Matrosov theorem for parameterized families of discrete-time systems

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Abstract

A version of Matrosov's theorem for parameterized discrete-time time-varying systems is presented. The theorem is a discrete-time version of the continuous-time result in [2]. Our result facilitates controller design for sampled-data nonlinear systems via their approximate discrete-time models. An application of the theorem to establishing uniform asymptotic stability of systems controlled by model reference adaptive controllers designed via approximate discrete-time plant models is presented.

Key words: Adaptive Control; Discrete-time; Matrosov Theorem; Nonlinear Systems; Persistency of Excitation; Stability; Time-varying.

1 Introduction

The prevalence of computer controlled systems and the fact that the nonlinearities in the plant model can often not be neglected strongly motivate consideration of the class of nonlinear sampled-data models. One of the main difficulties in dealing with this class of models is that it is not clear what is the model one should use when designing the controller. For example, the exact discrete-time model of this class of systems is typically not available for the controller design and one can only use an approximate model for this purpose. However, it was shown in [4,13] that controllers that stabilize an approximate discrete-time model of the system may destabilize the exact discrete-time model for all sampling periods. Hence, a careful analysis and design are needed if one is using an approximate discrete-time model for controller design.

This has motivated us to present checkable conditions [4,13] on the controller, approximate model and the continuous-time plant model that guarantee that if the controller stabilizes the approximate model then it would also stabilize the exact discrete-time model in an appropriate sense (semiglobal-practical) for sufficiently small sampling periods. These results provide a framework for controller design for sampled-data nonlinear systems via their approximate discrete-time models. In particular, [4, Theorem 2] gives Lyapunov like conditions on the approximate model to provide such a framework. It is the purpose of this paper to extend the results in [4,13] in the following way. We prove a version of Matrosov's theorem for establishing uniform asymptotic stability using several Lyapunov like functions that typically have negative semi-definite first difference (in [4] and [13] we required a negative definite first difference of the Lyapunov function; moreover in [4] we considered only time-invariant systems).

Our results are important for the following reasons. Time-varying systems arise in a range of control applications, such as tracking control, adaptive control or when time-varying controllers are used (such as, stabilization of non-holonomic systems). Matrosov's theorem generalizes, in an appropriate sense, La Salle's theorem to time-varying systems and it is very important in situations when we have a negative semi-definite derivative of the Lyapunov function along the trajectories of the system dynamics (such as, when we use the storage function for the passivity property to establish stability). The classical Matrosov theorem establishes UGAS via two Lyapunov like functions (see [9], [15, Theorem 5.5], [12, Theorem 2.5] and [14, Appendix]). Certain extensions of Matrosov theorem can be found in [3] and more recently in [2] where it was shown how it is possible to combine an arbitrary number of Lyapunov like

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functions to test UGAS of time-varying continuous-time systems. Our main result extends the main result from [2] to parameterized families of time-varying discrete-time systems that naturally arise when an approximate discrete-time model is used to design a controller for a sample-data system. While our proofs follow similar steps as in the continuous-time case [2], they are complicated by the fact that we also require certain uniformity of the stability property with respect to the parameter, which is crucial for sampled-data applications. An application to systems for which a model reference adaptive controller is designed via an approximate discrete-time model is presented (a continuous-time analogue of this result can be found in [1]).

The paper is organized as follows. First, we present preliminaries and definitions in Section 2. The main result is presented and proved in Section 3. Section 4 explains how our results can be used for controller design of sampled-data nonlinear systems via their approximate discrete-time models. Finally, in Section 5 we show how our result can be used to analyze systems controlled by model reference adaptive controllers that are designed via their approximate discrete-time plant models.

2 Preliminaries

We denote sets of real and integer numbers respectively as $\mathbb{R}$ and $\mathbb{Z}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for all $t > 0$, $\beta(\cdot, t) \in \mathcal{K}$, for all $s > 0$, $\beta(s, \cdot)$ is decreasing to zero. We denote by $|\cdot|$ the Euclidean norm of vectors. Given arbitrary $L \geq 0$ and $T > 0$ we use the following notation: $\ell_{L,T} := \left\lfloor \frac{L}{T} \right\rfloor = \max\left\{z \in \mathbb{Z} : z \leq \frac{L}{T}\right\}$. Also, we denote $B_\Delta := \{x : |x| \leq \Delta\}$ and $\mathcal{H}(\delta, \Delta) := \{x : \delta \leq |x| \leq \Delta\}$. Consider the class of parameterized time-varying discrete-time systems:

$$x(k+1) = F_T(k, x(k)) \ . \tag{1}$$

These systems arise naturally when an approximate discrete-time model of a sampled-data plant is used to design a controller (see Section 4 and [4,13] for more details). The trajectory of the system (1) at a time k starting at an initial time $k_\circ$ and emanating from the initial state $x_\circ$ is denoted $\phi_T(k, k_\circ, x_\circ)$. Often $k_\circ$ and $x_\circ$ are clear from the context and we use the shorthand notation $\phi_T(k)$. $T > 0$ is a parameter that can be arbitrarily assigned (most of the time T represents the sampling period). The following definitions will be used in the sequel.

Definition 1 *The system (1) is uniformly semiglobally practically asymptotically stable (USPAS) if there exists $\gamma \in \mathcal{K}_\infty$ such that for each pair of strictly positive real numbers (r, ϵ) there exists $L = L(r, \epsilon) > 0$ and for each strictly positive (Δ, ν) there exists $T^* = T^*(\Delta, \nu) > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, $T \in (0, T^*)$ and $r \leq \Delta$ we have*

$$|x_\circ| \leq \Delta \Longrightarrow |\phi_T(k)| \leq \gamma(|x_\circ|) + \nu, \qquad \forall k \geq k_\circ \tag{2}$$

$$|x_\circ| \leq r \Longrightarrow |\phi_T(k)| \leq \epsilon + \nu, \qquad \forall k \geq k_\circ + \ell_{L,T} \ . \tag{3}$$

The system (1) is uniformly semiglobally asymptotically stable (USAS) if for each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$ such that $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, $T \in (0, T^*)$ and $r \leq \Delta$ the inequalities (2) and (3) hold with $\nu = 0$. The system (1) is uniformly globally asymptotically stable (UGAS) if there exists $T^* > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, $T \in (0, T^*)$ the inequality (2) holds for all $x_\circ \in \mathbb{R}^n$ with $\nu = 0$ and (3) holds for all $r > 0$ and with $\nu = 0$.

The stability definitions in Definition 1 are suitable for the purposes of our paper. However, in [4,13] we have used definitions that make use of $\mathcal{KL}$ functions. These are presented next. As Proposition 1 below points out, the properties in Definitions 1 and 2 are equivalent.

Definition 2 *The system (1) is β-uniformly semiglobally practically asymptotically stable (β-USPAS) if there exists $\beta \in \mathcal{KL}$ such that for each strictly positive (Δ, ν) there exists $T^* = T^*(\Delta, \nu) > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, with $|x_\circ| \leq \Delta$ and $T \in (0, T^*)$ we have*

$$|\phi_T(k)| \leq \beta(|x_\circ|, (k - k_\circ)T) + \nu, \qquad \forall k \geq k_\circ \tag{4}$$

The system (1) is β-uniformly semiglobally asymptotically stable (β-USAS) if for each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, with $|x_\circ| \leq \Delta$ and $T \in (0, T^*)$ the inequality (4) holds with $\nu = 0$. The system (1) is β-uniformly globally asymptotically stable (β-UGAS) if there exists $T^* > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, with $x_\circ \in \mathbb{R}^n$ and $T \in (0, T^*)$ the inequality (4) holds with $\nu = 0$.

The proof of the following result follows very similar steps to the proof of [17, Proposition 1] and is omitted.

Proposition 1 *The system (1) is:*
(1) UGAS iff there exists a continuous $\beta \in \mathcal{KL}$ such that the system (1) is β-UGAS;
(2) USAS iff there exists a continuous $\beta \in \mathcal{KL}$ such that the system (1) is β-USAS;
(3) USPAS iff there exists a continuous $\beta \in \mathcal{KL}$ such that the system (1) is β-USPAS;

We also need the following definition for the statement of our main result.

Definition 3 *The system (1) is said to be semiglobally bounded on compact time intervals (SB) if there exist $\gamma, \varphi \in \mathcal{K}_\infty$ such that for each triple of strictly positive real numbers (Δ, L, δ) there exists $T^* = T^*(\Delta, L, \delta) > 0$*

such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$ with $|x_\circ| \leq \Delta$ and $T \in (0, T^*)$ we have that trajectories of (1) satisfy:

$$|\phi_T(k)| \leq \gamma(|x_\circ|) + \varphi(\delta T(k - k_\circ)), \quad (5)$$

for all $k \in [k_\circ, k_\circ + \ell_{L,T}]$. If for each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$ with $|x_\circ| \leq \Delta$ and $T \in (0, T^*)$ the bound (5) holds with $\delta = 0$ and for all $k \geq k_\circ$, then the system is uniformly semiglobally stable (USS). If there exists $T^* > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$ with $x_\circ \in \mathbb{R}^n$ and $T \in (0, T^*)$ the bound (5) holds, with $\delta = 0$, then the system (1) is uniformly globally stable (UGS).

3 A type of Matrosov Theorem

The following theorem is an adaptation of the Matrosov Theorem given in [2] to parameterized discrete-time time-varying systems that naturally arise when one uses an approximate discrete-time model of the sampled-data plant to design a discrete-time controller. A comparison between the statement of this theorem and the statement of Corollary 2, the latter of which is essentially a translation of the result in [2] to discrete-time, indicates the complications that arise from considering parameterized systems.

Theorem 1 *The system (1) is USPAS if the following conditions hold:*

1. *The system (1) is SB.*
2. *There exist integers $j, m > 0$ and a real number $\tilde{T} > 0$ such that for each $T \in (0, \tilde{T})$ and $i \in \{1, 2, \ldots, j\}$ there exist functions $V_T^i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $\chi_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and continuous functions $Y_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and for each $\Delta > 0$ there exists $\mu > 0$ such that*

(a) for all $T \in (0, \tilde{T})$, $|x| \leq \Delta$ and $k \in \mathbb{Z}$:

$$\max\{|V_T^i(k, x)|, |\chi_T(k, x)|\} \leq \mu\,. \quad (6)$$

(b) For each integer $k \in \{1, \ldots, j\}$ we have that

$$\left\{\begin{array}{l} (z, \xi) \in B_\Delta \times B_\mu, \\ Y_i(z, \xi) = 0, \\ \forall i \in \{1, \ldots, k-1\} \end{array}\right\} \Longrightarrow Y_k(z, \xi) \leq 0\,. \quad (7)$$

(c) We have that

$$\left\{\begin{array}{l} (z, \xi) \in B_\Delta \times B_\mu, \\ Y_i(z, \xi) = 0, \\ \forall i \in \{1, \ldots, j\} \end{array}\right\} \Longrightarrow z = 0\,. \quad (8)$$

3. *For each pair of strictly positive real numbers (Δ, ν) there exists $T^* > 0$ such that for all $T \in (0, T^*)$, $i \in \{1, \ldots, j\}$, $|x| \leq \Delta$ and $k \in \mathbb{Z}$ we have that:*

$$\frac{\Delta V_T}{T} \leq Y_i(x, \chi_T(k, x)) + \nu\,, \quad (9)$$

where $\Delta V_T := V_T^i(k+1, F_T(k, x)) - V_T^i(k, x)$.

Note that Condition 2 of Theorem 1 concerns only the properties of the functions V_T^i, χ_T and Y_i whereas Conditions 1 and 3 also involve the system (1). The significance of Condition 2 is the consequence given in the next proposition. (The proof is the same as that for continuous-time systems, as in [2], and thus is omitted.) This consequence is typically much more difficult to check than Condition 2 (see Example 1 which follows the proposition, and also the discussion in [2]):

Proposition 2 *Let $\Delta > 0$ be given and let it generate $\mu > 0$ via Condition 2 of Theorem 1. Then, for each $\delta \in (0, \Delta)$ there exist $\alpha > 0$ and numbers $K_i > 0, i = 1, \ldots, j-1$ such that the function $Z(z, \xi) := \sum_{i=1}^{j-1} K_i Y_i(z, \xi) + Y_j(z, \xi)$ satisfies for all $(z, \xi) \in \mathcal{H}(\delta, \Delta) \times B_\mu$ the following $Z(z, \xi) \leq -\frac{\alpha}{2^{j-1}}$.*

Example 1 *The following functions Y_i will arise in the analysis of the model reference adaptive control problem:*

$$Y_1(z, \xi) = -\alpha_3(z_1) \quad (10)$$

$$Y_2(z, \xi) = |z_1|\rho_3(|z|) + [\rho_1(|z_1|) + \rho_2(|z_1|)]\,\rho_3^2(|z|) - |\xi|^2 \quad (11)$$

$$Y_3(z, \xi) = |\xi| + \exp(1)\rho_3(|z|)\rho_1(|z_1|) - \frac{1}{2}\alpha(|z_2|) \quad (12)$$

where $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are continuous, nondecreasing, $\rho_i(0) = 0$ for $i = 1, 2$, and α_3, α are continuous, positive definite. We note that, from the properties of α_3, $Y_1(z, \xi) \leq 0$. Moreover, $Y_1(z, \xi) = 0$ implies $z_1 = 0$. Thus, from the properties of ρ_1 and ρ_2, $Y_1(z, \xi) = 0$ implies $Y_2(z, \xi) = -|\xi|^2 \leq 0$. In turn, $Y_1(z, \xi) = Y_2(z, \xi) = 0$ imply $z_1 = 0$ and $\xi = 0$ and thus imply $Y_3(z, \xi) = -\frac{1}{2}\alpha(z_2) \leq 0$. Thus Condition 2(b) of the Theorem is satisfied. Finally, from the properties of α, $Y_1(z, \xi) = Y_2(z, \xi) = Y_3(z, \xi) = 0$ imply $z = 0$. So Condition 2(c) of the Theorem is satisfied. On the other hand, it is a non-trivial task to solve for the numbers $K_i > 0$ and $\alpha > 0$ indicated in the Proposition.

The following results are corollaries of the proof used to establish Theorem 1.

Corollary 1 *The system (1) is USAS if:*
(1) The system (1) is USS.
(2) Condition 2 of Theorem 1 holds.
(3) For each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$ such that for all $T \in (0, T^*)$, $i \in \{1, \ldots, j\}$, $|x| \leq \Delta$ and

$k \in \mathbb{Z}$ the bound in Condition 3 of Theorem 1 holds with $\nu = 0$.

The following corollary is applicable to non-parameterized discrete-time time-varying systems. Indeed, we can write any non-parameterized system $x(k+1) = f(k, x(k))$ as a particular system obtained from a parameterized family of systems (1) with a fixed value of T, such as $T = 1$. If all conditions of the below given theorem hold for the non-parameterized system then we can conclude its UGAS. We are not aware of a reference that presents Matrosov theorem for non-parameterized systems. This result is interesting in situations when the exact discrete-time model of the system is known. Hence, the following result can be also regarded as a discrete-time (non-parameterized) version of the continuous-time result [2, Theorem 1].

Corollary 2 *The system (1) is UGAS if:*
(1) The system (1) is UGS.
(2) Condition 2 of Theorem 1 holds.
(3) There exists $T^* > 0$ such that for all $T \in (0, T^*)$, $i \in \{1, \dots, j\}$, $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ the bound in Condition 3 of Theorem 1 holds with $\nu = 0$.

Proof of Theorem 1: We will establish the characterization of USPAS given in Definition 1. Namely, we will construct $\gamma \in \mathcal{K}_\infty$ (in fact it comes directly from Condition 1 of Theorem 1), and for each pair of strictly positive real numbers (r, ϵ) we will construct $L > 0$ and for each pair of strictly positive numbers (Δ, ν) we will construct $T^* > 0$ so that (2)-(3) hold. While the construction of $L > 0$ and $T^* > 0$ is somewhat technical (see the calculations below) the main idea is this: We use Condition 1 of Theorem 1 to derive a bound like (2) on finite time intervals, and we use Conditions 2 and 3 of Theorem 1 (and their consequence in Proposition 2) to establish that trajectories converge to a small neighborhood of the origin at the end of the finite horizon. After this, an induction argument can be used to derive the bounds (2)-(3) for all appropriate time steps.

Let γ come from Condition 1 of Theorem 1. For arbitrary $a > b > 0$ we define:

(a) $\tilde{\delta}(b) := 1/2\gamma^{-1}(b/2)$; $\tilde{r}(a) := \gamma(a) + 1$. Note that since $\gamma(s) \geq s$, we have that $\tilde{\delta}(b) \leq b/4, \forall b \geq 0$;
(b) Let $\tilde{r}(a)$ generate $\tilde{\mu}(a)$ via Condition 2 of Theorem 1, with $\Delta = \tilde{r}(a)$. Let $\tilde{r}(a), \tilde{\mu}(a)$ and $\tilde{\delta}(b)$ generate $\tilde{K}_i(a, b) > 0$ and $\tilde{\alpha}(a, b)$ via Proposition 2;
(c) Let $\tilde{\eta}(a, b) := \tilde{\mu}(a, b)\left(1 + \sum_{i=0}^{j-1} \tilde{K}_i(a, b)\right)$;
(d) Let $\tilde{L}(a, b)$ be such that $\tilde{L}(a, b) > \frac{2^{j+1}\tilde{\eta}(a,b)}{\tilde{\alpha}(a,b)} + 1$.

Note: For any positive numbers satisfying $0 < b_1 \leq b_2 < a_2 \leq a_1$ we have that $\mathcal{H}(b_2, a_2) \subseteq \mathcal{H}(b_1, a_1)$ and we can choose the above given numbers so that:

$$\begin{aligned} &\tilde{\alpha}(a_1, b_1) \leq \tilde{\alpha}(a_2, b_2); \qquad \tilde{\eta}(a_1, b_1) \geq \tilde{\eta}(a_2, b_2) \\ &\tilde{L}(a_1, b_1) \geq \tilde{\alpha}(a_2, b_2) . \end{aligned} \tag{13}$$

Using the above given functions we generate all the numbers needed in the proof.

(i) Let (r, ϵ) be strictly positive given numbers and without loss of generality assume $r > \epsilon$;
(ii) Let $\delta_1 := \tilde{\delta}(\epsilon)$; $r_1 := \tilde{r}(r)$, $\mu_1 := \tilde{\mu}(r)$, $\alpha_1 := \tilde{\alpha}(r, \epsilon)$, $K_i^1 := \tilde{K}_i(r, \epsilon)$, $\eta_1 := \tilde{\eta}(r, \epsilon)$ and $L_1 := \tilde{L}(r, \epsilon)$;
(iii) Let (Δ, ν) be given and let $\nu_1 := \nu/2$;
(iv) Let $\delta_2 := \tilde{\delta}(\nu_1)$; $\tilde{\Delta} := \tilde{r}(\Delta)$, $\mu_2 := \tilde{\mu}(\Delta)$, $\alpha_2 := \tilde{\alpha}(\Delta, \nu_1)$, $K_i^2 := \tilde{K}_i(\Delta, \nu_1)$, $\eta_2 := \tilde{\eta}(\Delta, \nu_1)$ and $L_2 := \tilde{L}(\Delta, \nu_1)$;
(v) Define $\tilde{\nu} := \min\{\alpha_2/2^j, \nu_1/2, 1\}$;
(vi) Let $(\tilde{\Delta}, \tilde{\nu})$ generate $T_1^* > 0$ via Condition 3 of the theorem;
(vii) Let $L := L_2 + 1$ and $\tilde{\nu}_1$ be such that $\tilde{\nu}_1 \leq \varphi^{-1}(\tilde{\nu})/L$, where φ comes from Condition 1 of the theorem;
(viii) Let $(\tilde{\Delta}, L, \tilde{\nu}_1)$ generate $T_2^* > 0$ via Condition 1 of the theorem;
(ix) Let $T^* := \min\{T_1^*, T_2^*, \tilde{T}, 1\}$, where $\tilde{T}$ comes from Condition 2 of the theorem.

Before we proceed with the proof, note that if $|x_\circ| \leq \delta_1 + \delta_2 \leq \Delta$, $T \in (0, T^*)$ and $k \in [k_\circ, k_\circ + \ell_{L,T}]$ the following holds:

$$\begin{aligned} |\phi_T(k)| &\leq \gamma(|x_\circ|) + \tilde{\nu} \;\leq\; \gamma(\delta_1 + \delta_2) + \tilde{\nu} \\ &\leq \gamma\left(\frac{1}{2}\gamma^{-1}(\epsilon/2) + \frac{1}{2}\gamma^{-1}(\nu_1/2)\right) + \frac{\nu_1}{2} \\ &\leq \frac{\epsilon}{2} + \nu_1 , \end{aligned} \tag{14}$$

where the last inequality follows from the weak triangle inequality for the class $\mathcal{K}$ function γ, i.e. $\gamma(a+b) \leq \gamma(2a) + \gamma(2b), \forall a, b \geq 0$.

As in Definition 1, $\Delta \geq r$, $T \in (0, T^*)$, $k_\circ \geq 0$, $x(k_\circ) = x_\circ$ with $|x_\circ| \leq r$ and the corresponding trajectory is denoted as $\phi_T(k)$. We consider cases $\epsilon \geq \nu_1$ and $\epsilon < \nu_1$.

Case 1: $\epsilon \geq \nu_1$. We consider two sub cases $|x_\circ| \leq \delta_1 + \delta_2$ and $x_\circ \in \mathcal{H}(\delta_1 + \delta_2, r)$.

Case 1a: $|x_\circ| \leq \delta_1 + \delta_2$
From Condition 1 and the definitions (a), (ii), (iv) and (v) and the inequality (14) we can write for all $k \in [k_\circ, k_\circ + \ell_{L,T}]$:

$$|\phi_T(k)| \leq \epsilon/2 + \nu_1 \leq \epsilon + \nu . \tag{15}$$

Case 1b: $x_\circ \in \mathcal{H}(\delta_1+\delta_2, r)$
From $r \le \Delta$, Condition 1 of the theorem, (vii) and (viii), we have for all $k \in [k_\circ, k_\circ + \ell_{L,T}]$:

$$
\begin{aligned}
|\phi_T(k)| &\le \gamma(|x_\circ|) + \varphi(\tilde{\nu}_1 T(k-k_\circ)) \ \le\ \gamma(|x_\circ|) + \varphi(\tilde{\nu}_1 L) \\
&\le \gamma(|x_\circ|) + \tilde{\nu} \ \le\ \gamma(|x_\circ|) + \min\left\{\frac{\nu_1}{2}, 1\right\} , \quad (16)
\end{aligned}
$$

which together with (iv), (v) and (vi) implies that $\phi_T(k) \in B_{r_1}, \forall k \in [k_\circ, k_\circ + \ell_{L,T}]$.

We claim there exists $k_1^* \in [k_\circ+1, k_\circ+\ell_{L_1,T}]$ such that

$$
|\phi_T(k_1^*)| \le \delta_1 + \delta_2 \ . \quad (17)
$$

To prove this define the function $W_T(k,x) := \sum_{i=1}^{j-1} K_i^1 V_T^i(k,x) + V_T^j(k,x)$. From Condition 2 and (ii) we have that for all $(k,x) \in \mathbb{Z} \times B_{r_1}$ that the following holds $|W_T(k,x)| \le \mu_1 \left(1 + \sum_{i=1}^{j-1} K_i^1\right) = \eta_1$. Moreover, since $0 < \nu_1 \le \epsilon < r \le \Delta$, we have that $\mathcal{H}(\delta_1, r_1) \subseteq \mathcal{H}(\delta_2, \tilde{\Delta})$ and we can always pick $\alpha_1, \alpha_2, L_1, L_2$ such that $\alpha_1 \ge \alpha_2$ and $L_2 \ge L_1$ (see (13) and the preceding text). Hence, for all $(k,x) \in \mathbb{Z} \times \mathcal{H}(\delta_1+\delta_2, r_1) \subset \mathbb{Z} \times \mathcal{H}(\delta_1, r_1)$ we have $Z(x, \chi_T(k,x)) \le -\frac{\alpha_1}{2^{j-1}}$ and consequently:

$$
\begin{aligned}
\frac{W_T(k+1, F_T(k,x)) - W_T(k,x)}{T} &\le Z(x, \chi_T(k,x)) + \tilde{\nu} \\
&\le\ -\frac{\alpha_1}{2^{j-1}} + \frac{\alpha_2}{2^j} \le -\frac{\alpha_1}{2^j}. \quad (18)
\end{aligned}
$$

Suppose for the purpose of showing contradiction to (17) that for all $k \in [k_\circ, k_\circ + \ell_{L_1,T}]$ we have $|\phi_T(k)| > \delta_1 + \delta_2 > \delta_1$. Then using (18) along the trajectories of the system we can write $\frac{W_T(k+1, \phi_T(k+1)) - W_T(k, \phi_T(k))}{T} \le -\frac{\alpha_1}{2^j}$, for all $k \in [k_\circ, k_\circ + \ell_{L_1,T}]$. Adding both sides of the above inequality for $k \in [k_\circ, k_\circ + \ell_{L_1,T} - 1]$ and rearranging it we obtain $T\frac{\alpha_1}{2^j}\ell_{L_1,T} \le W_T(k_\circ + \ell_{L_1,T}, \phi_T(k_\circ + \ell_{L_1,T})) - W_T(k_\circ, x_\circ) \le 2\eta_1$. From our choice $T^* \le 1$ and definition of $\ell_{L_1,T}$ we have that $L_1 - 1 \le T\ell_{L_1,T}$ and hence we obtain $\frac{\alpha_1}{2^j}(L_1 - 1) \le 2\eta_1$, which contradicts our choice of L_1 in (ii) with (d).

Now the proof of (3) for $\epsilon \ge \nu_1$ is completed using induction. We let for all initial states with $|x_\circ| \le \delta_1 + \delta_2$ that $k_1^* = 0$. There are two possibilities, either $|\phi_T(k)| \le \delta_1 + \delta_2 \le \epsilon/4 + \nu_1/4$ for all $k \ge k_1^*$ or there exists $\bar{k}_2 > k_1^*$ such that $\left|\phi_T(\bar{k}_2)\right| > \delta_1 + \delta_2$. For definiteness assume that $\bar{k}_2$ is minimum such integer. Hence, we have that $|\phi_T(k)| \le \delta_1 + \delta_2, \forall k \in [k_1^*, \bar{k}_2 - 1]$ and $\left|\phi_T(\bar{k}_2)\right| > \delta_1 + \delta_2$. Then there exists $k_2^* \in [\bar{k}_2 + 1, \bar{k}_2 + \ell_{L_1,T}]$ such that the following hold:

$$
\begin{aligned}
|\phi_T(k)| &\le \gamma(\delta_1 + \delta_2) + \frac{\nu_1}{2} \\
&\le \frac{\epsilon}{2} + \nu_1, \qquad \forall k \in [\bar{k}_2, \bar{k}_2 + \ell_{L_1,T}] \quad (19) \\
|\phi_T(k_2^*)| &\le \delta_1 + \delta_2 \ . \quad (20)
\end{aligned}
$$

Indeed, (19) is proved using the same calculations as in Case 1a starting at the initial state $\phi_T(\bar{k}_2 - 1)$ and the fact that the bound in (15) holds on intervals of length $\ell_{L,T}$ (note also that since $L = L_2 + 1 \ge L_1 + 1$ and $T^* \le 1$, we have that $\ell_{L,T} \ge \ell_{L_1,T} + 1$). Inequality (20) is proved using the same calculations as in Case 1b starting from the initial state $\phi_T(\bar{k}_2)$. One can generate a sequence of such intervals and by induction we prove that:

$$
\left\{\begin{array}{l} x_\circ \in B_r, \\ k \ge k_\circ + \ell_{L_1,T} \end{array}\right\} \Longrightarrow |\phi_T(k)| \le \epsilon/2 + \nu_1 \le \epsilon + \nu \quad (21)
$$

which shows that (3) holds since $k_1^* \le \ell_{L_1,T}$ and L_1 depends only on (r, ϵ).

Since (21) holds for all $\epsilon \ge \nu_1$ and $r \le \Delta$, it holds in particular for $r = \Delta$ and $\epsilon = \nu_1$. Hence, for this choice we have from $\nu_1 < \nu$ and previous calculations that:

$$
\begin{aligned}
&\left\{\begin{array}{l} |x_\circ| \le \Delta, \\ k \in [k_\circ, k_\circ + \ell_{L_2,T}] \end{array}\right\} \Longrightarrow |\phi_T(k)| \le \gamma(|x_\circ|) + \nu \\
&|x_\circ| \le \Delta, k \ge k_\circ + \ell_{L_2,T} \Longrightarrow |\phi_T(k)| \le 2\nu_1 = \nu \ ,
\end{aligned}
$$

which implies that (2) holds.

Case 2: $\epsilon < \nu_1$. Again since the above calculations hold for all $\epsilon \ge \nu_1$, they hold, in particular, for $\epsilon = \nu_1$. Let $L_3 := \tilde{L}(r, \nu_1)$. Then, since $\epsilon < \nu_1$ we can write using (14) that for all $k \ge k_\circ + \ell_{L_3,T}$ we have $|\phi_T(k)| \le \epsilon/2 + \nu_1 \le \nu_1/2 + \nu_1 < 2\nu_1 = \nu$. Note that the time $\ell_{L,T}$ needed for $\phi_T(k)$ to converge to the ball $B_{\nu+\epsilon}$ is smaller than the time $\ell_{L_3,T}$ needed to converge to the ball $B_{2\nu_1} = B_\nu$. This completes the proof of (3).

4 Sampled-data systems

In this section we state an extension of [4, Theorem 2] that can be used to deal with time-varying systems using Matrosov functions (time-invariant systems using Lyapunov functions were considered in [4, Theorem 2]). This result motivates the stability definitions that we use. Consider the class of time-varying systems:

$$
\dot{x}(t) = f(t, x(t), u(t)) \quad (22)
$$

where $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^m$ are respectively the state and control input. Using the assumption of sampler and zero order hold (the control signal is piecewise constant), we can write the exact discrete-time model of (22) whenever the solutions are well defined:

$$
\begin{aligned}
x(k+1) = x(k) &+ \int_{kT}^{(k+1)T} f(\tau, x(\tau), u(k)) d\tau \quad (23) \\
&=: F_T^e(k, x(k), u(k)) \ ,
\end{aligned}
$$

where we denoted by $x(t)$ the solution of the initial value problem (22) at time t with given $t_\circ = kT$, $x_0 = x(k)$ and $u(k)$. We emphasize that F_T^e is not known in most cases. Indeed, in order to compute F_T^e we have to solve the initial value problem (22) analytically and this is usually impossible since f in (22) is nonlinear. Hence, we will use an approximate discrete-time model of the plant to design a discrete-time controller for the original plant (22). Approximate discrete-time models can be obtained using different methods, such as a classical Runge-Kutta numerical integration scheme for the initial value problem (22) (for time-varying systems see [8,7] and for time-invariant systems see [10,11,16]. Approximate discrete-time models are denoted as

$$x(k+1) = F_T^a(k, x(k), u(k)) \ . \tag{24}$$

For instance, if f is locally Lipschitz in t and x, the Euler approximate model can be defined as $x(k+1) = x(k) + Tf(kT, x(k), u(k))$ and it can be shown to be an $O(T^2)$ approximation of the exact discrete-time model. On the other hand, if f is measurable in t, then a modified "Euler" model that is $O(T^2)$ approximation of the exact model is given by $x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(\tau, x(k), u(k))d\tau$ (see [8]). In our work, the sampling period T is assumed to be a design parameter which can be arbitrarily assigned. Since we are dealing with a family of approximate discrete-time models F_T^a, parameterized by T, in order to achieve a certain objective we need in general to obtain a family of controllers, parameterized by T. We consider a family of dynamic feedback controllers

$$z(k+1) = G_T(k, x(k), z(k)); u(k) = u_T(k, x(k), z(k)) \tag{25}$$

where $z \in \mathbb{R}^{n_z}$. We also denote $\tilde{x} = (x^T \ z^T)^T \in \mathbb{R}^{\tilde{n}}$, where $\tilde{n} = n_x + n_z$.

We emphasize that if the controller (25) stabilizes the approximate model (24) for all small T, this does not guarantee that the same controller would approximately stabilize the exact model (23) for all small T. Several examples that illustrate this phenomenon can be found in [4]. Two different results were presented in [4] for time-invariant systems that can be used to deduce about stability of the exact discrete-time model from the corresponding properties of the approximate discrete-time model. The following property that has been adapted from the numerical analysis literature, such as [16], plays a crucial role in our next result:

Definition 4 *We say that F_T^e is one-step consistent with F_T^a if for any pair of strictly positive numbers (Δ_x, Δ_u) there exist $T^* > 0$ and $\rho \in \mathcal{K}$ such that for all $k \geq 0$, $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ and $T \in (0, T^*)$ we have:*

$$|F_T^e(k, x, u) - F_T^a(k, x, u)| \leq T\rho(T) \ . \tag{26}$$

We emphasize that the one-step consistency property can be checked by using only the approximate model (the function $F_T^a(k, x, u)$) and the continuous-time plant model (the function $f(t, x, u)$) and hence it is a checkable property, although we typically can not compute $F_T^e(k, x, u)$ – see, [4] for a sufficient condition for one-step consistency of time-invariant systems; it is straightforward to change these conditions to cover also the time-varying systems that we consider by using results in [8,7].

The following result can be regarded as an extension of the Lyapunov result [4, Theorem 2] to time varying systems via Matrosov Theorem 1. In the sequel we use the notation $\Delta W_T^\star := W_T(k+1, F_T^\star(k, x, u_T(x, z)), G_T(k, x, z)) - W_T(k, x, z)$, where $\star \in \{a, e\}$.

Corollary 3 *Suppose that the following are true:*
(1) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that the following is true: for any (Δ, δ) there exists $T^* > 0$, $M > 0$ and $L > 0$ such that for any $T \in (0, T^*)$ there exists $W_T : \mathbb{Z} \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}_{\geq 0}$ so that for all $(x, z), (x_1, z), (x_2, z) \in B_\Delta$, $k \geq 0$ and $T \in (0, T^*)$ the following hold:

$$\begin{aligned} \alpha_1(|(x, z)|) \ &\leq \ W_T(k, x, z) \leq \alpha_2(|(x, z)|) \\ &\Delta W_T^a \leq T\delta \\ |W_T(k, x_1, z) - W_T(k, x_2, z)| &\leq L\,|x_1 - x_2| \\ &|u_T(k, x, z)| \leq M \ . \end{aligned} \tag{27}$$

(2) Conditions of Theorem 1 hold for the approximate closed loop system (24), (25) with the functions $V_T^i(k, x, z)$, $\chi_T(k, x, z)$ and continuous functions $Y_i(x, z, \xi)$ and for any $\Delta > 0$ there exist $T^* > 0$ and $L > 0$ such that for all $(x_1, z), (x_2, z) \in B_\Delta$, $k \geq 0$ and $T \in (0, T^*)$ we have:

$$\left|V_T^i(k, x_1, z) - V_T^i(k, x_2, z)\right| \leq L\,|x_1 - x_2| \ . \tag{28}$$

(3) F_T^e is one-step consistent with F_T^a.

Then, the exact discrete-time closed-loop system (23), (25) satisfies all conditions of Theorem 1 and in particular (23), (25) is USPAS.

Sketch of the proof: Conditions 1 and 3 of the corollary are used to show that the following holds for approximate model in a semiglobal practical sense$\Delta W_T^e \leq T\delta$. This in turn implies SB for the exact closed loop system and hence Condition 1 of Theorem 1 holds. Conditions 2 and 3 and boundedness of $u_T(k, x, z)$ are used in a similar way to show that Conditions 2 and 3 of Theorem 1 hold.

Remark 1 *We note that our results imply under weak conditions (such as the weak Lipschitzness of (22) uniformly in t) that if the the exact discrete-time closed-loop system (23), (25) is USPAS then the sampled-data closed loop system is also USPAS (see [5] for more details).*

Remark 2 *Notice that if the approximate model satisfies the Matrosov conditions that would guarantee its UGAS or USAS (see Corollaries 1 and 2), we can only conclude from Corollary 3 that the exact model is USPAS. This is because: (i) the Matrosov conditions may be too weak in general to guarantee preservation of the UGAS or USAS property for the exact model if the approximate model has the corresponding stability property; (ii) our definition of one-step consistency is too weak to guarantee in general the stronger properties of UGAS or USAS for the exact model if the corresponding property holds for the approximate model. Hence, in order to state UGAS and USAS properties for the exact model we would have to strengthen the conditions of Theorem 1, as well as strengthen the one-step consistency property in Definition 4. While this is possible to do, the conditions under which these results hold are normally more restrictive and we have not pursued this avenue in the current paper.*

5 MRAC via approximate models

We consider the problem of sampled-data, adaptive tracking control (sometimes called model reference adaptive control or MRAC for nonlinear systems in the form

$$\dot{\xi} = f(\xi) + g(\xi)\left[u + h(\xi)^\top \theta\right] \; . \tag{29}$$

The parameter vector θ is unknown. The functions f,g and h are supposed to be locally Lipschitz. We will assume we can find a family of certainty equivalence discrete-time, tracking feedback laws for the Euler approximation of the system (29):

$$\begin{aligned} \xi(k+1) = {} & \xi(k) + Tf(\xi(k)) \\ & + Tg(\xi(k))\left[u(k) + h(\xi(k))^\top \theta\right] \; . \end{aligned} \tag{30}$$

In other words, we will assume:

Assumption 1 *For a given family of uniformly bounded reference state trajectories $\xi_{r,T}$ satisfying*

$$\xi_{r,T}(k+1) = \xi_{r,T}(k) + Ts(\xi_{r,T}(k), k, T) \tag{31}$$

for some function s, there exists a family of feedbacks $\widetilde{u}(\xi, \xi_{r,T}, T)$ such that, with the definition,

$$\begin{aligned} F_T(k,e) := {} & -Ts(\xi_{r,T}(k), k, T) + e + Tf(e + \xi_{r,T}(k)) \\ & + Tg(e + \xi_{r,T}(k))\widetilde{u}(e + \xi_{r,T}(k), \xi_{r,T}(k), T) \end{aligned}$$

there exist $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_4 \in \mathcal{K}_\infty$ and a continuous positive definite $\widetilde{\alpha}_3 : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ such that for every pair of strictly positive real numbers $(\widetilde{\Delta}, \widetilde{\nu}, \widetilde{\mu})$ there exists $T^* > 0$ and for each $T \in (0, T^*)$ there exists $W_T : \mathbb{Z} \times \mathbb{R}^n$ such that the following hold:

$$\widetilde{\alpha}_1(|e|) \; \leq \; W_T(k,e) \leq \widetilde{\alpha}_2(|e|) \tag{32}$$

$$\frac{W_T(k+1, F_T(k,e)) - W_T(k,e)}{T} \leq -\widetilde{\alpha}_3(e) + \widetilde{\nu} \tag{33}$$

$$|\nabla_e W_T(k,e)| \leq \widetilde{\alpha}_4(|e|) + \widetilde{\mu} \; , \tag{34}$$

for all $T \in (0, T^*)$, $k \geq 0$, $|e| \leq \widetilde{\Delta}$.

Under this assumption, we will implement the following discrete-time controller

$$u(k) = \widetilde{u}(\xi(k), \xi_{r,T}(k)) - h(\xi(k))^\top \hat{\theta}(k) \tag{35}$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) - Th(\xi(k))g(\xi(k))^\top \nabla_e W_T(k) \; , \tag{36}$$

where $\nabla_e W_T(k) := \nabla_e W_T(k, F_T(k, \xi(k) - \xi_{r,T}(k)))$. For the purposes of obtaining USPAS, we will assume the following persistency of excitation condition:

Assumption 2 *There exists $T^* > 0$, $L > 0$ and $\mu > 0$ such that, for all $k \in \mathbb{Z}$ and $T \in (0, T^*)$, we have*

$$T \sum_{i=k}^{k+\ell_{L,T}} |g(\xi_{r,T}(i))h(\xi_{r,T}(i))^\top x_2| \geq \mu |x_2| \; . \tag{37}$$

Under Assumptions 1 and 2, and with the definitions $x_1 := \xi - \xi_{r,T}$, $x_2 := \theta - \hat{\theta}$, $x := [x_1^T \; x_2^T]^T$

$$\begin{aligned} A := A(T,k,x) &:= \frac{F_T(k,x_1) - x_1}{T} \\ B := B(T,k,x) &:= g(x_1 + \xi_{r,T}(k))h(x_1 + \xi_{r,T}(k))^\top x_2 \\ B_\circ := B_\circ(T,k,x_2) &:= B(T,k,x)|_{x_1=0} \\ C := C(T,k,x) &:= -h^\top(k)g(k)^\top \nabla_e W_T^\top(k+1, F_T(k,x_1)) \\ B_\circ^+ := B_\circ^+(T,k,x_2) &:= B_\circ(T,k+1,x_2 + TC(T,k,x)) \; , \end{aligned}$$

where $h(k) := h(x_1 + \xi_{r,T}(k))$, $g(k) := g(x_1 + \xi_{r,T}(k))$, the Euler approximation of the closed-loop system consisting of (30), (31), (35) and (36) in (x_1, x_2) coordinates has the form

$$\begin{aligned} x_1(k+1) &= x_1(k) + T[A(T,k,x(k)) + B(T,k,x(k))] \\ x_2(k+1) &= x_2(k) + TC(T,k,x(k)) \end{aligned} \tag{38}$$

and the following three properties hold:

Property 1 *There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite $\alpha_3 : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ such that for every pair of strictly positive real numbers (Δ, ν) there exists $T^* > 0$ and for each $T \in (0, T^*)$ there exists $V_T : \mathbb{Z} \times \mathbb{R}^n$ such that the following hold:*

$$\alpha_1(|x|) \; \leq \; V_T(k,x) \leq \alpha_2(|x|) \tag{39}$$

$$\frac{V_T(k+1, F_T(k,x)) - V_T(k,x)}{T} \leq -\alpha_3(x_1) + \nu \; , \tag{40}$$

for all $T \in (0, T^*)$, $k \geq 0$, $|x| \leq \Delta$.

Property 2 *There exist continuous nondecreasing functions $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, i = 1, 2, 3$ such that $\rho_i(0) = 0, i = 1, 2$ and for any strictly positive numbers $\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3$ there exists $\tilde{T} > 0$ such that for all $T \in (0, \tilde{T})$, $k \geq 0$ and $x \in \mathbb{R}^n$ we have:*

$$
\begin{aligned}
&\max\left\{|B_\circ|, |B_\circ^+|, \frac{|B_\circ^+ - B_\circ|}{T}\right\} \leq \rho_3(|x|) + \widetilde{\nu}_1 \\
&\frac{|B_\circ(T, k, x_2 + TC) - B_\circ|}{T} \leq \rho_3(|x|)\rho_1(|x_1|) + \widetilde{\nu}_2 \\
&|B - B_\circ| \leq \rho_3(|x|)\rho_1(|x_1|) + \widetilde{\nu}_2 \\
&\max\{|A|, |C|\} \leq \rho_3(|x|)\rho_2(|x_1|) + \widetilde{\nu}_3 \ .
\end{aligned}
\tag{41}
$$

Property 3 *There exists a continuous, positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and $\hat{T} > 0$ such that for all $x_2 \in \mathbb{R}^{n_2}$, $T \in (0, \hat{T})$ and $k \geq 0$ we have:*

$$
T \sum_{i=k}^{\infty} e^{(k-i)T} |B_\circ(T, i, x_2)| \geq \alpha(|x_2|) \ . \tag{42}
$$

The reason why we introduce Properties 1-3 is because the main result of this section (Theorem 2 given below) applies to more general problems than the MRAC problem we introduced in the begining of this section. Moreover, it is obvious that Assumption 2 implies Property 3 and we can further prove the following:

Proposition 3 *If Assumption 1 holds with W_T such that $\nabla_e W_T$ is locally Lipschitz, uniformly in small T. Then Property 1 holds for the system (30), (31), (35) and (36) (in (x_1, x_2) coordinates) with $V_T(k, x) = W_T(k, x_1) + \frac{1}{2} x_2^T x_2$.*

Proposition 4 *If Assumption 1 holds with W_T such that $\nabla_e W_T$ is locally Lipschitz, uniformly in small T. Then Property 2 holds for the system (30), (31), (35) and (36) (in (x_1, x_2) coordinates).*

Remark 3 *The condition stated in Property 3 is like the so called uniform δ persistency of excitation (uδ-PE) that was introduced in [6] (see also [2,1] for more recent developments using this notion).*

We can now establish the following result which covers the solution to the MRAC problem:

Theorem 2 *Suppose that Properties 1, 2 and 3 hold. Then the system (38) satisfies all conditions of Theorem 1 and hence it is USPAS.*

Proof of Theorem 2: We define $\chi_T(k, x) := B_\circ$; $V_T^1(k, x) := V_T(k, x)$; $V_T^2(k, x) := -x_1^T B_\circ$; $V_T^3(k, x) := -T \sum_{i=k}^{\infty} e^{(k-i)T} |B_\circ(T, i, x_2)|$; $Y_1(x, \chi_T(k, x)) := -\alpha_3(x_1)$; $Y_2(x, \chi_T(k, x)) := |x_1| \rho_3(|x|) + [\rho_1(|x_1|) + \rho_2(|x_1|)]\rho_3^2(|x|) - |B_\circ|^2$; $Y_3(x, \chi_T(k, x)) := |B_\circ| + \exp(1)\rho_3(|x|)\rho_1(|x_1|) - \frac{1}{2}\alpha(|x_2|)$.

Condition 1 of Theorem 1 follows from Property 1. It is not hard to see that Condition 2 of Theorem 1 also holds from the properties of the functions that we have chosen.

The only thing left to prove is Condition 3 of Theorem 1. Let arbitrary strictly positive (Δ, ν) be given. Let $\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3$ be such that the following holds: $\max\{2 \cdot [\Delta\widetilde{\nu}_1 + 2\widetilde{\nu}_1\rho_1(\Delta)\rho_3(\Delta) + \widetilde{\nu}_3\rho_3(\Delta) + \widetilde{\nu}_1\widetilde{\nu}_3 + \widetilde{\nu}_2\rho_3(\Delta) + \widetilde{\nu}_1\widetilde{\nu}_2], \exp(1)\widetilde{\nu}_2\} \leq \nu$. Let $\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3$ generate $\tilde{T} > 0$ come from Property 2. Let $\hat{T} > 0$ come from Property 3. Let (Δ, ν) generate $T_1^* > 0$ via Property 1. We define $T_2^* := \frac{\nu}{2[\rho_3(\Delta) + \widetilde{\nu}_1]^2}$. Let $T_3^* > 0$ be such that $\frac{1}{2} \leq \frac{1 - e^{-T}}{T}, \forall T \in (0, T_3^*)$. Define $T^* := \min\{1, \tilde{T}, \hat{T}, T_1^*, T_2^*, T_3^*\}$. Let $|x| \leq \Delta$, $k \geq 0$ and $T \in (0, T^*)$. We now show that the bound in Condition 3 of Theorem 1 holds for V_T^1, V_T^2 and V_T^3. Actually, it follows directly from Property 1 that

$$
\frac{\Delta V_T^1}{T} = \frac{\Delta V_T}{T} \leq -\alpha_3(x_1) + \nu \ . \tag{43}
$$

Using (41) we can write:

$$
\begin{aligned}
\frac{\Delta V_T^2}{T} &= \frac{-(x_1 + TA + TB)^\top B_\circ^+ + x_1^\top B_\circ}{T} \\
&= -\frac{x_1^\top (B_\circ^+ - B_\circ)}{T} - A^\top B_\circ^+ - B^\top B_\circ^+ \\
&= -\frac{x_1^\top (B_\circ^+ - B_\circ)}{T} - A^\top B_\circ^+ - (B - B_\circ)^\top B_\circ^+ \\
&\quad - B_\circ^\top (B_\circ^+ - B_\circ) - B_\circ^\top B_\circ \\
&\leq |x_1| \frac{|B_\circ^+ - B_\circ|}{T} + |A| \, |B_\circ^+| + |B - B_\circ| \, |B_\circ^+| \\
&\quad + |B_\circ| \, |B_\circ^+ - B_\circ| - |B_\circ|^2 \\
&\leq |x_1| [\rho_3(|x|) + \widetilde{\nu}_1] + [\rho_3(|x|) + \widetilde{\nu}_1] \cdot \\
&\quad [\rho_3(|x|)\rho_2(|x_1|) + \widetilde{\nu}_3] \\
&\quad + [\rho_3(|x|)\rho_1(|x_1|) + \widetilde{\nu}_2] \cdot [\rho_3(|x|) + \widetilde{\nu}_1] \\
&\quad + T[\rho_3(|x|) + \widetilde{\nu}_1]^2 - |B_\circ|^2 \\
&\leq |x_1| \rho_3(|x|) + \rho_3^2(|x|)\rho_2(|x_1|) + \rho_3^2(|x|)\rho_1(|x_1|) \\
&\quad - |B_\circ|^2 + \Delta\widetilde{\nu}_1 + 2\widetilde{\nu}_1\rho_1(\Delta)\rho_3(\Delta) \\
&\quad + \widetilde{\nu}_3\rho_3(\Delta) + \widetilde{\nu}_1\widetilde{\nu}_3 + \widetilde{\nu}_2\rho_3(\Delta) + \widetilde{\nu}_1\widetilde{\nu}_2 \\
&\quad + T[\rho_3(\Delta) + \widetilde{\nu}_1]^2 \\
&= Y_2(x, \chi_T(k, x)) + \nu \ ,
\end{aligned}
$$

where the last step follows from the fact that $\Delta\widetilde{\nu}_1 + 2\widetilde{\nu}_1\rho_1(\Delta)\rho_3(\Delta) + \widetilde{\nu}_3\rho_3(\Delta) + \widetilde{\nu}_1\widetilde{\nu}_3 + \widetilde{\nu}_2\rho_3(\Delta) + \widetilde{\nu}_1\widetilde{\nu}_2 \leq \frac{\nu}{2}$ and $T[\rho_3(\Delta) + \widetilde{\nu}_1]^2 \leq \frac{\nu}{2}$. This completes the proof of the bound in Condition 3 for V_T^2.

Since dependence of $B_\circ$ and $B_\circ^+$ on i is important in the calculations below, we slightly change the notation $B_\circ^+(i) := B_\circ(T,i,x_2+TC)$ and $B_\circ(i) := B_\circ(T,i,x_2)$. Also, in the calculations below we use the following fact:

$$T\sum_{i=k+1}^{\infty} e^{(k-i)T} \le e^{kT}\int_{kT}^{\infty} e^{-t}dt = 1 \ .$$

Using the above fact, our choice of T^* and (41) we show next that the appropriate bound holds for the function V_T^3:

$$\begin{aligned}
\frac{\Delta V_T^3}{T} &= -\sum_{i=k+1}^{\infty} e^{(k+1-i)T}\left|B_\circ^+(i)\right| + \sum_{i=k}^{\infty} e^{(k-i)T}\left|B_\circ(i)\right| \\
&= \left|B_\circ(k)\right| - \sum_{i=k+1}^{\infty} e^{(k+1-i)T}\left|B_\circ^+(i)\right| \\
&\quad + e^{-T}\sum_{i=k+1}^{\infty} e^{(k+1-i)T}\left|B_\circ(i)\right| \\
&= \left|B_\circ(k)\right| - \sum_{i=k+1}^{\infty} e^{(k+1-i)T}\left(\left|B_\circ^+(i)\right| - \left|B_\circ(i)\right|\right) \\
&\quad - \sum_{i=k+1}^{\infty} e^{(k+1-i)T}\left|B_\circ(i)\right| \qquad (44) \\
&\quad + e^{-T}\sum_{i=k+1}^{\infty} e^{(k+1-i)T}\left|B_\circ(i)\right| \\
&\le \left|B_\circ(k)\right| + T[\rho_3(|x|)\rho_1(|x_1|) + \widetilde{\nu}_2]e^T\sum_{i=k+1}^{\infty} e^{(k-i)T} \\
&\quad -(1-e^{-T})\sum_{i=k+1}^{\infty} e^{(k+1-i)T}\left|B_\circ(i)\right| \\
&\le \left|B_\circ(k)\right| + e^{T^*}[\rho_3(|x|)\rho_1(|x_1|) + \widetilde{\nu}_2] \\
&\quad -\frac{1-e^{-T}}{T}\alpha(|x_2|) \\
&\le \left|B_\circ(k)\right| + \exp(1)[\rho_3(|x|)\rho_1(|x_1|) + \widetilde{\nu}_2] - \frac{1}{2}\alpha(|x_2|) \\
&= Y_3(x, \chi_T(k,x)) + \nu \ ,
\end{aligned}$$

which completes the proof.

6 Conclusions

We presented a Matrosov theorem for parameterized discrete-time time-varying models that facilitates controller design for sampled-data nonlinear systems via their approximate discrete-time models. Our main theorem is an analogue of the continuous-time result in [2] that generalized the classical Matrosov theorem for continuous-time systems. We have also related stability conditions needed on an approximate model to the stability properties of the exact discrete-time model and for this we used the notion of one-step consistency that is adapted from the numerical analysis literature. We presented an application of our results to a class of models that arise when an approximate discrete-time model of the plant is used to design a model reference adaptive controller.

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References

[1] D. Popović A.Loria, E. Panteley and A. R. Teel. An extension of matrosov's theorem with application to stabilization of nonholonomic control systems. In *Proc. Conference on Decision and Control.*

[2] D. Popović A.Loria, E. Panteley and A. R. Teel. δ-persistency of excitation: a necessary and sufficient condition for uniform attractivity. page submitted, 2002.

[3] E. Panteley A.R. Teel and A. Loria. Integral characterizations of uniform asymptotic and exponential stability with applications. *Math. Contr. Sig. Syst.*, 12:177–201, 2002.

[4] A. Teel D. Nešić and P. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Syst. & Contr. Letters*, 38:259–270, 1999.

[5] A. Teel D. Nešić and E. Sontag. Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems. *Syst. & Contr. Letters Vol.*, 38:49–60, 1999.

[6] A. Loría. E. Panteley and A.R.Teel. Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Trans. Automat. Contr.*, 46:1874–1886, 2001.

[7] R. Ferretti. High order approximations of linear control systems via runge-kutta schemes. *Computing*, 58:351–364, 1997.

[8] L. Grune and P.E.Kloeden. Higher order numerical schemes for affinely controlled nonlinear systems. *Numer. Math.*, 89:669–690, 2001.

[9] V. M. Matrosov. On the stability of motion. *J. Appl. Math. Mech.*, 26:1337–1353, 1962.

[10] S. Monaco and D. Normand-Cyrot. On the sampling of a linear analytic control system. In *Proc. Conference on Decision and Control.*

[11] S. Monaco and D. Normand-Cyrot. On nonlinear digital control. In D. Normand-Cyrot A.J. Fossard, editor, *Nonlinear Systems*, volume 3, pages 111–136. Chapman & Hall, Chapman & Hall, 1995.

[12] P. Habets N. Rouche and M. Laloy. *Stability theory by Lyapunov's direct method.* Springer Verlag, New York, 1977.

[13] D. Nešić and A.R. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Trans. Automat. Contr.*, page submitted, 2002.

[14] B. Paden and R. Panja. Globally asymptotically stable pd+ controller for robot manipulators. *Int. J. Contr.*, 47:1697–1712, 1988.

[15] N. Rouche and J. Mawhin. *Ordinary differential equations II: Stability and periodical solutions.* Pitman publishing Ltd, London, 1980.

[16] A.M. Stuart and A.R. Humphries. *Dynamical systems and numerical methods.* Cambridge University Press, Cambridge, 1996.

[17] A. R. Teel and L. Praly. A smooth lyapunov function from a class $||l$ estimate involving two positive semidefinite functions. *ESAIM: Control, Optimization and Calculus of Variations*, 1999.

7 Appendix

Proof of Proposition 3: Consider the Lyapunov function: $V_T(k, x_1, x_2) := W_T(k, x_1) + \frac{1}{2}x_2^\top x_2$. To simplify notation below, we omit the arguments of all functions unless they are really needed in the proof. Notice that $x_1 + TA + TB = F_T + TB$. Let (Δ, ν) be given. Let Assumption 1 with $(\Delta, \nu/2)$ generate $T_1^* > 0$. Let $L > 0$ be the Lipschitz constant for $\nabla_e W_T$ on the set $|x_1| \leq \Delta + 1$. Hence, if $|x_1| \leq \Delta$ and $T \leq \frac{1}{|A|+|B|}$, then we have $\max\{|F_T|, |F_T + TB|\} \leq \Delta + 1$. Define

$$T_2^* = \min\left\{\frac{\nu}{2L|B| + |C|^2}, \frac{1}{|A| + |B|}\right\} ,$$

and finally $T^* = \min\{T_1^*, T_2^*\}$. Consider arbitrary $|(x_1, x_2)| \leq \Delta$ and $T \in (0, T^*)$. Using the Mean Value Theorem, we can write along the solutions of the closed loop system:

$$\begin{aligned}
\frac{\Delta V_T}{T} &= \frac{W_T(k+1, x_1 + TA + TB) - W_T(k, x_1)}{T} \\
&\quad + \frac{(x_2 + TC)^\top (x_2 + TC) - x_2^\top x_2}{2T} \\
&= \frac{W_T(k+1, F_T) - W_T(x_1)}{T} + C^T x_2 + \frac{T}{2}|C|^2 \\
&\quad + \frac{W_T(k+1, F_T + TB) - W_T(k+1, F_T)}{T} \\
&= \frac{W_T(k+1, F_T) - W_T(k, x_1)}{T} - \nabla_e W_T(k+1, F_T) ghx_2 \\
&\quad + \nabla_e W_T(k+1, F_T + c_1 TB) ghx_2 + \frac{T}{2}|C|^2 ,
\end{aligned}$$

where $c_1 \in (0, 1)$. Since $\nabla_e W_T$ is locally Lipschitz uniformly in small T, there exists $L > 0$ such that:

$$|-\nabla_e W_T(k+1, F_T) + \nabla_e W_T(k+1, F_T + c_1 TB)| \leq TL|B| .$$

Hence, we can write:

$$\begin{aligned}
\frac{\Delta V_T}{T} &\leq -\alpha_1(x_1) + \frac{\nu}{2} + TL|B| + \frac{T}{2}|C|^2 \\
&\leq -\alpha_1(x_1) + \nu . \qquad (45)
\end{aligned}$$