Brief paper

Observer design for wired linear networked control systems using matrix inequalities

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1. Introduction

Networked control systems (NCS) are a paradigm in which the plant, controller and spatially distributed sensors/actuators communicate via a network. In particular, packet based networks are typically used in NCS applications such as drive-by-wire or fly-by-wire vehicles. Advantages of this setup come from easier installation and maintenance, lower cost, weight and volume. Instead, we focus here on the observer and protocol design for NCSs under packet based communication constraints. An algorithm for design of round robin (RR) protocols\(^1\) for LNCs that preserve observability of the underlying plant is proposed in Zhang and Hristu-Varsakelis (2005). Relying on such protocols, then the existing theory of periodic, linear time–varying systems is applied to derive an observer. Recently, Gupta, Chung, Hassibi, and Murray (2006) proposed a class of stochastic protocols and observers that under communication constraints minimize an upper bound on the estimation error covariance.

Our goal here is to design an observer and a TOD protocol to asymptotically reconstruct the plant states. We derive sufficient

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\(^1\) TOD protocols were proposed in Walsh et al. (2002a,b), Walsh and Ye (2001) and Walsh et al. (2001).

\(^2\) They are also referred to token ring in the literature.
conditions in terms of matrix inequalities for existence of an observer-protocol pair in the considered class that asymptotically reconstructs the plant states. Our model of LNCSs is inspired by the approach introduced in Walsh et al. (2002a,b), Walsh and Ye (2001) and Walsh et al. (2001), and further developed in Nešić and Teel (2004a,b), and allows us to relate observer design for NCSs under communication constraints to observer design for switched-systems, (Alessandri, Baglietto, & Battistelli, 2005; Alessandri & Coletta, 2003; Babaali, Egerstedt, & Kamen, 2004; Chen & Saif, 2004; Ji, Wang, & Xie, 2003; Ji, Xie, & Hao, 2004; Li, Wen, & Soh, 2003; Sun, 2006). We are thus able to exploit ideas already developed in the switched-system framework for observer design under communication constraints. However, in addition, use structural properties of NCSs not present in the switched-system framework, which allows us to relax the assumption on plant’s observability.

In Section 2 we introduce the class of LNCSs of interest and describe the communication constraints, while in Section 3 we define the class of observers and protocols which we use to estimate the plant states. Section 4 contains sufficient conditions in terms of matrix inequalities for existence of an observer/protocol pair in the considered class that guarantee asymptotically converging estimates of the plant states. Two examples are given in the same section. Summary is presented in Section 5. We define $N_l \triangleq \{1, \ldots, l\}$, $\mathbb{R}^+$ $\triangleq \{x \in \mathbb{R} : x \geq 0\}$, and $x(t^-)$ $\triangleq \lim_{t \to t^-} x(t).$

2. LNCS description

The schematic description of LNCSs we consider is given on Fig. 1. We consider LTI plants of the form

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t),$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and the state and output.

Assumption 1. The pair $(C, A)$ is observable.  

Assumption 1 can be relaxed by allowing that the pair $(C, A)$ is detectable, but we do not pursue this direction.

Plant (1) is equipped with $m$ sensors grouped into $1 \leq l \leq m$ nodes, where the output component measured in node $j \in N_l$ is denoted by

$$y_j = C_j x + \sum_{i=1}^{l} m_i = m.$$  

and $\sum_{j=1}^{l} m_j = m$. The nodes $\{1, \ldots, l\}$ can send packets with their data and receive packets from the network, and we refer to them as sensor nodes. In addition the network may contain arbitrary number of “passive” nodes that can only receive packets. Such nodes can be used to host distant observers and/or actuators for plants with inputs, and without loss of generality we assume that there is only one passive node in the network, see Fig. 1. All nodes have sufficient computational power to run dynamical systems, which can be interpreted as local observers. Depending on the host node, we denote their states with $\bar{x}_j$ $j \in N_l$, and interpret them as estimates of the plant states $x$.

The sequence of instants at which network transmissions occur is denoted by $\{t_i\}_{i=1}^\infty$, where $t_i$ is the instant of transmission $i$. To focus on the communication constraints we assume that the transmissions are instantaneous, the communication channel is noiseless, and that there are no packet dropouts. We note that packet dropouts due to collisions are prevented in our scenario by the virtue of using deterministic (TOD) protocols. Furthermore, we let the transmissions be periodic, that is, $t_i = (i - 1)T$, $i \in \mathbb{N}$, where $T > 0$ is the transmission interval. At each transmission we assume that exactly one sensor node transmits a packet with the current value of its output component. The assignment of the transmissions to the sensor nodes is specified by a piece-wise constant function $\sigma : \mathbb{R}^+ \to N_l$. To assign transmission $i$ to node $j$ we set $\sigma(t) = j$ for $t \in [t_i, t_{i+1})$, which results in sending $y_{\sigma(i)}(t) = y_j(t)$ over the network. We treat the function $\sigma$ as a control input and refer to the algorithm based on which $\sigma(t)$ takes values in the set $N_l$ as the protocol.

Due to the spatial distribution of plant (1) and the presence of the network, signals available in different network nodes is different. According to Fig. 1, we assume that at time $t \in [t_i, t_{i+1})$ the signals available to a sensor node $j \in N_l$ and a passive node $j = l + 1$ consist of

$$\text{sensor node } j : \sigma(t_i), y_{\sigma(i)}(t), \bar{x}_j(t), y_j(t),$$

$$\text{passive node } j : \sigma(t_i), y_{\sigma(i)}(t), \bar{x}_j(t).$$

Our goal is to design a protocol and an observer in the passive node that are capable of asymptotically reconstructing the plant states under information constraints [3]. Hence, input into the observer in node $l + 1$ is allowed to depend only on the signals $\sigma(t_i)$ and $y_{\sigma(i)}(t)$, implying that the ability to reconstruct plant states hinges not only on observability of plant (1) but also on the protocol.

Networks controlled by TOD protocols are necessarily wired networks. In such networks all sensor nodes simultaneously try to transmit packets consisting of the packet priority and the data part, while the network hardware allows the transmission only to the node whose packet has the highest priority. TOD protocols can be viewed as distributed algorithms implemented in all sensor nodes for computing packets’ priorities as a function of locally available signals. Thus, in contrast to the observer, the protocol is allowed to utilize all output components $y_j, j \in N_l,$ see (2). We note that manipulation of packets’ priorities is possible on some commercial networks such as controller area networks (CANs), (Walsh et al., 2002a,b; Walsh & Ye, 2001; Walsh et al., 2001).

We consider TOD protocols that determine priority of a packet in sensor node $j \in N_l$ as a function of the discrepancy between the output component $y_j$ and its estimate based on the observer states in the passive node $l + 1$, $\bar{y}_{l+1} \triangleq \sum_{j=1}^{l} C_j \bar{x}_j$. The idea behind such protocols is to assign the transmission to the node whose corresponding estimate is the least accurate, thus avoiding unnecessary transmissions. A consequence of such approach is that each sensor node requires a synchronized copy of the observer running in the passive node $l + 1$ to locally compute packets’ priorities. Hence, the resulting LNCS does not contain a single observer, but one observer copy per node.

3. Protocol and observer structure

In this section we specify a class of observers and TOD protocols used to estimate the plant states. To that end we define the matrix
We also exploit that all pairs $(C_i, A_i)$ are observable. Such assumption for NCSs corresponds to requiring that plant (1) be observable from any output component $y_j$ which significantly limits applicability of these observers. Instead we let the dwell time, that is TI, be constant and design the switching signal, that is the protocol. We also exploit that all pairs $(C_j, A_j)$ among which the protocol switches have the same matrix $A_j$, while the matrices $C_j$ are rows of the matrix $C$. This allows us to replace the above assumption on the plant observability by a less stringent Assumption 1.

We make the following assumptions on observers (4).

**Assumption 2.** Initial conditions of observers (4) are identical, $x_i(0) = x_0$, $\forall j \in \mathbb{N}_{l+1}$, $x_0 \in \mathbb{R}^n$.

Since observers (4) have the same input signal $y(t_i)$ and the same initial conditions, their trajectories satisfy

$$\dot{x}(t) = \bar{x}(t), \quad y_k(\bar{x}(t)) = \bar{y}(t).$$

In Example 2 we show that under Assumption 2 equality (6) remains satisfied even when plant (1) is subject to measurement noise and/or disturbances, but we note that equality (6) is violated when one of observers (4) is affected by a computational glitch. In the rest of the paper we suppress superscripts in the observer estimates and only write $\bar{x}(t)$ with the understanding that this signal is available in all nodes.

**Remark 2.** Synchronization assumptions, such as Assumption 2, are common in the NCS literature due to the spatial distribution of plant’s sensors and actuators. For example, in the schemes with “zoom in–zoom out” quantizers all nodes are assumed to possess a synchronized copy of the zoom parameter, see Liberzon (2003). Another example is in Gupta et al. (2006), where implementation of the protocols and observers constructed there requires that each node possesses a synchronized copy of a random number generator.

**Remark 3.** Robustness of observer-protocol pair (4), (7) to computational glitches is still an open problem and the focus of our current research. We will illustrate via simulations that observer-protocol pair (4), (7) indeed possesses a certain level of robustness to such disturbances. A “brute force” solution to this issue is to send additional information over the network. Namely, every $r \in \mathbb{N}$ transmits the complete state $\bar{x}$ of the observer to which the current transmission is assigned can in addition be sent over the network, to which value then all other observers reset their states. We note that integer $r$ is arbitrary and provides a tradeoff between robustness with respect to synchronization disturbances and the network throughput required for estimation.

Sensor node $j \in \mathbb{N}$ computes the priority of its packet by relying on the output component $y_j$ and the states of its observer $\bar{x} = \bar{x}$. We consider TOD protocols of the form

$$\sigma(t_i) = \Omega(y(t_i^-) - \bar{y}(t_i^-)).$$

$$\Omega(y - \bar{y}) \equiv \arg \max_{j \in \mathbb{N}} (y_j - \bar{y})(y_j - \bar{y}),$$

where $\bar{y}$ is the matrix $Q = Q^T = \text{diag}(Q_1, \ldots, Q_j)$ is to be determined. To implement protocol (7) each node sets the transmission priority of its packet to $Q_{ij}$ the network hardware assigns the transmission to the node with the largest transmission priority. It may happen that nodes $j_1, j_2 \in \mathbb{N}$ have packets with the same transmission priority in which case protocol (7)–(8) can not uniquely assign the transmission. To avoid this ambiguity we reserve several least significant bits of the priority field to encode the address of the sender node while the remainder encodes the transmission priority of the packet. Then even if two packets have the same transmission priority, the protocol will assign the transmission to node $\max(j_1, j_2)$ with the larger address. This mechanism can be implemented on CANs, which allow up to 29 bits for encoding packet priorities, and it prevents packet dropouts due to collisions. Implementation of protocol (7) requires that the sensor nodes be equipped with an
observer. This differs from the requirements on the sensor nodes in Dačić and Nešić (2007) and in examples considered in Nešić and Teel (2004a,b), where they are only equipped with buffers that store the most recently transmitted value of the output component from that node, and thus have a simpler structure. In the next example we demonstrate that TOD protocols (7) that use the most recently transmitted values of output components instead of their current estimates to assign the transmissions may lead to the loss of observability in LNCS (1), (5) for arbitrarily small TIs.

**Example 1.** We consider plant (1) with
\[
A = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
(9)
where \( \lambda > 0 \), that satisfies Assumption 1. The plant is equipped with two sensors collocated into separate nodes each measuring an output component \( y_i = C_i x, i = 1, 2 \). The nodes are equipped with buffers, where the values stored in them are denoted by \( \hat{y}_1, \hat{y}_2 \) and their evolution is given by
\[
\dot{\hat{y}}(t) = 0, \quad t \in [t_i, t_{i+1}),
\]
(10)
and
\[
\dot{\hat{y}}(t) = (I - \Psi_{y_i(t_i)})(\hat{y}(t_i)) + \Psi_{y_i(t)}(\hat{y}(t_i)),
\]
(11)
Dynamics (10) represent evolution of the stored values between two consecutive transmissions, while dynamics (11) govern update in one of the buffers due to the transmissions. If node \( j \) transmits a packet at \( t = t_i \), then the value in its buffer is set to the value \( y_j(t_i) \) while the value in the other buffer remains unchanged. Suppose that the network transmissions are assigned using the following TOD protocol
\[
\sigma(t_i) = \Omega_Q(\hat{y}(t_i) - \hat{y}(t_i)),
\]
(12)
which is obtained from (7) by substituting current estimates of output components \( \hat{y} \) with their most recently transmitted values \( \hat{y} \). We show that for any \( T > 0 \), protocol matrix \( Q = \text{diag}(q_1, q_2) \geq 0 \), and buffer initial conditions \( \hat{y}(0) = \begin{bmatrix} \hat{y}_1(0) \\ \hat{y}_2(0) \end{bmatrix} \), there exist initial conditions \( x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \) of plant (1), (9) for which protocol (12) never assigns a transmission to the second node. Consequently, it is impossible to build an observer that would provide exponentially converging estimates of the states \( x \). Introducing the error coordinates \( \hat{e} = \begin{bmatrix} \hat{e}_1^T \\ \hat{e}_2^T \end{bmatrix} = \begin{bmatrix} \hat{y}_1 - y_1, \hat{y}_2 - y_2 \end{bmatrix} \) and integrating the dynamics of LNCS (1), (9) over \( [t_i, t_{i+1}) \) we obtain
\[
x_{i+1} = A x_i, \quad \hat{e}_{i+1} = (A_{\hat{e}} - I) x_i + (I - \Psi_{e_i}) \hat{e}_i,
\]
(13)
where \( x_i = x(t_i), \hat{e}_i = \hat{e}(t_i), \sigma_i = \sigma(t_i), \) and \( A_{\hat{e}} = \text{diag}(\exp(\lambda T), 1) \).
Substituting \( \hat{e} \) into protocol (12) we get that \( \sigma = 1 \) if \( q_1 \hat{e}_1^2 > q_2 \hat{e}_2^2 \), and otherwise \( \sigma = 2 \). Let the initial conditions of system (13) satisfy
\[
q_1 \hat{e}_{1,0}^2 > q_2 \hat{e}_{2,0}^2,
\]
(14)
and
\[
q_1(1 - \exp(\lambda T))^2 \hat{e}_{1,0}^2 > q_2 \hat{e}_{2,0}^2,
\]
(15)
where the second subscript denotes the time instant. It is straightforward to show that for any \( \hat{y}(0) \) it is possible to select \( x(0) \) to satisfy conditions (14) and (15). We show by induction that \( \sigma_i = 1, \forall i \in \mathbb{N} \). From (14) it follows that \( \sigma_1 = 1 \), hence
\[
\hat{e}_{1,1} = (\exp(\lambda T) - I) x_{1,0}, \quad \text{and} \quad \hat{e}_{2,1} = \hat{e}_{2,0}.
\]
From (15) we get that \( \sigma_2 = 1 \), hence \( \hat{e}_{1,2} = \exp(\lambda T)(\exp(\lambda T) - I) x_{1,0}, \) and \( \hat{e}_{2,2} = \hat{e}_{2,1} = \hat{e}_{2,0} \). Now let \( q_1 \hat{e}_{1,i}^2 > q_2 \hat{e}_{2,i}^2 \) for some \( i \in \mathbb{N} \), and thus \( \sigma_i = 1 \). From (13) we get
\[
\hat{e}_{i+1} = \exp(\lambda T)(\exp(\lambda T) - I) x_{i,0} < \exp(\lambda T)(\exp(\lambda T) - I) x_{i,0} = \hat{e}_{i,1} \quad \text{and} \quad \hat{e}_{i+2,1} = \hat{e}_{i,2,1}.
\]
Then it follows that
\[
q_1 \hat{e}_{i+1,1}^2 > q_2 \hat{e}_{i+1,2}^2 > q_2 \hat{e}_{i+2,1}^2 = q_2 \hat{e}_{i+2,1}^2.
\]
Consequently, protocol (12) never assigns a network transmission to the second node.

### 4. Simultaneous observer and protocol design

Given \( T > 0 \) we now derive sufficient conditions for the existence of an observer-protocol pair (4), (7) that ensures asymptotically converging estimates of the plant states. More precisely, we derive sufficient conditions in terms of matrix inequalities (MIs) for the existence of the gain \( L \) in (4) and the matrix \( Q \) in (7) such that the resulting estimation error dynamics (EED) are quadratically stable (QS). Substituting
\[
e(t) \triangleq \hat{x}(t) - x(t)
\]
(16)
into (4), integrating over \( [t_i, t_{i+1}) \), and using Assumption 2 we get the discrete-time equivalent of the EED
\[
e_{i+1} = \tilde{A}_0 e_i \triangleq (A_d + \tilde{A}_d L_0 C_0) e_i,
\]
(17)
where \( A_d \triangleq \exp(A T), \tilde{A}_d \triangleq \int_0^T \exp(A r) d r, \sigma_i \triangleq \sigma(t_i), \) and \( e_i \triangleq e(t_i) \). It can be shown that the matrix \( \tilde{A}_d \) is nonsingular for any \( T > 0 \). Substituting (16) into (8) we obtain the TOD protocol dependent on the estimation errors
\[
\sigma = \Omega_Q(CE) = \arg \max_{j \in \mathbb{N}} e^T \bar{C}_j Q \bar{C}_j e,
\]
(18)
whose equivalent representation, more amenable for computations, is given by
\[
\forall e \in \mathcal{S}_j, \quad \Omega_Q(CE) = j,
\]
(19)
where the closed sets \( \mathcal{S}_j = \{ e \in \mathbb{R}^n : e^T (C_j^T Q_j C_j - C_i^T Q_i C_i) e \geq 0, \forall k \in \mathbb{N} \} \) have a common boundary, and for \( e \in \mathcal{S}_j \cap \mathcal{S}_k \) protocol (19) can not uniquely assign a transmission. We avoid this ambiguity by assigning the transmission to the node with the highest address \( j \) that ensures the estimation error, \( \mathcal{S}^* = \max \{ j \in \mathbb{N} : e \in \mathcal{S}_j \} \).

**Definition 1.** EED (17) and (18) are QS if for an observer gain \( L = \{ L_1, \ldots, L_j \} \) and a matrix \( Q = Q^T = \text{diag}(Q_1, \ldots, Q_j) > 0 \), there exist matrices \( P = P^T > 0 \) and \( H = H^T > 0 \), such that the quadratic Lyapunov function \( V(e) \triangleq e^T Pe \) satisfies for all \( e \neq 0 \)
\[
V(\tilde{A}_d e) - V(e) \leq -e^T H e.
\]
EED (17)–(18) are observer quadratically stabilizable (oQS) if for a given matrix \( Q \) there exists an observer gain \( L \) such that the resulting EED are QS. EED (17)–(18) are protocol and observer quadratically stabilizable (poQS) if there exists a matrix \( Q \) such that the resulting EED with protocol (18) are oQS.

We note that QS of EED (17) and (18) implies exponential convergence of the estimation errors, that is, the existence of constants \( d, \lambda > 0 \) satisfying \( ||e|| \leq d ||e_0|| \exp(-\lambda t) \).

**Remark 4.** Actual EED of LNCS on Fig. 2 are not of the form (17)–(18), but rather
\[
e_{i+1} = \tilde{A}_0 e_i, \quad j \in \mathbb{N} \cup \{ 0 \}, \quad \sigma = \arg \max_{j \in \mathbb{N}} (C_i e_i)^T Q_j C_i e_i,
\]
(20)
(21)
where \( e_i \triangleq x_i \), \( i \in N \), \( \forall i \in N \), and \( \forall i \in N \). We achieve QS of system (20)–(21), we achieve QES of EED (17)–(18) and then exploit that all observers (4) have identical estimates, that is, \( e_i = e_i^\ast, \forall j, i \in N \) and \( \forall i \in N, \) see Assumption 2.
Theorem 1. EED (17)-(18) are poQS if there exist matrices $P = P^T > 0$, $H = H^T > 0$, $M_j, Q = Q^T = \text{diag}(Q_j, \ldots, Q_k) > 0$, and constants $\tau_{jk} \geq 0$, $j, k \in \mathbb{N}_i$, such that $\forall j \in \mathbb{N}_i$

\[
\begin{bmatrix}
P - H - \frac{1}{\tau_{jk}} (C_j^T Q_j C_j - C_k^T Q_k C_k) & \star \\
PA_j + M_j C_j & P
\end{bmatrix} > 0.
\]  
(22)

Then the solutions of EED (17)-(18) satisfy

\[
J \triangleq \sum_{i=0}^{\infty} e_i^T P^{-1} e_i \leq e_i^T P e_0 \leq \|P\| \|e_0\|^2.
\]
(23)

where the observer gain is $L_j = (P \bar{A}_d)^{-1} M_j$.

Proof. Using Definition 1, protocol (19), and $\cup_{j=1}^{\infty} \delta_j = \mathbb{R}^n$, we deduce that EED (17)-(18) are poQS if there exist matrices $P, Q, H, L$ satisfying

\[
e_i^T [(A_d + \bar{A}_d L_j) P (A_d + \bar{A}_d L_j) - P + H] e < 0,
\]
for all $j \in \mathbb{N}_i$ and $e \in \delta_j, e \neq 0$. Applying the S-procedure and the Schur complement (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), and suppressing the dependence on $e$, the above inequality is satisfied if

\[
e_i^T [(A_d + \bar{A}_d L_j) P (A_d + \bar{A}_d L_j) - P + H] e + \sum_{k=1}^{\infty} \tau_{jk} e_i^T (C_j^T Q_j C_j - C_k^T Q_k C_k) e < 0,
\]

\[
\Leftrightarrow \begin{bmatrix}
P - H - \frac{1}{\tau_{jk}} (C_j^T Q_j C_j - C_k^T Q_k C_k) & \star \\
PA_j + M_j C_j & P
\end{bmatrix} > 0.
\]

Introducing the change of variables $M_j = P \bar{A}_d L_j$, the first claim of Theorem 1 follows. Along the solutions of EED (17) we have

\[
V(e_0) = \sum_{i=0}^{\infty} e_i^T (P - \bar{A}_d \bar{H} P \hat{A}_d) e_i.
\]

Substituting protocol (18), $\hat{A}_d = A_d + \bar{A}_d L_j C_i$, and MIs (22) into the above expression, we get

\[
\|P\| \|e_0\|^2 \geq V(e_0) \geq \sum_{i=0}^{\infty} e_i^T (P - \bar{A}_d \bar{H} P \hat{A}_d) e_i - \sum_{i=0}^{\infty} e_i^T \left( \sum_{k=1}^{\infty} \tau_{jk} (C_j^T Q_j C_j - C_k^T Q_k C_k) \right) e_i \\
\geq \sum_{i=0}^{\infty} e_i^T \bar{H} e_i,
\]

thus proving the second claim of Theorem 1.

Corollary 1. EED (17)-(18) are oQS if for a given matrix $Q = Q^T = \text{diag}(Q_1, \ldots, Q_k) > 0$, there exist matrices $P = P^T > 0$, $H = H^T > 0$, $M_j$, and constants $\tau_{jk} \geq 0$, $j, k \in \mathbb{N}_i$, such that MIs (22) are feasible for $\forall j \in \mathbb{N}_i$. Then the columns of the observer gain are given by $L_j = (P \bar{A}_d)^{-1} M_j$, and the solutions of EED (17)-(18) satisfy (23). ■

Theorem 2. Let the pair $(C, A)$ satisfy Assumption 1. Then for any matrix $Q = Q^T = \text{diag}(Q_1, \ldots, Q_k) > 0$ there exist matrices $P = P^T > 0$, $H = H^T > 0$, $L = [L_1 \ldots L_i]^T$, and constants $\tau_{jk} \geq 0$, $j, k \in \mathbb{N}_i$, such that MIs (22) are feasible for sufficiently small $T \in T$.

Proof. Let a matrix $Q = \text{diag}(Q_1, \ldots, Q_k)$ and a scalar $s > 0$ be given. For small $T > 0$ the matrices $A_d$ and $\bar{A}_d$ can be approximated by

$A_d = I + AT + O(T^2), \quad \bar{A}_d = IT + O(T^2)$.

(24)

We define the following matrices: $X \triangleq [I 0]^T$, $\bar{X} \triangleq [I 0]^T$, $Y_j \triangleq [C_j 0]$, and $\bar{Y}_j \triangleq [C_j I]$, where the matrix $\bar{C}_j$ is the orthogonal complement of the matrix $C_j, j \in \mathbb{N}_i$. Note that rank$[X \bar{X}] = 2n$, rank$[Y_j \bar{Y}_j] = 2n, \bar{X}^T X = 0$ and $\bar{Y}_j \bar{Y}_j = 0$. Then MI (22) can be rewritten as

\[
L_j + X M_j Y_j + Y_j^T M_j^T X > 0,
\]
(25)

where $L_j \triangleq \begin{bmatrix} P - H - \sum_{k=1}^{\infty} \tau_{jk} (\bar{Q}_j - \bar{Q}_k) & \star \\
PA_j & P \end{bmatrix}$.

are feasible for the same matrices $P, H$, and constants $\tau_{jk}$. Let the matrices $\bar{P} = \bar{P}^T > 0$, $\bar{H} = \bar{H}^T > 0$, and $\bar{L} \triangleq [\bar{L}_1 \ldots \bar{L}_i]$ satisfy the Lyapunov inequality:

\[
(A + i \bar{C})^T \bar{P} + \bar{P} (A + i \bar{C}) < -sl - \bar{H}.
\]

(27)

Such matrices always exist due to Assumption 1. We now show that MIs (26) are feasible for sufficiently small $T$ and parameters

\[
P = \bar{P}, \quad H = \bar{H}, \quad \tau_{jk} = \epsilon T, \quad \forall j, k \in \mathbb{N}_i,
\]
(28)

where $\epsilon \triangleq \frac{2}{\| \sum_{j=1}^{\infty} \bar{P} L_j Q_j^{-1} \bar{P} \|}$. The first MI in (26) holds for parameters (28) and sufficiently small $T$ because $\tau_{jk} = O(T), H = O(T)$, and $\bar{P} > 0$.

Substituting (24) and (28) into the second MI in (26), and applying the Schur complement we rewrite it as

\[
\bar{C}_j \bar{P} - HT - (I + TA) \bar{P} (l + TA)
\]

\[+ T \epsilon \bar{C}_j \bar{P} Q_k C_k \bar{C}_j^T + O(T^2) > 0.
\]

Extracting the terms linear in $T$ we deduce that the above MI is feasible for sufficiently small $T$ if

\[
\epsilon \bar{C}_j \left( \sum_{k=1}^{i} \bar{L}_j C_k - A^T \bar{P} - \bar{P} A - \bar{H} \right) \bar{C}_j^T > 0,
\]

which combined with the identity $\bar{L} C = \sum_{k=1}^{i} \bar{L}_k C_k$ and (27) is equivalent to

\[
\epsilon \bar{C}_j \left( sl + \sum_{k=1}^{i} \left( \epsilon C_k^T Q_k C_k + C_k^T \bar{L}_k \bar{P} + \bar{P} \bar{L}_k C_k \right) \right) \bar{C}_j^T > 0.
\]
Thus the second MI in (26) holds for parameters (28) and sufficiently small £T. □

**Theorem 2** proves that observer-protocol class (4), (7) always contains a pair capable of asymptotically reconstructing the plant states for sufficiently small TIs. In other words, LNCS (1), (5) does not lose observability for sufficiently small TIs under protocols (7) which assign the transmissions relying on local observers and their current output estimates. This is in contrast to protocols (12) which assign the transmissions relying on buffers and the most recently transmitted values of the output components.

Using MILs (22) we now formulate an optimization for computation of the observer gain £ and the protocol matrix Q to minimize an upper bound on cost (23). We first formulate such an optimization to compute observer gain £ for a fixed protocol matrix Q, in which case (22) has the form of linear matrix inequalities (LMIs). Then, using the formulated optimization we propose an iterative algorithm to simultaneously compute observer gain £ and protocol matrix Q in which case (22) has the form of bilinear matrix inequalities (BMIs) due to the multiplication of the unknowns £jk and Qk. Given a matrix $H = H^T > 0$ in (23) and a protocol matrix $Q = Q^T = diag(Q_1, ..., Q_l) > 0$, we solve the following optimization

$$\min \kappa : \kappa I \geq P,$$

$$\left[ P - H - \sum_{k=1}^{l} \tau_{jk}(C^T Q_{jk} - C^T Q_k C_k) \right] X + \sum_{k=1}^{l} \tau_{jk} Q_j C_k \geq 0,$$  (29)

for the matrices $P = P^T > 0, M_l$ and the constants $\kappa, \tau_{jk} \geq 0, j, k \in N_l$. If optimization (29) is feasible, the columns of the observer gain are given by $L_j = (PA_d)^{-1} M_j$. By minimizing $\kappa$ via (23) we minimize an upper bound on the cost $J$. This approach to the optimization of the observer gain was proposed in Alessandri and Coletta (2003) under different assumptions, see Remark 1.

**Remark 5.** The matrix $H$ was assumed to be unknown in Theorem 2, while it is given in optimization (29). Theorem 2 is thus not applicable to optimization (29), which raises the issue of its well-posedness. It can be shown that if the matrix $H$ is of the form $H = \tilde{H}^T$, where $\tilde{H} = \tilde{H}^T > 0$ is independent of $T$, then optimization (29) is always feasible for sufficiently small $T$ and any matrices $Q$ and $\tilde{H}$. □

Given matrix $H$, we simultaneously compute the observer gain $L$ and the protocol matrix $Q$ via the following algorithm based on linearization of BMIs (29).

**Linearized BMI algorithm (LBMI A)**

**Step 1:** Set $Q^1 = I$, and solve optimization (29). If it is feasible, set $k^1 = k$ and $\tau_{jk}^1 = \tau_{jk}, j, k \in N_l, i = 1$, and pick constants $1 \gg \epsilon > 0, 1 \gg \Delta k > 0$.

**Step 2:** Solve the following minimization

$$\min \kappa^{i+1} : \kappa^{i+1} I \geq P^{i+1} > 0,$$

$$\Delta \tau_{jk} \leq \sqrt{\frac{\epsilon}{2l}}, \quad C^T_j \kappa Q \leq \sqrt{\frac{\epsilon}{2l}} I,$$

$$\tau_{jk} + \Delta \tau_{jk} \geq 0, \quad Q_j + \Delta Q_j > 0,$$

$$\left[ P^i - H - X^i_j - Y^i_j - Z^i_j \right] P^i \geq \epsilon I,$$  (30)

$$X^i_j \triangleq \sum_{j=1}^{l} \tau_{jk}(C^T_j Q_j C_j - C^T_j Q_j C_k),$$

$$Y^i_j \triangleq \tau_{jk}(C^T_j Q_j C_j - C^T_j Q_j C_k),$$

$$Z^i_j \triangleq \tau_{jk}(C^T_j Q_j C_j - C^T_j Q_j C_k),$$

for matrices $P^i = (P^i)^T > 0, M_l, \delta Q \triangleq [\delta Q_1, ..., \delta Q_l], \delta Q_j \triangleq \delta Q^T_j, \Delta Q_j$, and constants $\delta \tau_{jk}, j, k \in N_l$.

**Step 3:** If minimization (30) is feasible, and $\kappa^{i+1} \leq \kappa - \Delta \kappa$ set $\kappa^{i+1} = \kappa$ and $\tau_{jk}^{i+1} = \tau_{jk}^i + \Delta \tau_{jk}$. $\Delta \tau_{jk}^{i+1} = \epsilon^T + \delta \tau_{jk}$, let $i = i + 1$, and go to Step 2. Otherwise, stop iterating.

**Remark 6.** The increments $\delta Q_j$ and $\delta \tau_{jk}$ are bounded by a function of $\epsilon$ so that feasibility of optimization (30) for matrices $P^i, M_l, \delta Q$ and constants $\delta \tau_{jk}$ implies feasibility of optimization (29) for matrices $P^i, M_l, \delta Q + \Delta Q$ and constants $\delta \tau_{jk} + \Delta \tau_{jk}$. We require that each iteration of LBMI A reduces $\kappa^i$ for at least $\Delta \kappa$ to ensure that it terminates after finitely many iterations. When it terminates, the resulting $\kappa^i$ is smaller or equal than $\kappa$ stemming from minimization (29), since LBMI A uses the solution of minimization (29) as its initial condition. However, the actual value of the cost $J$ along solutions of EED (17)-(18) is not guaranteed to be smaller for the observer-protocol pair designed via LBMI A, since $\kappa$ only determines its upper bound. □

**Example 2.** In this example we consider the plant

$$\dot{x} = Ax, \quad y = Cx + n, \tag{31}$$

whose output is corrupted by measurement noise $n \triangleq [n^T n^T n^T]$. We show that Assumption 2 implies (6) despite presence of the measurement noise, that is, that the measurement noise does not destroy synchronization of observers (4). Substituting the estimation errors $\epsilon^i = \bar{x}^i - x$ in (4) and integrating over $[0, T]$ we obtain the discrete-time equivalent EED for observer $j \in N_l$

$$e^i_{j+1} = (A_d + \tilde{A}_d L_n C_n) e^i_j - F_n n_i, \tag{32}$$

where $n_i \triangleq n(t_i)$ and $F_n \triangleq \int_0^T e^{ji} dL_j$. From (32) we deduce that the measurement noise affects all observers (4) in the same way because the term $F_n n_i$ is independent of $j$. Combining this observation with Assumption 2 it follows that (6) holds. We thus suppress the superscripts in the estimates for the reminder of this example. Note that in presence of measurement noise $n$ protocol (7)-(8) becomes

$$\sigma_i = \arg \max_{j \in N_l} (C_j e_i + \psi_j n_i)^T Q_j (C_e_i + \psi_j n_i) \tag{33}$$

where $\psi_j = \arg \max (\delta (j - 1) I_{m_l}, ..., \delta (j - l) I_{m_l})$. We now show that under Theorem 1 EED (32)-(33) are input-to-state stable (ISS) with respect to (wrt) the measurement noise. The increment of Lyapunov function $V(e) = e^T P e$ along the solutions of EED (32)-(33) is

$$V(e_{i+1}) - V(e_i) = e^T_i (A^T_j P A_d - P) e_i + 2e^T_i P F_n n_i + n^T_i F^T_n P F_n n_i.$$
Consider plant (Remark 2 illustrates a tradeoff between the desired value of cost J and TI (that is, network throughput) required to estimate the plant states (Dačić & Nešić, 2007; Nešić & Teel, 2004a,b; Walsh et al., 2002a,b; Walsh & Ye, 2001; Walsh et al., 2001). Fig. 3 illustrates a tradeoff between the desired value of cost J and TI (that is, network throughput) required to ensure it. In Fig. 4 we compare trajectories of EED (17)-(18) corresponding to: (a) LBMIA design, (b) design via minimization (29), and (c) design in Gupta et al. (2006). We choose $T = 0.05$ and for matrix $H(T)$ we obtain protocol matrices $Q_\text{a} = \text{diag}(0.7, 1.2, 1.9)$, $Q_\text{b} = \text{diag}(1.1, 1.1)$ and

$$L_a = \begin{bmatrix} 5 & -14.8 & 20.9 \\ -26.2 & 21.3 & -30.1 \\ 31 & -43.4 & 32.9 \\ -35.5 & 45.8 & -55.7 \end{bmatrix},$$

$$L_b = \begin{bmatrix} 10.1 & -13 & 11.2 \\ -31.4 & 15.8 & -14.9 \\ 34.8 & -38 & 17.5 \\ -39.4 & 41.2 & -40 \end{bmatrix}.$$

where subscripts a and b denote the design via LBMIA and the design via (29), respectively. The protocols in Gupta et al. (2006) are stochastic, specified by probabilities with which the transmissions are assigned to the nodes, where the probabilities are computed by minimizing an upper bound on the estimation error covariance. In contrast, the protocols designed here are deterministic, designed neglecting the measurement noise to minimize an upper bound on the transient of the estimation errors (23). We note that both designs require synchronization of the sensor nodes. In our design we impose Assumption 2 while the design in Gupta et al. (2006) requires that each sensor node possesses a synchronized random number generator. For plant (31), (34), and Gaussian noise with zero mean and variance 0.01, $n_{k,j} \sim N(0, 0.01)$, $j \in \mathbb{N}_3$, we obtain from Gupta et al. (2006) the stochastic protocol in which the probability of assigning a transmission to node $j$ is $p_j$, where $[p_1, p_2, p_3] = [0.28, 0.32, 0.4]$. The corresponding observer gains are

$$[L_{11} L_{22} L_{33}] = \begin{bmatrix} 11.8 & -4.1 & 4.9 \\ -27.7 & 6.1 & -7.2 \\ 32.6 & -25.3 & 10.7 \\ -37.8 & 29 & -30.1 \end{bmatrix}.$$

where $L_{ij}$ is the observer gain when the transmission is assigned to node $j \in \mathbb{N}_3$. We set initial conditions such that $e(0) = [2 2 3 1.9]^T$ and assume that $n_{k,j} \sim N(0, 0.01)$. As observed from Fig. 4, the fastest convergence rate of the estimates (for the same network’s TI) are produced by LBMIA. This indicates that the observer-protocol pairs computed via LBMIA ensure that the actual performance, and not only its worst case value, is better than the actual performance of the corresponding pairs computed via (29).

The convergence rate of the design in Gupta et al. (2006) is the slowest due to using a stochastic protocol. On the other hand, the design in Gupta et al. (2006) produces the smallest “steady-state” estimation error covariance, which was to be expected because the measurement noise was neglected in the designs developed in this paper.

Finally in Fig. 5 we demonstrate robustness of the observer/protocol pairs developed in this paper to mismatch in the observer initialization. In this case EED have the form (20)-(21) which we simulate for $T = 0.05$, observer gain $L_b$, protocol matrix $Q_\text{b}$, and measurement noise $n_{k,j} \sim N(0, 0.01)$. We set the initial conditions of EED (20) to $e(0) = [2 3 1 3]^T$, $e^2(0) = [1.5 3.5 0.75 2.25]^T$, $e^3(0) = [2.5 2.5 1.25 3.5]^T$. As shown in Fig. 5, all observers have bounded estimates despite a 25% mismatch in their initial conditions and the presence of the measurement noise. We note that an analytical proof of this robustness is still an open problem.

Example 3. Consider plant (31) with

$$A = \text{diag}(1, 2, 3, 4), \quad C = [C_1^T C_2^T C_3^T]^T,$$

$$C_1 = [1 1 0 0], \quad C_2 = [0 1 1 0], \quad C_3 = [0 0 1 1],$$

(34)

which has four unstable eigenvalues and satisfies Assumption 1, but none of the pairs $(C_j, A), j \in \mathbb{N}_3$, are observable (see Remark 2). The plant is equipped with three sensors each collocated in a separate network node, where node $j$ measures output component $y_j = C_j x + n_j, j \in \mathbb{N}_3$. We disregard the measurement noise in the observer/protocol design, but illustrate its effect in simulations. We increment $T$ from $T_1 = 0.01$ to $T_5 = 0.06$ with step $\Delta T = 0.0025$ and compute the discrete-time model of plant (31), (34), $x_{i+1} = A_0(T_i)x_i, y = Cx$, where $A_0(T) = \exp(AT)$. For each $T_i, k \in \mathbb{N}_5$, we use the matrix $H(T_k) = T_kH, H = I$, to compute an upper bound $\kappa(T_k)$ on cost $J$ in (23) via: (a) LBMIA for parameters $\epsilon = \Delta \kappa = 0.01$, and (b) minimization (29) for fixed protocol matrix $Q = I$. As shown in Fig. 3 the bound $\kappa(T_k)$ on cost $J$ of the observer-protocol pair computed via LBMIA is smaller than the bound $\kappa(T_k)$ obtained by fixing a protocol matrix and computing the observer gain from (29). Maximal TI for which minimization (29) has a feasible solution (and thus LBMIA also) for plant (34) and matrix $H(T)$ is $T^* = 0.063$. We interpret this value as a bound on maximum allowable transmission interval (MATI), which is inversely proportional to the network throughput.

Fig. 3. Dependence of $\kappa$ on TI: (a) LBMIA algorithm, (b) Minimization (29) for protocol matrix $Q = I$. 

- $\leq -\sum_{j=1}^{l} e_j^T(C_j^T Q_\alpha C_j - C_j^T Q_\beta C_j)e_j$
- $e_j^T He_j + 2e_j^T PF_j n_j + n_j^T F_j^T P F_j n_j$
- $\leq -\sum_{j=1}^{l} e_j^T He_j + n_j^T \left( \sum_{j=1}^{l} \left( \Psi_j Q_\alpha \Psi_j - \Psi_j Q_\beta \Psi_j \right) \right) n_j + n_j^T F_j^T P F_j n_j$
- $+ 2n_j^T \left( \sum_{j=1}^{l} \left( \Psi_j Q_\alpha C_j - \Psi_j Q_\alpha C_j \right) \right) e_j + 2e_j^T P F_j n_j$
- $\leq -\frac{h}{2} \| e_j \|^2 + \left( r^* + \frac{2}{h} (s^*)^2 \right) \| n_j \|^2,$
- $r^* \max_{j \in \mathbb{N}_3} \left( F_j^T P F_j + \sum_{k=1}^{l} \left( \Psi_k Q_\alpha \Psi_k - \Psi_k Q_\alpha Q_\beta \right) \right),$
- $s^* \max_{j \in \mathbb{N}_3} \left( F_j P + \sum_{k=1}^{l} \left( \Psi_k Q_\beta \Psi_k - \Psi_k Q_\beta C_k \right) \right),$

where to obtain the second and the third inequality we exploited (22) and (33), respectively. The resulting inequality certifies that EED (32)-(33) is ISS wrt the measurement noise. We note that the same conclusion is true if plant (31) is in addition affected by external disturbances. 

Fig. 4.
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5. Conclusion

We developed a simultaneous design of an observer and a TOD protocol to asymptotically reconstruct the states of an LTI plant under the network induced communication constraints. For a parameterized class of observers and protocols, we derived sufficient conditions for existence of an observer-protocol pair within the considered class that asymptotically reconstructs the plant states.

Acknowledgements

This work is supported by the Australian Research Council under the Discovery Project and Australian Professorial Fellow program.

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