e, d_2) \\ \dot{e} &=& g(x_2, e, d_2) \\ \dot{\tau} &=& 1 \end{array}\right\} \tau \in [0, T^*]$$
$$\left.\begin{array}{rcl} x^+ &=& x \\ e^+ &=& 0 \\ \tau^+ &=& 0 \end{array}\right\} \tau \in [\varepsilon, \infty) \ ,$$

where $f_2(x_2, e, d_2) := -2x_2 + d_2 x_2^2 - x_2^3 - 2e$ and $g(x_2, e, d_2) := 2e + 2x_2 - d_2 x_2^2 + x_2^3$. To study the global asymptotic stability of the origin $[x_2, e] = [0, 0]$ we consider the function $W(e) = |e|$ which satisfies for all $e \neq 0$

$$\begin{aligned} &\left\langle \tfrac{\partial W(e)}{\partial e}, g(x_2, e, d_2) \right\rangle \\ &\quad = \operatorname{sign}(e) g(x, e, d_2) \ \leq 2\, W(e) + H(x_2, d_2) \ , \end{aligned}$$

where $H(x_2, d_2) := |2x_2 - d_2 x_2^2 + x_2^3|$ and $L = 2$. Hence, (8) and (9) hold globally. Next, we show that (10) holds globally. Consider the Lyapunov function (the value of α, β and σ are strictly positive numbers that will be chosen later)

$$V(x_2) = \sigma^2 \left[\alpha \frac{x_2^2}{2} + \beta \frac{x_2^4}{4}\right] \tag{23}$$

and the time derivative of V along f_2 is

$$\begin{aligned} \dot{V} &= \sigma^2 \left[\alpha\, {x_2}^3 d_2 - \alpha\, {x_2}^4 - 2\,\alpha\, {x_2}^2 - 2\,\alpha\, x_2\, e + \right. \\ &\quad \left. \beta\, {x_2}^5 d_2 - \beta\, {x_2}^6 - 2\,\beta\, {x_2}^4 - 2\,\beta\, {x_2}^3 e\right] \\ &\leq \sigma^2 \left[\left(2\,\alpha^2 + 2\,\beta^2\right) e^2 + (-\beta + 1/2)\, {x_2}^6 + \beta\, {x_2}^5 d_2 \right. \\ &\quad \left. + (-\alpha - 2\,\beta)\, {x_2}^4 + \alpha\, {x_2}^3 d_2 + (-2\,\alpha + 1/2)\, {x_2}^2\right] \ . \end{aligned} \tag{24}$$

We add and subtract the terms $H(x, d_2)^2$, $\sigma^2 \varepsilon x_2^2$ and $\sigma^2 \varepsilon e^2$ to the right hand side of (24) yielding:

$$\begin{aligned} \dot{V} &\leq -\sigma^2 \varepsilon x_2^2 - \sigma^2 \varepsilon e^2 - H(x_2, d_2)^2 + \sigma^2 (2\alpha^2 + 2\beta^2 + \varepsilon) e^2 \\ &\quad + \sigma^2 x_2^2 p(x_2, \sigma, \alpha, \beta) \ , \end{aligned} \tag{25}$$

where (we used the fact that $|d_2| \leq 1$)

$$\begin{aligned} &p(x_2, \sigma, \alpha, \beta) := -2\,\alpha + 1/2 + 4\,\sigma^{-2} + \varepsilon + \\ &\left| x_2 \left(\alpha - 4\,\frac{1}{\sigma^2}\right)\right| + {x_2}^2 \left(-\alpha - 2\,\beta + \frac{5}{\sigma^2}\right) \\ &\quad + \left| x_2^3 \left(-2\,\frac{1}{\sigma^2} + \beta\right)\right| + {x_2}^4 \left(-\beta + 1/2 + \sigma^{-2}\right) \ . \end{aligned} \tag{26}$$

We note that if we can choose α, β, σ so that $p(x_2, \sigma, \alpha, \beta) \leq 0$ for all x_2 then Assumption 1 holds globally with $\varrho(s) := \sigma^2 \varepsilon s^2$ and

$$\gamma = \sigma \sqrt{(2\alpha^2 + 2\beta^2 + \varepsilon)} \ .$$

The parameters α, β, σ and ε are obtained numerically $[\sigma, \alpha, \beta, \varepsilon] = [2, 0.77, 0.77, 0.01]$ and this leads to $\gamma = 3.086$. Using the computed γ and L we compute MASP $= 0.368$sec via (14). The numerical simulations of the sampled data system (17)–(18), for several values of d_1 and d_2, show that sampling periods lower than 1 sec lead to UGAS. The upper bounds for MASP proposed in [6] (eq. (4.7) and (4.11)) using one Lyapunov function and two Lyapunov functions are 0.09sec and 0.1428sec, respectively. We note, however, that both our approach and the approach in [6] are quite flexible and it is unlikely that our bounds would be less conservative in all possible cases.

V. Conclusions

We have presented an explicit formula for the maximum allowable sampling period that guarantees stability of sampled-data nonlinear systems with emulated controllers. Our results are simple consequences of the results and proofs in [2]. While we concentrated only on sampled-data systems without disturbances, we already indicated that much more general results for networked control systems are available in [2], [10] that involve network protocol scheduling and exogenous disturbances. Moreover, extensions of our results to deal with computational delays and scheduling due to controller implementation on embedded computer systems seem possible (see [1]). Finally, similar sampling schemes arise in certain classes of hybrid systems (see [4]) and extending our results in that direction is an interesting topic for further research.

References

[1] K.E. Arzen, A. Cervin and D. Henriksson, “Implementation-aware embeded control systems”, in Handbook of networked control systems (Ed.) Birkhäuser, Boston, 2005, pp. 377-394.

[2] D. Carnevale, A.R. Teel and D. Nešić, “A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems”, IEEE Trans. Automat. Contr., vol. 52 (2007), pp. 892-897.

[3] T. Chen and B.Francis, “Input-output stability of sampled-data systems”, *IEEE Trans. Automat. Contr.*, vol. 36, No. 1 (1991), pp. 50-58.

[4] Y. Gao, J. Lygeros, M. Quincapoix and N. Seube, “On the control of uncertain impulsive sytems: approximate stabilization and controlled invariance”, Int. J. Contr., vol. 77 (2004), pp. 1393-1407.

[5] R. Goebel and A.R. Teel, “Solutions to hybrid inclusions via set and graphical convergence with stability theory applications”, Automatica, vol. 42 (2006), pp. 573-587.

[6] I. Karafyllis and C. Kravaris, “Global stability results for sytems under sampled-data control”, Submitted to Int. J. Robust and Nonlinear Control, 2007. `available on-line http://front.math.ucdavis.edu/0612.5233`

[7] I. Karafyllis and Z.P. Jiang, “A small-gain theorem for a wide class of feedback systems with control applications”, SIAM J. Contr. Appl., vol. 46, (2007), pp. 1483-1517.

[8] H. K. Khalil, *Nonlinear Control Systems (2^{nd} Ed.)* Prentice Hall, 1996.

[9] D.S. Laila, D. Nešić and A.R. Teel, “Open and closed loop dissipation inequalities under sampling and controller emulation”, Europ. J. Contr., vol. 18 (2002), pp. 109-125.

[10] D. Nešić and A.R.Teel, “Input output stability properties of networked control systems”, IEEE Trans. Automat. Contr., vol. 49 (2004), pp. 1650-1667.

[11] D. Nešić and A.R. Teel, “Input to state stability of networked control systems”, Automatica, vol. 40 (2004), pp. 2121-2128.

[12] A. Loria and D. Nešić, “On uniform boundedness of parameterized discrete-time cascades with decaying inputs: applications to cascades”, Syst. Contr. Lett., vol. 49 (2003), pp. 163-174.

[13] M. Tabbara, D. Nešić and A.R. Teel, “Stability of wireless and wireline control systems”, IEEE Trans. Automat. Contr., vol. 52, (2007), pp. 1615-1630.

[14] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks", IEEE Trans. Automat. Contr., vol. 52, (2007), pp. 1680-1685.

[15] G.C. Walsh, H. Ye and L.G. Bushnell, "Stability analysis of networked control systems", IEEE Trans. Contr. Syst. Techn., vol 10 (2002), pp. 438-446.

[16] Y. Zheng, D.H. Owens and S.A. Billings, "Fast sampling and stability of nonlinear sampled-data systems: Part 2. sampling rate estimation", IMA J. Math. Contr. Inf., vol. 7 (1990), pp. 13-33.

VI. Appendix

Let $\phi : [0, \widetilde{\mathcal{T}}] \to \mathbb{R}$ be the solution to

$$\dot{\phi} = -2L\phi - \gamma(\phi^2 + 1) \qquad \phi(0) = \lambda^{-1} \ , \tag{27}$$

where $\lambda \in (0,1)$. In order to prove Theorem 1, we first recall the following result from [2].

Claim 1: $\phi(\tau) \in [\lambda, \lambda^{-1}]$ for all $\tau \in [0, \widetilde{\mathcal{T}}]$. Moreover, we have that $\phi(\widetilde{\mathcal{T}}) = \lambda$ for $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(\lambda, \gamma, L)$ defined as

$$\widetilde{\mathcal{T}} := \begin{cases} \dfrac{1}{Lr}\arctan\left(\dfrac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma > L \\ \dfrac{1}{L}\dfrac{1-\lambda}{1+\lambda} & \gamma = L \\ \dfrac{1}{Lr}\text{arctanh}\left(\dfrac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma < L \ , \end{cases} \tag{28}$$

and r is defined in (15).

A. *Proof of Theorem 1*

First, we prove the result when Assumption 1 holds globally. Let $T^* < \mathcal{T}$, where $\mathcal{T}$ comes from (14). We will use the definitions $\xi := (x, e, \tau)$ and $F(\xi) := (f(x,e), g(x,e), 1)$. Note that $\mathcal{T}$ in (14) and $\widetilde{\mathcal{T}}$ in (28) satisfy $\mathcal{T}(\gamma, L) = \widetilde{\mathcal{T}}(0, \gamma, L)$. Moreover, for any fixed L and γ we have that $\widetilde{\mathcal{T}}(\cdot, \gamma, L)$ is a strictly decreasing function. Hence, since the conditions of the theorem require T^* to be strictly smaller than $\mathcal{T}(\gamma, L)$, there exists $\lambda \in (0,1)$ such that $T^* = \widetilde{\mathcal{T}}(\lambda, \gamma, L)$. Let these (λ, γ, L) generate ϕ via Claim 1 and define

$$U(\xi) := V(x) + \gamma\phi(\tau)W^2(e) \ . \tag{29}$$

Hence, from Claim 1 we have that $\phi(\tau) \in [\lambda, \lambda^{-1}]$ and there exist two functions $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$ such that for all x, e, τ we have:

$$\varphi_1(|(x,e)|) \le U(\xi) \le \varphi(|(x,e)|) \ . \tag{30}$$

We now consider the quantity $\langle \nabla U(\xi), F(\xi) \rangle$. We first note that

$$\begin{array}{rcl} U(\xi^+) & = & V(x^+) + \gamma\phi(\tau^+)W^2(e^+) \\ & = & V(x) + \gamma\phi(0)W^2(h(e)) \\ & \le & V(x) + \gamma\lambda W^2(e) \le U(\xi) \ . \end{array} \tag{31}$$

We also have, for all τ and almost all (x,e),

$$\begin{aligned} \langle \nabla U(\xi), F(\xi) \rangle &\le -\varrho(|x|) - \varrho(W(e)) - H^2(x) + \gamma^2 \\ &\quad W^2(e) + 2\gamma\phi(\tau)W(e)(LW(e) + H(x)) \\ &\quad - \gamma W^2(e)(2L\phi(\tau) + \gamma(\phi^2(\tau)+1)) \\ &\le -\varrho(|x|) - \varrho(W(e)) - H^2(x) + 2\gamma\phi(\tau)W(e)H(x) \\ &\quad - \gamma^2 W^2(e)\phi^2(\tau) \\ &\le -\varrho(|x|) - \varrho(W(e)) \ . \end{aligned}$$

Since ϱ is positive definite, V is positive definite and radially unbounded, and Claim 1 holds, it follows that there exists a continuous, positive definite function $\widetilde{\varrho}$ such that

$$\langle \nabla U(\xi), F(\xi) \rangle \le -\widetilde{\varrho}(U(\xi)) \ . \tag{32}$$

Then, by standard results for continuous-time systems, we have the existence of $\beta \in \mathcal{KL}$ satisfying

$$\beta(s, t_1 + t_2) = \beta(\beta(s, t_1), t_2) \quad \forall (s, t_1, t_2) \in \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \tag{33}$$

and such that

$$U(\xi(t,j)) \le \beta(U(\xi(t_j, j)), t - t_j) \quad \forall (t_j, j) \preceq (t, j) \in \text{dom } \xi \tag{34}$$

where $(t_j, j) \preceq (t, j)$ means that $t_j \le t$. From (31) it follows that

$$U(\xi(t_{j+1}, j+1)) \le U(\xi(t_{j+1}, j)) \tag{35}$$

for all j such that $(t,j) \in \text{dom } \xi$ for some $t \ge 0$. Combining (33)-(35), we get

$$U(\xi(t,j)) \le \beta(U(\xi(0,0)), t) \qquad \forall (t,j) \in \text{dom } \xi \ . \tag{36}$$

Next, since $t \ge \varepsilon j$ for all $(t,j) \in \text{dom } \xi$, it follows that

$$U(\xi(t,j)) \le \beta(U(\xi(0,0)), 0.5t + 0.5\varepsilon j) \qquad \forall (t,j) \in \text{dom } \xi \ . \tag{37}$$

Then, using that V is positive definite and proper, using (8), Claim 1, and the definition of U in (29), uniform global asymptotic stability of the set $\{(x, e, \tau) : x = 0, e = 0\}$ follows.

Finally, note that if Assumption 1 holds locally, then one can find an invariant set $|(x,e)| \le c$ for some $c > 0$ on which we have that (31) and (32) hold. The conclusion on UAS follows using the standard Lyapunov arguments.

B. *Sketch of Proof of Theorem 2*

Under the assumptions made in the theorem to guarantee uniform global exponential stability, it follows that we can take φ_1, φ_2 in (30) to be quadratic, $\widetilde{\varrho}$ can be taken to be quadratic and, hence, β can be taken to be of the form $\beta(s,t) = Ms\exp(-\lambda t)$. The local result follows trivially from our assumptions.