A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models

D.Nešić and A.R.Teel

Abstract

A unified framework for design of stabilizing controllers for sampled-data differential inclusions via their approximate discrete-time models is presented. Both fixed and fast sampling are considered. In each case, sufficient conditions are presented which guarantee that the controller that stabilizes a family of approximate discrete-time plant models also stabilizes the exact discrete-time plant model for sufficiently small integration and/or sampling periods. Previous results in the literature are extended to cover: (i) continuous-time plants modeled as differential inclusions; (ii) general approximate discrete-time plant models; (iii) dynamical discontinuous controllers modeled as difference inclusions; (iv) stability with respect to closed arbitrary (not necessarily compact) sets.

I. Introduction

A. Background

In the vast literature on nonlinear control design, an area that has received scant attention is sampled-data control. In this problem, a continuous time plant is typically controlled by a discrete-time feedback algorithm. A sample and hold device provides the interface between continuous time and discrete-time. One way to address sampled-data control is to implement a continuous time control algorithm with a sufficiently small sampling period. However, the hardware used to sample and hold the plant measurements or compute the feedback control action may make it impossible to reduce the sampling period to a level that guarantees acceptable closed-loop performance. In this case, it becomes interesting to investigate the application of sampled-data control algorithms based on a discrete-time model of the process.

One reason that the sampled-data nonlinear control problem is difficult is because exact discrete-time models of continuous time processes are typically impossible to compute. So the typical procedure is to:

1. develop a parameterized family of approximate discrete-time models, where the family of approximate models approaches the exact model as the parameter (e.g., integration and/or sampling period) converges to zero;
2. design a corresponding family of discrete-time controllers;
3. pick the modeling parameter small enough to guarantee stability of the exact nonlinear sampled-data system.

While it is clear that the first two steps pose very interesting and challenging problems, the issues associated with the third, seemingly innocuous, step are much more subtle and will be investigated in this paper. Our contribution will be to provide criteria that can be used in the first two steps to guarantee that the third step is possible. In the next section we motivate our work by providing three examples where it is not possible to accomplish the third step in the above procedure. We also provide an example that motivates using algorithms based on discrete-time models rather than simply implementing a continuous time algorithm by sample and hold.

Research supported by the Australian Research Council under the large grants scheme. Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3052, Victoria, Australia.

Research supported in part by the AFOSR under grant F49620-00-1-0106 and the NSF under grant ECS-9988813. CCEC, Electrical and Computer Engineering Department, University of California, Santa Barbara, CA, 93106-9560, USA.

B. Motivating examples

B.1 When things go wrong

In the following three examples, we design control laws for a continuous time process based on an approximate discrete-time model. The approximate models are "consistent" in that they approach the exact discrete-time model in the limit as a modeling parameter tends to zero. Moreover, the control laws, which are also parameterized by the discrete-time modeling parameter, globally exponentially or asymptotically stabilize the origin of the approximate model. Nevertheless, the origin of the closed-loop using the exact discrete-time model is exponentially unstable, or at least not approximately attractive, no matter how small the modeling parameter is. The problem is that the family of discrete-time closed loop systems does not have the proper robustness to account for the mismatch between the exact and approximate discrete-time plant models. Each example which we discuss has a different indicator of insufficient robustness. Our main contribution will be to show that if these indicators are ruled out then robustness to the mismatch between approximate and discrete-time models can be guaranteed.

Control with excessive force

We consider the sampled data control of the triple integrator (this example was taken from [36])

$$\dot{x}_1 = x_2 \ , \quad \dot{x}_2 = x_3 \ , \quad \dot{x}_3 = u \ . \tag{1}$$

While the exact discrete-time model of this system can be computed, we base our control algorithm on the family of Euler approximate discrete-time models in order to illustrate possible pitfalls in control design based on approximate discrete-time models. The family of Euler approximate discrete-time models is

$$x_1^+ = x_1 + Tx_2 \ , \quad x_2^+ = x_2 + Tx_3 \ , \quad x_3^+ = x_3 + Tu \ . \tag{2}$$

A minimum time dead beat controller for the Euler discrete-time model is given by

$$u = \alpha_T(x) = \left(-\frac{x_1}{T^3} \ -\frac{3x_2}{T^2} \ -\frac{3x_3}{T} \right) \ . \tag{3}$$

The closed loop system (2)-(3) has all poles equal to zero for all $T > 0$ and hence this discrete-time Euler-based closed loop system is asymptotically stable for all $T > 0$. On the other hand, the closed loop system consisting of the exact discrete-time model of the triple integrator and controller (3) has a pole at ≈ -2.644 for all $T > 0$. Hence, the closed-loop sampled-data control system is unstable for all $T > 0$ and the third step of the proposed control design procedure is impossible to accomplish. The approximate closed-loop system contains two indicators that its robustness may not be sufficient to account for the mismatch between the approximate and exact discrete-time plant models:

- *Nonuniform bound on overshoot:* The solutions of the family of approximate models with the given controller satisfy for all $T > 0$ an estimate of the following type: $|\phi_T(k, x_\circ)| \leq b_T e^{-kT} |x_\circ|$, $\forall k \in \mathbb{N}$ where $b_T \to \infty$ as $T \to 0$. Hence, the overshoot in the stability estimate for the family of approximate models is not uniformly bounded in T.

- *Nonuniform bound on control:* The control is not uniformly bounded on compact sets with respect to the parameter T and in particular we have for all $x \neq 0$ that $|\alpha_T(x)| \to \infty$ as $T \to 0$.

Control with excessive finesse

Consider the system $\dot{x} = x + u$. Again the exact discrete-time model can be computed, but we consider control design based on the "partial Euler" model $x^+ = (1+T)x + (e^T - 1)u$. The control

$$u = \alpha_T(x) = -\frac{T(1+\frac{1}{2}T)x}{e^T - 1} \tag{4}$$

stabilizes the family of approximate models (for $T \in (0,2)$) by placing the pole of the closed-loop at $1 - \frac{1}{2}T^2$. On the other hand, the pole of the exact discrete-time closed-loop is located at $e^T - T - \frac{1}{2}T^2 > 1$, $\forall T > 0$. Hence, the third step of the proposed control procedure is not possible to accomplish. The approximate closed-loop system contains the following indicator that its robustness may not be sufficient to account for the mismatch between the approximate and exact discrete-time plant models:

- *Nonuniform attraction rate:* For all $T > 0$, the family of approximate discrete-time models satisfies $|\phi_T(k, x_\circ)| \leq be^{-kT^2}|x_\circ|$, $\forall k \in \mathbb{N}$, where $b > 0$ is independent of T. Therefore the overshoot is uniformly bounded with T. However, if we think of $kT = t$ as "continuous time", then as $T \to 0$, the rate of convergence of solutions satisfies that for any $t > 0$ we have $e^{-tT} \to 1$. In other words, the rate of convergence in continuous time is not uniform in the parameter T.

Control without a continuous Lyapunov function certificate

Consider the single integrator $\dot{x} = u$ and a fixed sampling period T. We build a controller based on the approximate discrete-time model $x^+ = x + (T + \varepsilon)u$ where $\varepsilon > 0$. As $\varepsilon \to 0$ we approach the exact discrete-time model. We choose

$$u = \alpha_\varepsilon(x) = \frac{1}{T+\varepsilon}\left[-x + f(x)\right] \tag{5}$$

where f is defined as follows $f(0) = 0$, $f(x) = \mathrm{sgn}(x)j(x) \qquad x \neq 0$ and $j(x)$ is the integer j satisfying $j < |x| \leq j+1$. The approximate closed-loop system is

$$x^+ = f(x) \tag{6}$$

the origin of which is locally exponentially and globally asymptotically stable. The exact discrete-time closed-loop system is

$$x^+ = f(x) + \frac{\varepsilon}{T+\varepsilon}\left[x - f(x)\right] \tag{7}$$

which has the set $\mathbb{R}_{>j} := \{x \in \mathbb{R} : x > j\}$, for each positive integer j, forward invariant for all $\varepsilon > 0$. Thus the origin of the exact discrete-time closed-loop system is not even approximately attractive, and so the third step of the proposed control procedure is not possible to accomplish. The problem is that the asymptotic stability of the origin for (6) has no robustness. While it would be easy to attribute this to the discontinuous nature of the right-hand side of (6) we would like to develop results that permit discontinuous controllers. This is because discontinuous controllers are sometimes necessary for stabilization. (See, for example [30].) Moreover, controllers generated using the techniques in [7] with sample and hold (see [42] or [8]) are typically discontinuous in the state and yet produce some robustness because they come with a continuous Lyapunov function that certifies asymptotic stability. While discontinuous Lyapunov functions are generic (see, for example, [33]), continuous Lyapunov functions are not. It can be shown that the system (6) admits no continuous Lyapunov function. We take this as the indicator in the approximate closed-loop system that the robustness (it actually doesn't have any) may not be sufficient to account for the mismatch between the approximate and exact discrete-time plant models:

- *No continuous Lyapunov function certificate:* If there existed a continuous Lyapunov function for (6) then the origin of (7) would be semiglobally practically asymptotically stable in ε. Since this is not the case, there does not exist a continuous Lyapunov function that certifies the asymptotic stability of the origin of (6).

Ruling out these indicators

In our work, we will rule out all of the indicators that we have seen in the above examples by assuming that the feedback control is uniformly bounded in the modeling parameter and that there exists a parameterized family of Lyapunov functions that are continuous, positive definite, decrescent and decreasing along trajectories with all of these properties uniform, in an appropriate sense, in the modeling parameter(s). The example in the next section illustrates what is sufficient.

B.2 When things go right

In our last example, which is taken from [37], we demonstrate how the procedure we have described can out-perform sample and hold implementation of a continuous time control algorithm. Consider the continuous-time plant:

$$\dot{\eta} \;=\; \eta^2 + \xi; \qquad \dot{\xi} \;=\; u \; . \tag{8}$$

First, we design the continuous-time backstepping controller based on results in [24]. Note that the first subsystem can be stabilized with the "control" $\phi(\eta) = -\eta^2 - \eta$ with the Lyapunov function $W(\eta) = \frac{1}{2}\eta^2$. Using this information and applying [24, Lemma 2.8 with $c = 1$], we obtain:

$$u^{ct}(\eta, \xi) = -2\eta - \eta^2 - \xi - (2\eta + 1)(\xi + \eta^2) \; , \tag{9}$$

which globally asymptotically stabilizes the continuous-time model (8).

Assume now that the plant (8) is between a sampler and a zero order hold and consider its Euler approximate model:

$$\eta(k+1) \;=\; \eta(k) + T(\eta^2(k) + \xi(k)); \qquad \xi(k+1) \;=\; \xi(k) + Tu(k) \; . \tag{10}$$

Again, the control law $\phi(\eta) = -\eta^2 - \eta$ globally asymptotically stabilizes the η-subsystem of (10) with the Lyapunov function $W(\eta) = \frac{1}{2}\eta^2$. Using results in [37], we obtained the controller:

$$u_T^{Euler}(\eta, \xi) = u^{ct}(\eta, \xi) - T[0.5\eta^2 + 0.5\xi - 0.5\eta + (\xi + \eta^2)^2] \; , \tag{11}$$

which semi-globally practically asymptotically stabilizes the Euler model (10). This can be proved with the Lyapunov function $V(\xi, \eta) = \frac{1}{2}\eta^2 + \frac{1}{2}(\xi + \eta + \eta^2)^2$ which serves as a Lyapunov certificate of asymptotic stability that is uniform in T in an appropriate way. Note that the term $-T[0.5\eta^2 + 0.5\xi - 0.5\eta + (\xi + \eta^2)^2]$ can be regarded as a modification of the controller (9). Moreover, for $T = 0$ we have that $u_0^{Euler}(\eta, \xi) = u^{ct}(\eta, \xi)$. We have compared the performance of the sampled-data systems with the two different controllers and have observed that u_T^{Euler} consistently yielded at least 4 times larger domain of attraction than u^{ct} for all tested sampling periods. Estimates of domains of attraction (DOA) with the two controllers for the sampling period $T = 0.5$ *sec* were obtained using simulations (for more details see [37]) and are given in Figure 1 where "$*$" and "$\triangle$" indicate respectively the boundary of the estimates of DOA for the closed loop with u^{ct} and u_T^{Euler}. Not all backstepping controllers that are based on Euler model will stabilize the exact model. Indeed, the controller (3) in the first example can be obtained using a backstepping procedure similar to the one used in this example.

C. Literature review

In this paper we concentrate on the discrete-time controller design approach and in particular on the third step of the procedure presented in the first section. Before we outline our contributions, we overview some results from the literature that are related to the procedure we have outlined above.

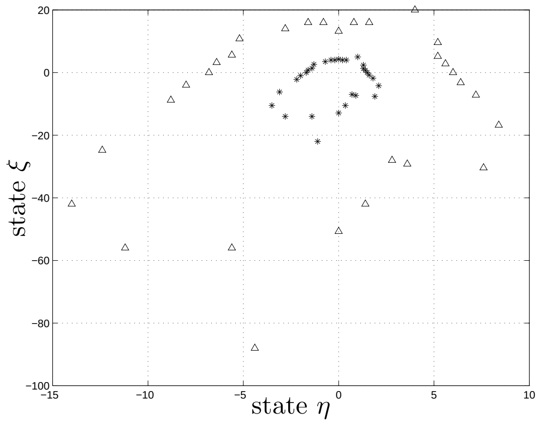

Fig. 1. Estimates of DOA with with $T = 0.5$ *sec.*

There is a range of different methods for approximate discretization of control-free systems which can be found in numerical analysis textbooks, such as [43]. Most of these methods can be adapted in a straightforward manner to controlled systems. The results in [31] illustrate this very well. We also point out recent results in [14] that can be used to produce approximate discrete-time models for control systems with measurable disturbances. All of these results can be used in generating families of approximate discrete-time plant models in the first step of the typical procedure.

We emphasize that a large body of nonlinear discrete-time literature that discusses controller design *assumes* that the exact discrete-time model of the plant is known (see for instance [2], [4], [9], [23] and references defined therein). However, as we have already pointed out, this is typically not true even if the continuous-time plant model is known exactly. Our first three motivating examples illustrate that one can not blindly apply these results in the second step of the typical procedure since the outcome may be a destabilizing controller for the sampled-data system. Hence, in order to carry out the typical procedure successfully one needs to perform controller design carefully, making sure that the stability of the approximate model will be robust to perturbations induced by the underlying numerical approximation.

Controller design based on approximate discrete-time plant models was pursued in several control applications [10], [12], in the context of trajectory approximation based adaptive control in [28], [40] and in the context of backstepping in [37]. All of these references use the Euler approximate model for controller design and they consider a particular class of plants and control laws. Popularity of the Euler approximate model is due to the fact that it is the simplest approximate model that preserves the structure of the continuous-time model and hence it is easy to use for controller design in the second step of the typical procedure. All of these results require fast sampling and in general they produce semiglobally practically stabilizing control laws. We note that fast sampling results suffer the same drawback as discretization of continuous time controllers, since the required sampling rate may not be implementable due to hardware limitations.

A more realistic approach is to assume that the sampling period is fixed (or has positive a lower bound). The three step procedure described above can still be carried out but in this case the modeling parameter is (typically) the integration period of the underlying numerical integration scheme used to generate the family of approximate models. Moreover, semiglobal stabilization is not possible in general in this case due to possible finite escape times that can occur for large initial states. This approach was used in illustrative examples in several references (see for instance Section V in [23], Example 1 in [2] and Section V in [9]) but none of these references presents a rigorous analysis of this approach.

Recently, a rigorous analysis of the third step of the typical procedure was carried out in [33] for a large class of plants, approximate models and static state feedback control laws. We emphasize these results are *not constructive* since they do not provide controller design methods. Indeed, we can say that the results in [33] are *prescriptive* since they can be used to guide one when designing a controller based on an approximate discrete-time plant model. These results were further generalized in [35] and [38] to deal respectively with input-to-state stabilization and integral versions of input-to-state stabilization for sampled-data nonlinear systems with disturbances. We have already remarked that these results require *fast sampling* which means that they may not be implementable in practice in cases when the required sampling period is too small to be realized with the available hardware. The reference [37] contains an example illustrating how these results can be used as a guide in designing backstepping controllers based on the Euler approximate model of strict feedback sampled-data plants. Our fourth motivating example has illustrated possible advantages of this approach.

D. Contributions

In this paper, we extend the results of [33] in several directions by considering: (i) stability with respect to arbitrary closed, not necessarily compact, sets ([33] only considers stability of the origin); (ii) plants that are modeled as a differential inclusion ([33] only considers plants described by differential equations) ; (iii) dynamic control laws that may be discontinuous and modeled as a difference inclusion ([33] only considers static state feedback control laws); (iv) both fast sampling and fixed sampling problems ([33] only considers fast sampling). Our motivation for considering the problem in such generality is discussed next.

D.1 Why stabilization of (not necessarily compact) sets?

Many interesting nonlinear stabilization problems are difficult, if not impossible, to reduce to the problem of stabilizing a point. For example, the problem of stabilizing a periodic orbit (see, for example, [15], [16]) is most naturally understood as a set stabilization problem. The problem of causing an output to track the *nonsmooth* output of a slowly varying exosystem and the problem of stabilizing a manifold defined implicitly through a nonsmooth equation can be naturally viewed as a set stabilization problems while casting these problems as a point stabilization problems is fraught with difficulties due to the nonsmoothness. This point has also been made in [44], for example. In general, the set stabilization viewpoint is a natural way to interpret problems where certain quantities should converge to zero while other quantities may be allowed to evolve freely and even become unbounded. This type of problem arises in the design of observers for nonlinear systems and, with the right viewpoint, in the control of time-varying systems. See, for instance [27, Chapter 5]. In particular, if we are given a time-varying control system $\dot{x} = f(t, x, u)$ and we wish to force $x(t) \to \alpha(t)$, we can consider a time-invariant system with state $\tilde{x} := (x^T \ p)^T$ where $\dot{x} = f(p, x, u)$, $\dot{p} = 1$ and stabilize the set $\mathcal{A} := \left\{\tilde{x} \in \mathbb{R}^{n+1} : x = \alpha(p)\right\}$. Related results in the numerical analysis literature on global properties of attractors under discretization can be found in [43, Chapter 7] and [21].

D.2 Why stabilization of differential inclusions?

Differential inclusions generalize differential equations. They are the most accurate way to model differential equations with discontinuous right-hand sides, due to effects like stiction, etc. See [11]. They also can be used to model systems with disturbances in which case the problem of achieving input-to-state stability is transformed to the problem of achieving asymptotic stability. Consider, for instance, the plant with disturbances $\dot{x} = f(x, u, w)$, where u and w are respectively control and disturbance inputs to the system. Suppose also that there exists a function $\gamma \in \mathcal{K}_\infty$ such that the auxiliary system $\dot{x} \in F(x, u)$, where $F(x, u) := \overline{co}\{f(x, u, \gamma(|x|)d) : |d| \leq 1\}$ can be stabilized using a sampled-data static state feedback or, alternatively, the auxiliary system $\dot{x} \in F_\gamma(x, u)$ where $F_\gamma(x, u) := \overline{co}\{f(x, u, d) : |d| \leq \gamma\}$ can be driven to a ball of radius related to the size of γ. Then, it can be shown that the same controller achieves input-to-state stability of the original system with disturbances. An illustration of this idea can be found in Section VI-C where an input-to-state stabilizing controller is designed for a nonholonomic integrator via its approximate discrete-time model. These results can be regarded as an alternative to results on input-to-state stabilization proved in [35] that use approximate discrete-time model for systems with inputs instead of an auxiliary differential inclusion.

D.3 Why stabilization by dynamic discontinuous control laws?

Discontinuous stabilizers are generic in stabilization algorithms based on optimal control formulations. Dynamic control algorithms arise when using observers in output feedback problems. Hybrid control algorithms, where the continuous variables of the plant interact with the discrete-valued logic variables of a controller, are both dynamic and discontinuous. We illustrate how our results apply to a hysteresis switching control law in Section VI-D.

D.4 Overview of main results

We now present a nontechnical overview of our main results. Section II may be used to clarify any unfamiliar notation.

We will consider nonlinear control systems of the form

$$\dot{x} \in F(x,u) \tag{12}$$

where $x \in \mathbb{R}^n$ and the set-valued map $F(\cdot,u)$ is assumed to have enough regularity to guarantee existence (but not necessarily uniqueness) of solutions. (See Assumption 1 in Section IV). We will assume, for the family of models

$$x^+ \in F^a_{T,h}(x,u) \ , \tag{13}$$

which approximate the exact discrete-time model of (12), that a family of possibly discontinuous discrete-time controllers

$$z^+ \ \in \ G_{T,h}(z,x); \qquad u \ \in \ H_{T,h}(z,x) \ , \tag{14}$$

where $z \in \mathbb{R}^{n_c}$, has been designed to (approximately) asymptotically stabilize a nonempty closed set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$. The parameter T represents the sampling period used when measuring the plant. The parameter h is used to enhance the accuracy between the approximate discrete-time model (13) and the exact discrete-time model

$$x^+ \in F^e_T(x,u) \ , \tag{15}$$

where $F^e_T(x,u)$ is the set of values the solutions to (12) can take at time T when starting at x and with the constant input u applied. We will consider both the case where the sampling period T is fixed and the case where the sampling period can be adjusted to arbitrarily small values. In the latter case, (15) represents a family of systems. In some cases where T is adjustable, we will take $h = T$. The question is then:

Under which conditions does the family of controllers (14) also (approximately) asymptotically stabilizes the set $\mathcal{A}$ for the (family of) exact discrete-time model(s) (15) for sufficiently small values of h and/or T?

Motivated by our previous examples, we will show that the approximate discrete-time closed loop system has enough robustness to account for the mismatch between the approximate and exact discrete-time plant model when, over the set of states in which we expect to operate and over the parameter values we expect to use, we have:

1. [**Lyapunov certificates of asymptotic stability**] a family of Lyapunov functions with upper and lower bounds uniform in the modeling parameters and a decrease, which depends on T but is essentially uniform in small h, along trajectories of the approximate discrete-time closed-loop model. That is, there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, α_3 continuous, positive definite and a family of functions $V_{T,h} : \mathbb{R}^{n+n_c} \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1\left(|\tilde{x}|_{\mathcal{A}}\right) \leq V_{T,h}(x,z) \leq \alpha_2\left(|\tilde{x}|_{\mathcal{A}}\right) \tag{16}$$

where $\tilde{x} := (x^T \ z^T)^T$ and for all (w_1, w_2) such that $w_1 \in F^a_{T,h}(x, H_{T,h}(x,z))$, $w_2 \in G_{T,h}(x,z)$ we have

$$V_{T,h}(w_1,w_2) - V_{T,h}(x,z) \leq -T\left[\alpha_3\left(|\tilde{x}|_{\mathcal{A}}\right) - \delta_1(h)\right] \tag{17}$$

with $\delta_1(h) \to 0^+$ as $h \to 0^+$.

2. [**Continuity of Lyapunov certificates and modeling consistency**] the family of Lyapunov functions has strong enough continuity properties and the approximate discrete-time plant models are close enough to the (family of) exact discrete-time model(s) so that the Lyapunov function still decreases along the trajectories of the exact discrete-time closed-loop system. In other words, for all (w_1^e, w_2) such that $w_1^e \in F_T^e(x, H_{T,h}(x,z))$, $w_2 \in G_{T,h}(x,z)$ we have

$$\inf_{w_1^a \in F_{T,h}^a(x, H_{T,h}(x,z))} |V_{T,h}(w_1^a, w_2) - V_{T,h}(w_1^e, w_2)| \le T\delta_2(h) \tag{18}$$

with $\delta_2(h) \to 0^+$ as $h \to 0^+$.

In order to provide modularity between plant modeling and control design it is useful to regroup these conditions, with a slight loss of generality, in the following way: Over the set of states in which we expect to operate and the parameter values we expect to use, we have

Property 1 [**Continuous (or Lipschitz) Lyapunov certificates of asymptotic stability**] the Lyapunov certificates of asymptotic stability described above are *continuous (or Lipschitz if $h = T$) in their first argument* uniformly in the states and parameters of interest;

Property 2 [**Uniformly bounded controls**] the control values are uniformly bounded; i.e., there exists $M > 0$ such that $|v| \le M$ for all $v \in H_{T,h}(x,z)$ and all x, z, T and h of interest.

Property 3 [**Modeling consistency**] the approximate discrete-time plant models are sufficiently close to the (family of) exact discrete-time model(s); i.e., $F_T^e(x,u) \subseteq F_{T,h}^a(x,u) + T\widetilde{\delta}_2(h)\overline{\mathcal{B}}_n$, for all x, u, T and h of interest and with $\widetilde{\delta}_2(h) \to 0^+$ as $h \to 0^+$. (Modeling consistency, which is very natural to assume, can be checked without explicit knowledge of the exact discrete-time model. For more details, see Section V).

With these pseudo-definitions, our main results can be paraphrased as follows:

Property 1 for approximate + Property 2 + Property 3 $\Longrightarrow$ Property 1 for exact

It is instructive to note that none of the three examples illustrating when things go wrong had continuous Lyapunov certificates of asymptotic stability, in the sense specified above, for the family of approximate closed-loop systems. They all had modeling consistency and the second and third examples had uniformly bounded controls.

The rest of our paper is organized as follows: In Section II we fix notation. In Section III we present results on the use of Lyapunov functions to establish asymptotic stability of sets for difference inclusions. In Section IV we present our main results, expressed in terms of modeling consistency and Lyapunov certificates. In Section V we justify the modeling consistency assumption and in Section VI we justify the Lyapunov certificates assumption.

II. Notation

The sets of natural and real numbers are respectively denoted as $\mathbb{N}$ and $\mathbb{R}$. A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ is of class-$\mathcal{L}$ if it is continuous and strictly decreasing to zero. A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ is of class-$\mathcal{M}$ if it is continuous and nonincreasing. A continuous function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \ge 0$ and $\beta(s, \cdot)$ is of class-$\mathcal{L}$ for each $s > 0$. Given an arbitrary set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, we define $|x|_{\mathcal{A}} := \inf_{s \in \mathcal{A}} |x - s|$, where $|x|$ denotes the Euclidean norm of the vector x. The following result is needed in the sequel:

Lemma 1: [1, Lemma 4.1] Given any continuous positive definite function $\alpha : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, there exist functions $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{M}$ such that $\alpha(s) \ge \rho_1(s) \cdot \rho_2(s)$ for all $s \ge 0$. ■

For a closed set $\mathcal{A}$ and non negative real numbers $0 \le \delta \le \Delta$ we define $\mathcal{H}_{\mathcal{A}}(\delta, \Delta) := \{x \in \mathbb{R}^n : \delta \le |x|_{\mathcal{A}} \le \Delta\}$. Similarly, given two positive numbers ℓ_1, ℓ_2 with $\ell_1 \le \ell_2$ and a function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ we define $\mathcal{V}(\ell_1, \ell_2) :=$

$\{x \in \mathbb{R}^n : \ell_1 \leq V(x) \leq \ell_2\}$. Given two set valued maps F and H, we often write $F(H(x))$ to denote $\{F(w) : w \in H(x)\}$. Given $a \in \mathbb{R}_{\geq 0}$ we denote $\lfloor a \rfloor := \max_{b \in \mathbb{N}, b \leq a} b$.

III. Lyapunov's method for difference inclusions

We now present results, which are of independent interest, on the use of Lyapunov functions to establish regional, practical asymptotic stability of sets for difference inclusions

$$x^+ \in F(x) \ , \tag{19}$$

where $F : \mathbb{R}^n \to$ subsets of $\mathbb{R}^n$. We assume that for any $x \in \mathbb{R}^n$ the set valued map $F(\cdot)$ is well defined and its value $F(x)$ is a nonempty subset of $\mathbb{R}^n$. Hence, for any initial condition $x_\circ \in \mathbb{R}^n$, the solutions $\phi(k, x_\circ)$ of the system (19) exist for all $k \geq 0$.

For *differential* inclusions, the Lyapunov results are straightforward, modulo the possibility of finite escape times. Indeed, equipped with a continuously differentiable Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a continuous, positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\langle \nabla V(x), w \rangle \leq -\alpha(V(x)) \qquad \forall x \in \mathcal{V}(\ell_1, \ell_2) \ , \ w \in F(x) \tag{20}$$

where $0 \leq \ell_1 \leq \ell_2$, it is easy to deduce from classical comparison theorems (see also [41] or [20, Lemma 2.5]) that, for all $x_\circ \in \mathcal{V}(0, \ell_2)$ and all t where a given solution $\phi(\cdot, x_\circ)$ of $\dot{x} \in F(x)$ is defined, we have

$$V(\phi(t, x_\circ)) \leq \max\{\beta(V(x_\circ), t), \ell_1\} \tag{21}$$

where $\beta \in \mathcal{KL}$ is such that $\beta(s, \cdot)$ is the (maximal) solution of the differential equation $\dot{y} = -\alpha(y), \ y(0) = s$. When the bound in (21) is combined with extra information about V like: there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \tag{22}$$

then asymptotic stability of the set $\mathcal{A}$ can be deduced, modulo finite escape times, which are ruled out by (21) if escapes to infinity in finite time require the distance to $\mathcal{A}$ to grow without bound, e.g., $\mathcal{A}$ is compact.

For difference inclusions, the condition that appears to be analogous to (20) is

$$V(x^+) - V(x) \leq -\alpha(V(x)) \qquad \forall x \in \mathcal{V}(\ell_1, \ell_2) \ , \quad x^+ \in F(x) \ . \tag{23}$$

However, this condition is not even enough to guarantee that the function $V(\phi(\cdot, x_\circ))$, where $\phi(\cdot, x_\circ)$ is a solution of (19) with $x_\circ \in \mathcal{V}(\ell_1, \ell_2)$, remains bounded for all $k \geq 0$. This is illustrated by the system $x^+ = f(x)$ where $f(\cdot)$ is any continuous function satisfying

$$f(x) = \begin{cases} 2\ell_2 & |x| \leq \ell_1/2 \\ 2|x| & |x| \geq 2\ell_2 \\ 0 & \ell_1 \leq |x| \leq \ell_2 \ . \end{cases} \tag{24}$$

With, for example, $V(x) = |x|$, we have

$$V(f(x)) - V(x) = -V(x) \qquad \forall x \in \mathcal{V}(\ell_1, \ell_2) \tag{25}$$

yet $V(\phi(\cdot, x_\circ))$ is unbounded for all $x_\circ$ since every trajectory grows without bound. This example shows that some information about $V(f(x))$ is required even for values $x \in \mathcal{V}(0, \ell_1)$ in order to assert a bound on $V(\phi(\cdot, x_\circ))$. We can state the following result which is proved in the Appendix.

Proposition 1: Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, let $F : \mathbb{R}^n \to$ nonempty subsets of $\mathbb{R}^n$ and let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous positive definite function. Suppose $\ell_1, \ell_2, \ell_3 \in \mathbb{R}_{\geq 0}$ and $T > 0$ satisfy

$$\ell_1 + T\ell_3 \leq \ell_2 \tag{26}$$

$$V(x^+) - V(x) \leq -T\alpha(V(x)) \qquad \forall x \in \mathcal{V}(\ell_1, \ell_2), \quad x^+ \in F(x) \bigcap \mathcal{V}(\ell_1, \ell_2 + T\ell_3) \ , \tag{27}$$

$$V(x^+) - V(x) \leq T\ell_3 \qquad \forall x \in \mathcal{V}(0, \ell_2) \ , \quad x^+ \in F(x) \ . \tag{28}$$

Then for all $x_\circ \in \mathcal{V}(0, \ell_2)$, the solutions $\phi(\cdot, x_\circ)$ of the difference inclusion (19) satisfy

$$V(\phi(k, x_\circ)) \leq \max\left\{\beta(V(x_\circ), kT), \ell_1 + T\ell_3\right\} \qquad \forall k \in \{0, 1, 2, \dots\} \tag{29}$$

where $\beta \in \mathcal{KL}$ is defined as $\beta(s,t) := \tilde{\beta}(s, \rho_2(s)t)$ with $\tilde{\beta}(s, \cdot)$ the maximal solution of the differential equation $\dot{y} = -\rho_1(y)$, $y(0) = s \geq 0$ and $\rho_1 \in \mathcal{K}_\infty$, $\rho_2 \in \mathcal{M}$ such that $\alpha(s) \geq \rho_1(s) \cdot \rho_2(s)$ for all $s \geq 0$ (see Lemma 1). ■

An alternative set of Lyapunov conditions, which guarantee those of the previous proposition, is given next:

Proposition 2: Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, let $F : \mathbb{R}^n \to$ nonempty subsets of $\mathbb{R}^n$ and let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous, positive definite function. Suppose $\ell_1, \ell_2, \ell_3 \in \mathbb{R}_{\geq 0}$ and $T > 0$ satisfy

$$\ell_3 \leq \min_{s \in [\ell_1, \ell_2]} \alpha(s); \qquad \ell_1 + T\ell_3 \leq \ell_2 \tag{30}$$

$$V(x^+) - V(x) \leq -T\left[2\alpha(V(x)) - \ell_3\right] \qquad \forall x \in \mathcal{V}(0, \ell_2) \ , \quad x^+ \in F(x) \ . \tag{31}$$

Then the assumptions of Proposition 1, i.e., (26)-(28), hold. ■

Proof. The second condition in (30) is the same as (26). The condition (28) follows immediately from (31). The condition (27) follows from the combination of (31) and the first condition in (30). ■

The following proposition guarantees the conditions of Proposition 1 when the conditions of Proposition 2 hold for a nearby difference inclusion.

Proposition 3: Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, let $F^a : \mathbb{R}^n \to$ nonempty subsets of $\mathbb{R}^n$, $F : \mathbb{R}^n \to$ nonempty subsets of $\mathbb{R}^n$ and let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous positive definite function. Suppose $\ell_1, \ell_2, \ell_3, \widetilde{\ell}_1, \widetilde{\ell}_2 \in \mathbb{R}_{\geq 0}$ and $T > 0$ satisfy

$$\widetilde{\ell}_1 + \widetilde{\ell}_2 \leq \min_{s \in [\ell_1, \ell_2]} \alpha(s); \qquad \max\left\{\ell_1, T\left(\widetilde{\ell}_1 + \widetilde{\ell}_2\right)\right\} \leq T\ell_3; \qquad \ell_1 + T\ell_3 \leq \ell_2 \tag{32}$$

$$V(x_a^+) - V(x) \leq -T\left[2\alpha(V(x)) - \widetilde{\ell}_1\right] \qquad \forall x \in \mathcal{V}(0, \ell_2) \ , \quad x_a^+ \in F^a(x) \tag{33}$$

and for all $x \in \mathcal{V}(0, \ell_2)$ and any $x^+ \in F(x)$ with $V(x^+) \geq \ell_1$ there exists $x_a^+ \in F^a(x)$ such that

$$V(x^+) \leq V(x_a^+) + T\widetilde{\ell}_2 \ . \tag{34}$$

Then the assumptions of Proposition 1, i.e., (26)-(28), hold. ■

Proof of Proposition 3: The first condition in (32) and (33) imply

$$\begin{aligned} V(x_a^+) - V(x) &\leq -T\left[\alpha(V(x)) + \widetilde{\ell}_2\right] && \forall x \in \mathcal{V}(\ell_1, \ell_2) \ , \quad x_a^+ \in F^a(x) && (35) \\ V(x_a^+) - V(x) &\leq T\widetilde{\ell}_1 && \forall x \in \mathcal{V}(0, \ell_2) \ , \quad x_a^+ \in F^a(x) \ . && (36) \end{aligned}$$

It follows from (36), (34), second condition in (32) and $V(x) \geq 0$ that

$$V(x^+) - V(x) \leq \max\left\{\ell_1, T\left(\widetilde{\ell}_1 + \widetilde{\ell}_2\right)\right\} \leq T\ell_3 \qquad \forall x \in \mathcal{V}(0,\ell_2)\ , \qquad x^+ \in F(x)\ , \tag{37}$$

i.e., (28) holds. To see that (27) holds, we consider $x \in \mathcal{V}(\ell_1,\ell_2)$ and $x^+ \in F(x) \bigcap \mathcal{V}(\ell_1, \ell_2 + T\ell_3)$. For such values, it follows from (35) and (34) that $V(x^+) - V(x) \leq -T\alpha(V(x))$, i.e., (27) holds. ■

The connection between these propositions and uniform asymptotic stability of sets is made in the following:

Corollary 1: Let $\mathcal{A}$ be a nonempty closed set, let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, let $F : \mathbb{R}^n \to$ nonempty subsets of $\mathbb{R}^n$ and let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous positive definite function. Suppose $\ell_1, \ell_2, \ell_3 \in \mathbb{R}_{\geq 0}$ and $T > 0$ satisfy the conditions (26)-(28) of Proposition 1. Also suppose that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (22) holds. Then, for all $x_\circ \in \mathcal{V}(0,\ell_2)$, the solutions $\phi(\cdot, x_\circ)$ of the difference inclusion (19) satisfy:

$$|\phi(k,x_\circ)|_{\mathcal{A}} \leq \max\{\alpha_1^{-1}(\beta(\alpha_2(|x_\circ|_{\mathcal{A}}), kT)), \alpha_1^{-1}(\ell_1 + T\ell_3)\}, \qquad \forall k \in \mathbb{N}$$

where $\beta \in \mathcal{KL}$ was defined in Proposition 1. ■

IV. Main results

In this section we present sufficient conditions for stabilization of sampled-data nonlinear inclusions via their approximate discrete-time models. Our main results specify checkable conditions on the continuous-time plant model, the approximate discrete-time plant model and the controller that guarantee that we can pick the modeling parameter small enough so that the controllers that stabilize approximate model would also approximately stabilize the exact model (i.e., the third step of the typical procedure presented in the introduction can be successfully carried out). These conditions can be used as guidelines for controller design based on approximate models. In particular, these conditions can be used to discard "bad" controllers such as the ones used in the first three motivating examples of the introduction. In Subsection IV-A we consider the case when $T = h$ and T can either be fixed or varying.

A starting point in our investigation are sampled-data differential inclusions (12) with the following assumption for the set-valued map $F(\cdot, u)$:

Assumption 1: For each $u \in \mathbb{R}^m$, the set-valued map $F(\cdot,u)$ satisfies the following basic conditions: 1) it is upper semi-continuous, i.e., for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $|\xi - x| \leq \delta$ we have $F(\xi,u) \subseteq F(x,u) + \varepsilon\overline{\mathcal{B}}_n$, where $\overline{\mathcal{B}}_n$ denotes the closed unit ball in $\mathbb{R}^n$, 2) for each $x \in \mathbb{R}^n$ the set $F(x,u)$ is nonempty, compact and convex. ■

This assumption guarantees that for each fixed u there exists at least one solution to (12) (see [3]). We will use $\mathcal{S}(x,u)$ to denote the set of solutions to (12) starting at x with constant input u. For a given $t > 0$ and $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ we use the following notation $F_t^e(x,u) := \{\xi \in \mathbb{R}^n : \xi = \phi(t,x,u),\ \phi \in \mathcal{S}(x,u)\}$.

We consider plants (12) for which we assume that the control u is held constant between the sampling instants kT, where $T > 0$ is the *sampling period*, and the exact discrete-time model (when it exists) can be written in the form:

$$x^+ \in F_T^e(x,u)\ . \tag{38}$$

Note that (38) is in general well defined only for a subset of $\mathbb{R}^n \times \mathbb{R}^m$ and it is in general unknown on the set where it is defined. Hence, we introduce a family of approximate discrete-time models that can be written in the form:

$$x^+ \in F_{T,h}^a(x,u)\ , \tag{39}$$

where $h > 0$ is the modeling parameter that can be arbitrarily adjusted.

Remark 1: The modeling parameter can have various sources. A common situation is when h represents the *integration period* of a numerical integration routine (Euler, Runge-Kutta, etc.) used to approximate F^e_T. In this case, and with

$$x^+ \in f_h(x,u) \tag{40}$$

representing one step of the numerical integration routine (for forward Euler we have $f_h(x,u) := x + hF(x,u)$) we can generate a family of numerically integrated approximate models $F^a_{T,h}(\cdot,\cdot)$ by defining

$$\begin{aligned} f^1_h(x,u) &:= f_h(x,u) \\ f^{i+1}_h(x,u) &:= f_h(f^i_h(x,u),u)\ , \quad i = 1,2,\dots\ , && (41) \\ F^a_{T,h}(x,u) &:= f^N_h(x,u)\ , && (42) \end{aligned}$$

where $N = N(T,h) := \lfloor T/h \rfloor$. More details on numerical methods for differential inclusions that can be used to generate different approximate models $F^a_{T,h}$ can be found in [45].

Another, or additional, source for the modeling parameter h is when the control is constructed from a numerical optimization procedure where the state and input space are quantized. In this case, h may specify the size of the quantization levels used in the numerical computation of the optimal controller. This idea was used, for example, in [23], and it is also relevant in the application of discrete-time based receding horizon control (see, for example, the survey paper [29]) for continuous-time plants.

When both of these sources are presented, it is natural to treat h as a modeling parameter vector. The reader should be able to easily see how the main results stated below can also be stated for a modeling parameter vector. ■

Remark 2: There are problems where it is reasonable to consider that the exact discrete-time model depends on a modeling parameter. For example, in order to treat the case of *large* sampling period T in the framework of fixed sampling periods, we can define a new time scale $\tau = t/T =: ht$ and perhaps also consider an input transformation $u = h\widetilde{u}$ so that the exact discrete-time model with sampling period T_τ in the τ time scale becomes

$$x^+ \in F^e_{T_\tau,h}(x,\widetilde{u}) = \{\xi \in \mathbb{R}^n : \xi = \phi(T_\tau/h, x, h\widetilde{u}), \phi \in \mathcal{S}(x, h\widetilde{u})\}\ . \tag{43}$$

We will do this for the nonholonomic integrator example in Section VI-C. For this reason, we will state the results in this section allowing modeling parameter dependence in the exact discrete-time model. ■

Next, we assume that a family of discrete-time controllers

$$\begin{aligned} z^+ &\in G_{T,h}(z,x) \\ u &\in H_{T,h}(z,x) && (44) \end{aligned}$$

has been designed based on (39), where $z \in \mathbb{R}^{n_c}$ is the controller state variable and $G_{T,h}, H_{T,h}$ are set values maps. We denote $(x,z) := (x^T\ z^T)^T$. In the sequel we investigate stability of the system (39), (44) or (38), (44) with respect to a nonempty closed set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$. We also make use of the following "projection" set:

$$P(\mathcal{A}) := \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^{n_c} \text{ such that } (x,z) \in \mathcal{A}\}\ . \tag{45}$$

In particular, we use the following definitions.

Definition 1: [**Uniformly bounded controls**] Let strictly positive real numbers (T, Δ_1, Δ_2) and a nonempty closed set $\mathcal{A} \in \mathbb{R}^{n+n_c}$ be given. If

$$\sup_{\{(x,z)\in\mathcal{H}_{\mathcal{A}}(0,\Delta_1)\ ,\ w\in H_{T,h}(z,x)\ ,\ h\in(0,h^*]\}} |w| \le \Delta_2\ , \tag{46}$$

for some $h^* > 0$, then we say that the family of controllers (44) is $(T, \Delta_1, \Delta_2, \mathcal{A})$-uniformly bounded. ■

The following "consistency" property is central in our developments and it is an appropriate adaptation and generalization of consistency property used in the numerical analysis literature (see [43]):

Definition 2: [**Modeling consistency**] Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ and a triple of strictly positive numbers (T, Δ_1, Δ_2) be given and suppose that for any $\epsilon > 0$ there exists $h^* > 0$ such that for all $(x, u) \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1) \times \Delta_2 \overline{\mathcal{B}}_m$ and all $h \in (0, h^*]$ we have $F^e_{T,h}(x, u) \subseteq F^a_{T,h}(x, u) + T\epsilon\overline{\mathcal{B}}_n$. Then we say that the family $F^a_{T,h}$ is $(T, \Delta_1, \Delta_2, \mathcal{A})$-upper semi-consistent with $F^e_{T,h}$. ■

Sufficient checkable conditions for $(T, \Delta_1, \Delta_2, \mathcal{A})$-upper semi-consistency are presented in Section V.

Definition 3: [**Partially quasi-continuous Lyapunov certificates of asymptotic stability**] Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$, a pair of strictly positive real numbers (T, D), a family of functions $V_{T,h} : \mathbb{R}^{n+n_c} \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ a positive definite function $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a nonnegative real number δ_0 be given. Suppose for any pair of strictly positive real numbers (δ_1, δ_2) with $\delta_2 < D$ there exist $h^* > 0$ and $c > 0$ such that for all $(x, z) \in \mathcal{H}_{\mathcal{A}}(0, D)$, $h \in (0, h^*]$, we have

$$\alpha_1(|(x,z)|_{\mathcal{A}}) \leq V_{T,h}(x,z) \leq \alpha_2(|(x,z)|_{\mathcal{A}}) \tag{47}$$

$$\sup_{\substack{w_1 \in F^a_{T,h}(x, H_{T,h}(x,z)),\\ w_2 \in G_{T,h}(x,z)}} V_{T,h}(w_1, w_2) - V_{T,h}(x,z) \leq -T\alpha_3(|(x,z)|_{\mathcal{A}}) + T(\delta_0 + \delta_1) \ , \tag{48}$$

and, for all $(x_1, z), (x_2, z) \in \mathcal{H}_{\mathcal{A}}(\delta_2, D)$, with $|x_1 - x_2| \leq c$ we have

$$|V_{T,h}(x_1, z) - V_{T,h}(x_2, z)| \leq \delta_1 \ . \tag{49}$$

Then we say that $(V_{T,h}, \alpha_1, \alpha_2, \alpha_3, \delta_0)$ provides a $(T, D, \mathcal{A})$-partially quasi-continuous family of Lyapunov certificates for the family (39), (44). ■

The notion of partially quasi-continuous Lyapunov function will be further clarified and illustrated in Section VI.

A. Case 1: T is independent of h

In this section we consider the case when the sampling period T is not equal to the modeling parameter h. Our results apply to situations when T is either fixed or it can be arbitrarily assigned whereas h is always possible to arbitrarily assign. When the sampling period is fixed, finite escape times may occur between sampling instants and hence the achievable region of attraction is usually bounded. When T is varying then we can state semiglobal practical stabilization results. To shorten notation, we introduce $\tilde{x} = (x, z)$ and

$$\mathcal{F}^a_{T,h}(\tilde{x}) := \begin{pmatrix} F^a_{T,h}(x, H_{T,h}(z,x)) \\ G_{T,h}(z,x) \end{pmatrix} \ , \ \ \mathcal{F}^e_{T,h}(\tilde{x}) := \begin{pmatrix} F^e_{T,h}(x, H_{T,h}(z,x)) \\ G_{T,h}(z,x) \end{pmatrix} \ .$$

The main result of this section is presented next.

Theorem 1: Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$, strictly positive real numbers (T, D, M), the family of functions $V_{T,h} : \mathbb{R}^{n+n_c} \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a positive definite function $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a nonnegative real number δ_0 be such that the following conditions hold:

1. $(V_{T,h}, \alpha_1, \alpha_2, \alpha_3, \delta_0)$ provides a $(T, D, \mathcal{A})$-partially quasi-continuous family of Lyapunov certificates for the family (39), (44);
2. The family of controllers (44) is $(T, D, M, \mathcal{A})$-uniformly bounded;

3. The family $F^a_{T,h}$ is $(T, D, M, P(\mathcal{A}))$-upper semi-consistent with F^e_T;

Let $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{M}$ be generated by α_3 via Lemma 1. Let $\alpha(s) := \frac{1}{2}\rho_1 \circ \alpha_2^{-1}(s) \cdot \rho_2 \circ \alpha_1^{-1}(s)$. Then, for any strictly positive $\ell_1, \ell_2, \widetilde{\ell}_1, \widetilde{\ell}_2$ where $\widetilde{\ell}_1 > \delta_0$ and

$$\alpha_2(2\alpha_1^{-1}(4T\delta_0)) \; < \; \ell_1 \; < \; \ell_2 \;\; \leq \;\; \alpha_1(D) \; , \tag{50}$$

there exists $h^* > 0$ such that for all $h \in (0, h^*]$ we have that

1. for all $\tilde{x} \in \mathcal{V}_{T,h}(0, \ell_2)$ and $\tilde{x}^+_a \in \mathcal{F}^a_{T,h}(\tilde{x})$ we have

$$V_{T,h}(\tilde{x}^+_a) - V_{T,h}(\tilde{x}) \leq -T\left(2\alpha(V_{T,h}(\tilde{x})) - \widetilde{\ell}_1\right)$$

2. for all $\tilde{x} \in \mathcal{V}_{T,h}(0, \ell_2)$ and all $\tilde{x}^+_e \in \mathcal{F}^e_{T,h}(\tilde{x})$ with $V_{T,h}(\tilde{x}^+_e) \geq \ell_1$ there exists $\tilde{x}^+_a \in \mathcal{F}^a_{T,h}(\tilde{x})$ such that

$$V_{T,h}(\tilde{x}^+_e) \leq V_{T,h}(\tilde{x}^+_a) + T\widetilde{\ell}_2 \; .$$

■

Proof of Theorem 1: Let the functions $\alpha, \rho_1, \rho_2, \alpha_1, \alpha_2, \alpha_3$, the nonnegative number δ_0, the set $\mathcal{A}$ and numbers T, D, M come from the theorem. Let $\ell_1, \ell_2, \widehat{\ell}_1, \widetilde{\ell}_2$ be arbitrary positive real numbers satisfying conditions (50) and define $\widetilde{\ell}_1 := \widehat{\ell}_1 + \delta_0$. Let δ_1, δ_2 be strictly positive real numbers such that:

$$\begin{aligned} \delta_1 \;\; &\leq \;\; \min\left\{\frac{1}{2}\alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right), \widehat{\ell}_1, T\widetilde{\ell}_2\right\} && (51)\\ \delta_2 \;\; &\leq \;\; \min\left\{\alpha_2^{-1}\left(\frac{1}{4}\alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right)\right), D\right\} && (52) \end{aligned}$$

Let the pair (δ_1, δ_2) generate, using item 1 of the theorem, the numbers $h^*_1 > 0$ and $c > 0$. Let T, D, M generate $h^*_2 > 0$ from item 2 of the theorem. Let $\epsilon_1 > 0$ be such that

$$T\epsilon_1 \leq \frac{1}{4}\alpha_2^{-1}(\ell_1) \; . \tag{53}$$

Take $\epsilon = \min\{\epsilon_1, \delta_1, c/T\}$. Let ϵ, T, D, M generate $h^*_3 > 0$ using item 3 of the theorem. Define $h^* := \min\{h^*_1, h^*_2, h^*_3\}$. Let $h \in (0, h^*]$ be arbitrary but fixed. With our choice of ℓ_2 to satisfy (50), we have that:

$$\mathcal{V}_{T,h}(0, \ell_2) \subset \mathcal{H}_{\mathcal{A}}(0, D) \; . \tag{54}$$

Using (47) and definition of α it follows from (54), conditions of the theorem and our choice of h^* and δ_1 (in particular (51)) that for all $\tilde{x} \in \mathcal{V}_{T,h}(0, \ell_2)$ and $\tilde{x}^+_a \in \mathcal{F}^a_{T,h}(\tilde{x})$:

$$\begin{aligned} V_{T,h}(\tilde{x}^+_a) - V_{T,h}(\tilde{x}) \;\; &\leq \;\; -T\alpha_3(|\tilde{x}|_{\mathcal{A}}) + T(\delta_0 + \delta_1) \;\leq\; -T\rho_1 \circ \alpha_2^{-1}(V_{T,h}(\tilde{x})) \cdot \rho_2 \circ \alpha_1^{-1}(V_{T,h}(\tilde{x})) + T(\delta_0 + \widehat{\ell}_1)\\ &= \;\; -T\left[2\alpha(V_{T,h}(\tilde{x})) - \widetilde{\ell}_1\right] \; , && (55) \end{aligned}$$

which proves the first part of the theorem. Consider now an arbitrary $\tilde{x} \in \mathcal{V}_{T,h}(0, \ell_2)$ and any $\tilde{x}^+_e \in \mathcal{F}^e_{T,h}(\tilde{x})$ with $V_{T,h}(\tilde{x}^+_e) \geq \ell_1$. That implies $|\tilde{x}^+_e|_{\mathcal{A}} \geq \alpha_2^{-1}(\ell_1) > \frac{1}{2}\alpha_2^{-1}(\ell_1)$. Using this and consistency it follows that there exists an $\tilde{x}^+_a \in \mathcal{F}^a_{T,h}(\tilde{x})$ such that (using the definition of ϵ and (53)):

$$\left|\tilde{x}^+_a\right|_{\mathcal{A}} \;\; \geq \;\; -\left|\tilde{x}^+_e - \tilde{x}^+_a\right| + \left|\tilde{x}^+_e\right|_{\mathcal{A}} \;\geq\; -T\epsilon + \alpha_2^{-1}(\ell_1) \;\geq\; -\frac{1}{4}\alpha_2^{-1}(\ell_1) + \alpha_2^{-1}(\ell_1) \;>\; \frac{1}{2}\alpha_2^{-1}(\ell_1) \; . \tag{56}$$

Moreover, using the definition of δ_1, (56), (47), (48) and (50) we have:

$$\begin{aligned} \frac{1}{2}\alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right) \;\; &= \;\; \alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right) - \frac{1}{2}\alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right) \;\leq\; \alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right) - T\delta_1\\ &< \;\; \alpha_1(\left|\tilde{x}^+_a\right|_{\mathcal{A}}) - T\delta_1 \;\leq\; V_{T,h}(\tilde{x}) + T\delta_0 \;\leq\; \alpha_2(|\tilde{x}|_{\mathcal{A}}) + T\delta_0 \; , && (57) \end{aligned}$$

which implies from (52) that

$$|\tilde{x}|_{\mathcal{A}} \geq \alpha_2^{-1}\left(\frac{1}{2}\alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right) - T\delta_0\right) \;\geq\; \alpha_2^{-1}\left(\frac{1}{4}\alpha_1\left(\frac{1}{2}\alpha_2^{-1}(\ell_1)\right)\right) \;\geq \delta_2\ . \tag{58}$$

Hence, using (54) and (58) we have that $\tilde{x} \in \mathcal{V}_{T,h}(0,\ell_2)$, $\tilde{x}_e^+ \in \mathcal{F}^e_{T,h}(\tilde{x})$ and $V_{T,h}(\tilde{x}_e^+) \geq \ell_1$ imply $\tilde{x} \in \mathcal{H}_{\mathcal{A}}(\delta_2, D)$ and from (49) and definition of ϵ (in particular our choice of $\epsilon \leq \min\{\delta_1, c/T\}$) we can write that for any $\tilde{x} \in \mathcal{V}_{T,h}(0,\ell_2)$ and $(x_e^+, z) \in \mathcal{F}^e_{T,h}(\tilde{x})$ with $V_{T,h}(x_e^+, z) \geq \ell_1$ there exists $(x_a^+, z) \in \mathcal{F}^a_{T,h}(\tilde{x})$ such that $|x_a^+ - x_e^+| \leq c$ and :

$$V_{T,h}(x_e^+, z) \quad\leq\quad V_{T,h}(x_a^+, z) + \delta_1 \;\leq\; V_{T,h}(x_a^+, z) + T\widetilde{\ell}_2\ , \tag{59}$$

which completes the proof of theorem. ■

By combining Theorem 1 with Propositions 1 and 3 we obtain the following corollary:

Corollary 2: Suppose that all conditions of Theorem 1 hold. Let $\ell_1, \ell_2, \ell_3, \widetilde{\ell}_1, \widetilde{\ell}_2$ be arbitrary positive numbers satisfying the conditions (32), $\widetilde{\ell}_1 > \delta_0$, and $\ell_2 + T\ell_3 \leq \alpha_1(D)$. Then, there exists $h^* > 0$ such that for all $h \in (0, h^*]$ we have:

1.

$$V_{T,h}(\tilde{x}_e^+) - V_{T,h}(\tilde{x}) \leq -T\alpha(V_{T,h}(\tilde{x})) \qquad \forall \tilde{x} \in \mathcal{V}_{T,h}(\ell_1,\ell_2), \tilde{x}_e^+ \in \mathcal{F}^e_{T,h}(\tilde{x}) \bigcap \mathcal{V}_{T,h}(\ell_1, \ell_2 + T\ell_3) \tag{60}$$

$$V_{T,h}(x_e^+) - V_{T,h}(\tilde{x}) \leq T\ell_3 \qquad \forall \tilde{x} \in \mathcal{V}_{T,h}(0,\ell_2)\ , \tilde{x}_e^+ \in \mathcal{F}^e_{T,h}(x)\ . \tag{61}$$

2. with α_3 generating α using Theorem 1 and α generating $\beta \in \mathcal{KL}$ using Proposition 1, for all $x_\circ \in \mathcal{V}_{T,h}(0,\ell_2)$, the solutions $\phi(\cdot, x_\circ)$ of the family (38), (44) satisfy

$$|\phi(k, \tilde{x}_\circ)|_{\mathcal{A}} \leq \max\left\{\alpha_1^{-1}(\beta(\alpha_2(|\tilde{x}_\circ|_{\mathcal{A}}), kT)), \alpha_1^{-1}(\ell_1 + T\ell_3)\right\} \qquad \forall k \in \{0,1,2,\dots\}\ . \tag{62}$$

■

Remark 3: Note that since α is positive definite and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, given any $D > 0$ there always exist $\ell_1, \ell_2, \ell_3, \widetilde{\ell}_1, \widetilde{\ell}_2$ satisfying (32) and (50). Moreover, ℓ_1 and ℓ_3 in Corollary 2 can be taken arbitrarily small. Consequently, if $\delta_0 = 0$ then the residual set to which trajectories of the exact model converge can be made arbitrarily small (i.e., we achieve "practical" stabilization). ■

Remark 4: We emphasize that no regularity assumptions on $H_{T,h}$ and $G_{T,h}$ are needed in Theorem 1. In particular, $H_{T,h}$ and $G_{T,h}$ may be discontinuous in x and z. This allows us to consider hysteresis switching control laws in Section VI-D. Moreover, we do not need continuity of $V_{T,h}$ in z. This is because we are using the same control law for exact and approximate models. If we wanted to consider situations where we apply different control laws (that are "close" in some sense) to exact and approximate models, then we would in general need stronger continuity assumptions of the Lyapunov function $V_{T,h}$ in (x,z) and not only in x in order to prove similar results. Similar observations hold for results presented in the next section. ■

Remark 5: Theorem 1 provides a general framework for controller design for sampled-data differential inclusions based on their approximate discrete-time plant models. The theorem indicates that besides stabilization of the approximate discrete-time model (item 1), the controller should also posses extra properties (items 2 and 3) in order to be stabilizing for the exact discrete-time plant model (see the first motivating example where item 1 holds whereas items 2 and 3 do not hold). Note that item 2 of Theorem 1 is relatively easy to check since we know the control law. Subsection V is dedicated to presenting checkable sufficient conditions for consistency property that is used in item 3 of Theorem 1. These conditions are important since they do not require the knowledge of F_T^e to check the consistency. Indeed, these conditions use only the information about the continuous-time plant, the control law and the approximate model to check the consistency property. ■

Remark 6: We emphasize that the conditions of Theorem 1 may hold in a semiglobal sense (in the parameter T). In this case, we may be able to achieve semiglobal practical stabilization by appropriately choosing T and h. We did not state these results for space reasons. In this case h^* depends on T and in particular smaller T will normally require smaller h^*. Roughly speaking we first achieve "semiglobal" stabilization by choosing T sufficiently small and then with the fixed T we achieve "practical" stabilization by choosing h sufficiently small. ■

B. Case 2: T is equal to h

All the references that we are aware of [10], [12], [28], [37], [40] that deal with controller design via approximate discrete-time models exploit the Euler approximate model in controller design. The Euler approximate model can be regarded as a special case of (39) where $T = h$ and $F^a_{T,h}(x,u) = x + Tf(x,u)$. This case is of particular importance since it is best suited for the design of explicit control laws (see, for instance, [37]). Hence, we discuss in this section the special case when $T = h$. We consider families (39), (44) and (38), (44) with $T = h$ and we use the notation:

$$F^a_T(x,u) := F^a_{T,T}(x,u),\ G_T(z,x) := G_{T,T}(z,x), H_T(z,x) := H_{T,T}(z,x)\ .$$

Since in this case we need to achieve both semiglobal and practical stabilization by reducing T, the conditions that we use are slightly different from the ones used in the previous sections. For instance, we use a different notion of consistency from the one used in Theorem 1. Also, we will need stronger continuity of the Lyapunov function than the partial quasi-continuity property which was used in Theorem 1. Next we define the properties used in this section:

Definition 4: **[Uniformly bounded controls]** Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$ be given. If for any strictly positive real numbers (Δ_1, Δ_2) there exists $T^* > 0$ such that for any $T \in (0, T^*)$ we have

$$\sup_{\{\tilde{x}\in\mathcal{H}_{\mathcal{A}}(0,\Delta_1)\ ,\ w\in H_T(z,x)\}} |w| \le \Delta_2\ , \tag{63}$$

then we say that the family of controllers (44) is $\mathcal{A}$-uniformly bounded. ■

Definition 5: **[Modeling consistency]** Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ be given. If for any pair of strictly positive numbers (Δ_1, Δ_2) there exist $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that for any $(x,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta_1) \times \Delta_2\overline{\mathcal{B}}_m$ and all $T \in (0,T^*)$ we have $F^e_T(x,u) \subseteq F^a_T(x,u) + T\rho(T)\overline{\mathcal{B}}_n$, then we say that the family F^a_T is $\mathcal{A}$-one-step upper semi-consistent with F^e_T. ■
Sufficient checkable conditions for one step upper semi-consistency are presented in Section V.

Definition 6: **[Partially Lipschitz Lyapunov certificates of asymptotic stability]** Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^{n\times n_c}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a positive definite function $\alpha_3 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and a nonnegative real number δ_0 be given. Suppose for any triple of strictly positive real numbers (D, δ_1, δ_2) with $\delta_2 < D$ there exist $T^* > 0$ and $L > 0$ and for all $T \in (0,T^*)$ there exists $V_T : \mathbb{R}^{n+n_c} \to \mathbb{R}_{\ge 0}$ such that for all $(x,z) \in \mathcal{H}_{\mathcal{A}}(0,D)$ $T \in (0,T^*)$, we have

$$\alpha_1(|(x,z)|_{\mathcal{A}}) \le V_T(x,z) \le \alpha_2(|(x,z)|_{\mathcal{A}}) \tag{64}$$

$$\sup_{\substack{w_1 \in F^a_T(x, H_T(x,z)),\\ w_2 \in G_T(x,z)}} V_T(w_1,w_2) - V_T(x,z) \le -T\alpha_3(|(x,z)|_{\mathcal{A}}) + T\delta_1 + \delta_0\ , \tag{65}$$

and, for all $(x_1,z), (x_2,z) \in \mathcal{H}_{\mathcal{A}}(\delta_2, D)$ we have

$$|V_T(x_1,z) - V_T(x_2,z)| \le L\,|x_1 - x_2|\ . \tag{66}$$

Then, we say that $(V_T, \alpha_1, \alpha_2, \alpha_3, \delta_0)$ provides an $\mathcal{A}$-partially Lipschitz family of Lyapunov certificates for the family (39), (44). ■

Proofs of results in this section are very similar to proofs of results for fixed sampling periods and are omitted (complete proofs for the case $\delta_0 = 0$ can be found in the conference paper [34]).

Theorem 2: Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$, the family of functions $V_T : \mathbb{R}^{n+n_c} \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a positive definite function $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a nonnegative real number δ_0 be such that the following conditions hold:
1. $(V_T, \alpha_1, \alpha_2, \alpha_3, \delta_0)$ provides an $\mathcal{A}$-partially Lipschitz family of Lyapunov certificates for the family (39), (44);
2. The family of controllers (44) is $\mathcal{A}$-uniformly bounded;
3. The family F_T^a is $P(\mathcal{A})$-one-step upper semi-consistent with F_T^e.

Let $\alpha_1, \alpha_2, \alpha_3$ come from the definition of V_T in item 1. Let $\rho_1, \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{M}$ be generated by α_3 via Lemma 1. Let $\alpha(s) := \frac{1}{2}\rho_1 \circ \alpha_2^{-1}(s) \cdot \rho_2 \circ \alpha_1^{-1}(s)$. Then, for any strictly positive $\ell_1, \ell_2, \widetilde{\ell}_1, \widetilde{\ell}_2$, where $\widetilde{\ell}_1 > \delta_0$, there exists $T^* > 0$ such that for any $T \in (0, T^*)$ we have that
1. For all $\tilde{x} \in \mathcal{V}_T(0, \ell_2)$ and $\tilde{x}_1^+ \in \mathcal{F}_T^a(\tilde{x})$ we have $V_T(\tilde{x}_1^+) - V_T(\tilde{x}) \leq -T\left(2\alpha(V_T(\tilde{x})) - \widetilde{\ell}_1\right)$.
2. For all $\tilde{x} \in \mathcal{V}_T(0, \ell_2)$ and all $\tilde{x}_2^+ \in \mathcal{F}_T^e(\tilde{x})$ with $V_T(\tilde{x}_2^+) \geq \ell_1$ there exists $\tilde{x}_2^+ \in \mathcal{F}_T^a(\tilde{x})$ such that $V_T(\tilde{x}_2^+) \leq V_T(\tilde{x}_1^+) + T\widetilde{\ell}_2$.

■

Corollary 3: Suppose that all conditions of Theorem 2 hold. Let $\ell_1, \ell_2, \ell_3, \widetilde{\ell}_1, \widetilde{\ell}_2$ be arbitrary positive numbers satisfying conditions (32), $\widetilde{\ell}_1 > \delta_0$. Then, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ we have:
1.

$$V_T(\tilde{x}_2^+) - V_T(\tilde{x}) \leq -T\alpha(V_T(\tilde{x})) \qquad \forall \tilde{x} \in \mathcal{V}_T(\ell_1, \ell_2), \tilde{x}_2^+ \in \mathcal{F}_T^e(\tilde{x}) \bigcap \mathcal{V}_T(\ell_1, \ell_2 + T\ell_3) \tag{67}$$

$$V_T(x_2^+) - V_T(\tilde{x}) \leq T\ell_3 \qquad \forall \tilde{x} \in \mathcal{V}_T(0, \ell_2)\ , \tilde{x}_2^+ \in \mathcal{F}_T^e(x)\ . \tag{68}$$

2. with α_3 generating α using Theorem 1 and α generating $\beta \in \mathcal{KL}$ using Proposition 1, for all $x_\circ \in \mathcal{V}_T(0, \ell_2)$, the solutions $\phi(\cdot, x_\circ)$ of the family (38), (44) satisfy

$$|\phi(k, \tilde{x}_\circ)|_{\mathcal{A}} \leq \max\left\{\alpha_1^{-1}(\beta(\alpha_2(|\tilde{x}_\circ|_{\mathcal{A}}), kT)), \alpha_1^{-1}(\ell_1 + T\ell_3)\right\} \qquad \forall k \in \{0, 1, 2, \dots\}\ . \tag{69}$$

■

Remark 7: Results of this section cover the case when the approximate model is the Euler approximation. The Euler approximate model is best suited for controller design since it has the same structure as the continuous-time model. In Section VI we show several non-trivial situations where the Lyapunov conditions of Theorem 2 can be verified. Systematic methods for design of controllers based on approximate models so that the closed loop approximate model satisfies Lyapunov conditions of Theorem 2 were addressed in [37] for strict feedback systems via backstepping and in [25] using the ideas of changing supply rates of input-output to state stable systems. Finding systematic controller design procedures for other classes of systems and approximate models so that the conditions of Theorems 1 or 2 are satisfied is an important area for further research. ■

Remark 8: We emphasize that besides stability of the family of approximate models we also need consistency and boundedness of the control law in order to guarantee that the family of exact models would be stabilized. In particular, consistency conditions help us discard bad controllers like the ones presented in the first three motivating examples. ■

Remark 9: We emphasize that in Theorem 2 and Corollary 3 the number ℓ_2 can be arbitrarily large and ℓ_1 and ℓ_3 can be arbitrarily small. Hence, we have conditions for semiglobal practical stabilization of the family of exact models. ■

Remark 10: Similar results to Theorem 2 have been proved for systems with disturbances for input-to-state stability in [35] and integral versions of input-to-state stability property in [38]. It is not hard to see that appropriate versions of Theorem 1 can also be proved for the input-to-state stability properties considered in [35], [38]. ■

V. Consistency

The purpose of this section is to present checkable sufficient conditions for the consistency property of Definition 2. We do this by introducing two new consistency properties between the numerical integration scheme (40), defined by f_h, and the exact discrete-time model defined by F_h^e. Motivation for introducing new consistency properties (Definitions 7 and 8) is twofold. First, they can be used to prove sufficient conditions for consistency property defined in Definition 2 (see Corollary 4) that is used in Theorem 1. Second, new consistency properties are of interest in their own right and one of them is used in statement of Theorem 2.

In this section we use respectively notation $\mathcal{S}_h^a(x,u)$ and $\mathcal{S}^e(x,u)$ to denote sets of solutions of the difference inclusion (40) and differential inclusion (12). The solutions of the difference inclusion (40) and differential inclusion (12) are respectively denoted as $\phi_h^a(k,x,u)$ and $\phi^e(t,x,u)$, that is $\phi_h^a \in \mathcal{S}_h^a(x,u)$ and $\phi^e \in \mathcal{S}^e(x,u)$.

Definition 7: Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty closed set. The family f_h is said to be $\mathcal{A}$*-one-step upper semi-consistent* with F_h^e if for each pair of positive real numbers (Δ, M) there exist $\rho \in \mathcal{K}_\infty$ and $h^* > 0$ such that, for all $(x,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta) \times M\overline{\mathcal{B}}_m$ and all $h \in (0,h^*)$, we have

$$F_h^e(x,u) \subseteq f_h(x,u) + h\rho(h)\overline{\mathcal{B}}_n \tag{70}$$

■

A sufficient condition for one-step upper semi-consistency is stated below and proved in the appendix.

Proposition 4: If the following conditions hold:

1. for each $\widetilde{\Delta} \geq 0$ there exists $M > 0$ such that

$$\sup_{\left\{(x,u)\in\mathcal{H}_{\mathcal{A}}(0,\widetilde{\Delta})\times\widetilde{\Delta}\overline{\mathcal{B}}_m\ ,\ w\in F(x,u)\right\}} |w| \leq M\ , \tag{71}$$

2. there exists a set-valued map $\widetilde{F}(\cdot,\cdot)$ such that
   (a) for each $u \in \mathbb{R}^m$, the set-valued map $\widetilde{F}(\cdot,u)$ satisfies the basic conditions of Assumption 1,
   (b) f_h is $\mathcal{A}$-one-step upper semi-consistent with $\widetilde{f}_h^{Euler}(x,u) := x + h\widetilde{F}(x,u)$,
   (c) for each $\widetilde{\Delta} \geq 0$ there exists $\widetilde{\rho} \in \mathcal{K}_\infty$ such that

$$(x,\xi,u) \in \mathcal{H}_{\mathcal{A}}(0,\widetilde{\Delta}) \times \mathcal{H}_{\mathcal{A}}(0,\widetilde{\Delta}) \times \widetilde{\Delta}\overline{\mathcal{B}}_m \quad \Longrightarrow \quad F(\xi,u) \subseteq \widetilde{F}(x,u) + \widetilde{\rho}(|\xi - x|)\overline{\mathcal{B}}_n \tag{72}$$

then f_h is $\mathcal{A}$-one-step upper semi-consistent with F_h^e. ■

Remark 11: A candidate choice for $\widetilde{F}$ is $\widetilde{F}(x,u) = F(x,u)$. In this case, item 2c becomes a *uniform* continuity condition on $F(\cdot,u)$. If $F(\cdot,u)$ has this uniform continuity property and item 1 of the lemma holds then the Euler approximation $f_h(x,u) = x + hF(x,u)$ is one-step upper semi-consistent with F_h^e. Unfortunately, it is not sufficient to take $\widetilde{F}(x,u) = F(x,u)$ and assume that $F(x,u)$ is only upper semi-continuous, as the following example shows.

Example 1: Consider

$$F(x,u) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 10 & \text{if} \quad x > 0 \\ [1,10] & \text{if} \quad x = 0 \end{cases} \tag{73}$$

which satisfies Assumption 1 and let $f_h(x,u) = x + hF(x,u)$. Let $\mathcal{A} = \{1\}$, let $\Delta = 2$ and suppose there exist $\rho \in \mathcal{K}_\infty$ and $h^* > 0$ such that (70) holds for all $(x,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta) \times \Delta\overline{\mathcal{B}}_m$. Let $h > 0$ be such that $h < \min\left\{h^*, 1, \rho^{-1}\left(\frac{1}{2}\right)\right\}$. Let $x = -\frac{h}{2}$, so that $x \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$, and $u = 0$. Then $f_h(x,u) + \frac{1}{2}h\overline{\mathcal{B}}_1 = [0,h]$. On the other hand $F_h^e(x,u) = 5h$. It follows from the fact that $\rho(h) \leq \frac{1}{2}$ that $F_h^e(x,u) \not\subseteq f_h(x,u) + h\rho(h)\overline{\mathcal{B}}_1$. This contradicts (70).

We note that it is still possible to find a one-step upper semi-consistent approximate model for this system (see Remark 12 below). ■

Remark 12: In the absence of the uniform continuity condition, one option is to search for an inclusion $\widetilde{F}$ satisfying basic conditions such that: (i) $F(x,u) \subseteq \widetilde{F}(x,u)$; (ii) $\widetilde{F}(\cdot,u)$ is uniformly continuous, or uniformly locally Lipschitz; (iii) the Euler approximation of $\dot{x} \in \widetilde{F}(x,u)$ can be stabilized in the appropriate sense. In this case, $x + h\widetilde{F}(x,u)$ will be $\mathcal{A}$-one-step upper semi-consistent with F_h^e and we will be in a position to apply our results. ■

Definition 8: The family f_h is said to be multi-step upper semi-consistent with F_h^e if, for each 4-tuple of strictly positive real numbers $(T, \eta, \Delta_1, \Delta_2)$ there exist a function $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $h^* > 0$ such that, for all $h \in (0, h^*)$ we have

$$\left\{(x,u),(y,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta_1) \times \Delta_2 \overline{\mathcal{B}}_m \ , \ |x-y| \leq \delta\right\} \qquad \Longrightarrow \qquad F_h^e(x,u) \subseteq f_h(y,u) + \alpha(\delta,h)\overline{\mathcal{B}}_n \tag{74}$$

$$i \leq T/h \qquad \Longrightarrow \qquad \alpha^i(0,h) := \overbrace{\alpha\left(\cdots \alpha\left(\alpha\right.\right.}^{i}\left.\left.\left(0,h\right),h\right)\cdots,h\right) \leq \eta \ . \tag{75}$$

■

A sufficient condition for multi-step upper semi-consistency is given in the following:

Proposition 5: If, for each pair of strictly positive real numbers (Δ_1, Δ_2), there exist $K > 0$, $\rho \in \mathcal{K}_\infty$ and $h^* > 0$ such that for all $h \in (0, h^*)$ and all $(x,u),(y,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta_1) \times \Delta_2 \overline{\mathcal{B}}_m$ we have

$$F_h^e(x,u) \subseteq f_h(y,u) + \left[(1+Kh)\,|x-y| + h\rho(h)\right]\overline{\mathcal{B}}_n \tag{76}$$

then f_h is multi-step consistent with F_h^e. ■

Proof. Let $(T, \eta, \Delta_1, \Delta_2)$ be given. From the assumption of the lemma, let these numbers generate $K > 0$, $\rho \in \mathcal{K}_\infty$ and $h_1^* > 0$. Define

$$\alpha(\delta,h) := (1+Kh)\delta + h\rho(h); \qquad h^* := \min\left\{h_1^*, \rho^{-1}\left(\frac{\eta K}{\exp(KT)-1}\right)\right\} \ . \tag{77}$$

With these definitions, the condition (74) is satisfied. Also note that

$$\alpha^i(0,h) = h\rho(h)\sum_{j=0}^{i-1}(1+Kh)^j = \frac{\rho(h)}{K}\left[(1+Kh)^i - 1\right] \leq \frac{\rho(h)}{K}\left[\exp(KT)-1\right] \tag{78}$$

and so (75) is satisfied. ■

Remark 13: Relative to the one-step consistency condition, the condition of Lemma 5 is guaranteed by one-step consistency plus the following type of Lipschitz condition on either the family F_h^e or the family f_h: for each (Δ_1, Δ_2) there exist $K > 0$ and $h^* > 0$ such that for all $(x,u),(y,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta_1) \times \Delta_2 \overline{\mathcal{B}}_m$ and all $h \in (0, h^*]$,

$$f_h(x,u) \subseteq f_h(y,u) + (1+Kh)|x-y|\overline{\mathcal{B}}_n \ . \tag{79}$$

This condition is guaranteed for F_h^e when $F(x,u)$ is locally Lipschitz. The condition given in Lemma 5 for multi-step consistency is similar to conditions used in the numerical analysis literature (e.g., see conditions (i) and (iii) of Assumption 6.1.2 in [43, pg.429]). ■

In terms of trajectory error over "continuous-time" intervals with length of order one, multi-step consistency gives the following:

Proposition 6: If f_h is multi-step consistent with F_h^e then for each 4-tuple of strictly positive real numbers $(T, \eta, \Delta_1, \Delta_2)$ there exists $h^* > 0$ such that, if h, u and ξ satisfy

$$h \in (0, h^*) \ , \qquad |u| \le \Delta_2, \qquad \phi_h^a(i, \xi, u) \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1) \cap \mathcal{S}_h^a(x, u) \quad \forall i : ih \in [0, T], \tag{80}$$

then for any $\phi_h^a(i, \xi, u) \in \mathcal{S}_h^a(\xi, u)$ and $\phi^e(ih, \xi, u) \in \mathcal{S}^e(\xi, u)$ we have

$$|\phi^e(ih, \xi, u) - \phi_h^a(i, \xi, u)| \le \eta \qquad \forall i : ih \in [0, T] \ . \tag{81}$$

■

Proof of Proposition 6: Define $\overline{\Delta}_1 = \Delta_1 + \eta$. Since f_h is multi-step consistent with F_h^e, there exist a function $\alpha(\cdot, \cdot)$ and a strictly positive real number h^* such that (74) and (75) are satisfied for the 4-tuple $(T, \eta, \overline{\Delta}_1, \Delta_2)$. We now prove the result by induction. For any given $i \ge 0$ we consider arbitrary $\phi_h^a(i, \xi, u) \in \mathcal{S}_h^a(\xi, u)$ and $\phi^e(ih, \xi, u) \in \mathcal{S}^e(\xi, u)$. First we have $|\phi^e(0, \xi, u) - \phi_h^a(0, \xi, u)| = 0 \le \alpha^1(0, h) \le \eta$. Next, suppose $|\phi^e(ih, \xi, u) - \phi_h^a(i, \xi, u)| \le \alpha^i(0, i) \le \eta$ and $(i+1)h \in [0, T]$. Since $(i+1)h \in [0, T]$, it follows from the definition of $\overline{\Delta}_1$ that $\phi_h^a(i, \xi, u), \phi^e(ih, \xi, u) \in \mathcal{H}_{\mathcal{A}}(0, \overline{\Delta}_1)$. It then follows from (74) that $|\phi^e((i+1)h, \xi, u) - \phi_h^a(i+1, \xi, u)| \le \alpha^{i+1}(0, h)$. Since $(i+1)h \in [0, T]$ it follows from (75) that $\alpha^{i+1}(0, h) \le \eta$. ■

A simple consequence of the above lemma is a sufficient condition for consistency presented in Definition 2.

Corollary 4: Let f_h be multiple-step upper semi-consistent with F_h^e, let the family $F_{T,h}^a$ be defined using (41) and (42). Given any $(T, \epsilon, \Delta_1, \Delta_2)$ there exists $h^* > 0$ such that if the condition (80) holds then for all $(x, u) \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1) \times \Delta_2 \overline{\mathcal{B}}_m$ and all $h \in (0, h^*]$, we have $F_T^e(x, u) \subseteq F_{T,h}^a(x, u) + \epsilon T \overline{\mathcal{B}}_n$ ■

Remark 14: We note that in order to verify that consistency in Definition 2 holds, one needs to show that the condition (80) holds. This may be hard to do in general with a given triple (T, Δ_1, Δ_2). However, if the family f_h satisfies a Lipschitz condition, uniform in h, then given any (T, Δ_1, Δ_2) there exist $(\widetilde{\Delta}_1, \widetilde{\Delta}_2)$ and $h^* > 0$ such that if $(\xi, u) \in \mathcal{H}_{\mathcal{A}}(0, \widetilde{\Delta}_1) \times \widetilde{\Delta}_2 \overline{\mathcal{B}}_m$ and $h \in (0, h^*)$, then $\phi_h^a(k, \xi, u) \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1)$ for all k such that $kh \in [0, T]$. ■

VI. Lyapunov certificates of asymptotic stability

We have shown through our main results that having a family of Lyapunov certificates, in addition to bounded controls and modeling consistency, is sufficient to guarantee robustness to the mismatch between an approximate discrete-time model and the exact discrete-time model. We have also shown by example that the lack of Lyapunov certificates can suggest a lack of the appropriate robustness. In this way, we have emphasized the importance of a Lyapunov proof of asymptotic stability for discrete-time systems, especially those with discontinuous right-hand side. In this section we will further clarify and illustrate the notion of Lyapunov certificates.

A. Some general observations

Since checking the right continuity properties of Lyapunov certificates is hard in general, it is useful to explore situations when this procedure can be simplified. The right continuity is needed for our results to hold (see item 1 in Theorems 1 and 2). Several such situations are presented next.

In several common situations a family of Lyapunov certificates $V_{T,h}$ will exist and $\lim_{h \to 0^+} V_{T,h}(x)$ will exist for each x. The next two propositions, which are simple consequences of continuity, can be useful for relating the continuity of the limiting function (which is simpler to verify) to the continuity of the family of functions.

Proposition 7: Let $T > 0$ be fixed. Suppose the following conditions hold: (i) For each pair of strictly positive real numbers ϵ and b there exist $h^* > 0$ and $\delta > 0$ such that $h \in (0, h^*]$, $|x - y| \le \delta$, $\max\{|x|, |y|\} \le b$ implies that

$|V_{T,h}(x) - V_{T,h}(y)| \leq \epsilon$. (ii) For each $x \in \mathbb{R}^n$ we have that $V_T(x) := \lim_{h\to 0} V_{T,h}(x)$ is well defined. Then V_T is continuous. ■

Proposition 8: Let $T > 0$ be fixed. Suppose that the following conditions hold: (i) For each $x \in \mathbb{R}^n$ the limit $V_T(x) := \lim_{h\to 0} V_{T,h}(x)$ is well defined. (ii) V_T is continuous. (iii) For each $\epsilon > 0$ and $b > 0$ there exists $h^* > 0$ such that $h \in (0, h^*]$, $|x| \leq b$ implies that $|V_{T,h}(x) - V_T(x)| \leq \epsilon$. Then, for each $\tilde{\epsilon} > 0$ and $b > 0$ there exist $h^* > 0$ and $\delta > 0$ such that $h \in (0, h^*]$, $|x - y| \leq \delta$, $\max\{|x|, |y|\} \leq b$ implies that $|V_{T,h}(x) - V_{T,h}(y)| \leq \tilde{\epsilon}$. ■

Since we are dealing with families of systems parameterized by h, it may seem more natural to use Lyapunov certificates for asymptotic stability with respect to a family of nonempty closed sets $\mathcal{A}_h$ that are parameterized with h. However, under reasonable general conditions we can show that verifying stability with respect to a family of sets $\mathcal{A}_h$ can be done by considering stability with respect to a fixed set $\mathcal{A}$. The following proposition which is proved in the appendix makes this statement precise:

Proposition 9: Suppose

$$\alpha_1(|x|_{\mathcal{A}_h}) \leq V_h(x) \leq \alpha_2(|x|_{\mathcal{A}_h}) \ . \tag{82}$$

Let $\mathcal{A}$ be a nonempty closed set and suppose there exists $\delta_1 \in \mathcal{K}_\infty$ such that

$$|x|_{\mathcal{A}} - \delta_1(h) \leq |x|_{\mathcal{A}_h} \leq |x|_{\mathcal{A}} + \delta_1(h) \ . \tag{83}$$

Under these conditions, there exists $\delta \in \mathcal{K}_\infty$ and a family of functions $x \to \widetilde{V}_h(x)$ such that

$$\begin{aligned} \min\left\{|x|_{\mathcal{A}}, \alpha_1\left(\frac{1}{2}|x|_{\mathcal{A}}\right)\right\} &\leq \widetilde{V}_h(x) \leq |x|_{\mathcal{A}} + \alpha_2(2|x|_{\mathcal{A}}) && (84)\\ \widetilde{V}_h(x) &= V_h(x) \qquad\qquad \forall |x|_{\mathcal{A}} \geq \delta(h) && (85)\\ \widetilde{V}_h(x^+) - \widetilde{V}_h(x) &\leq V_h(x^+) - V_h(x) + \delta(h) \ . && (86) \end{aligned}$$

B. Illustrations

In the rest of this section we present two new nontrivial examples that further illustrate how Lyapunov certificates can be found and used in controller design based on approximate discrete-time plant models. The examples illustrate generality and rigor of our approach. In particular, since results of the present paper are much more general than the results of [33], we are able to rigorously tackle completely new situations which are illustrated below.

The first example addresses stabilization and input-to-state stabilization of a nonholonomic integrator. This example illustrates generality of our approach by: (i) using a non-standard approximate discrete-time model for the controller design; (ii) using a non-standard modeling parameter defined as $h = \frac{1}{kT}$ where T is the sampling period and $k > 0$ is a constant. As a result, *we can achieve semiglobal practical stabilization with arbitrarily large sampling period*; (iii) using appropriate differential inclusions we show that we can tackle the problem of input-to-state stabilization of a system with exogenous disturbances. This is an alternative approach to the approach for input-to-state stabilization taken in [35]; (iv) using a discontinuous control law. In the second example we combine local and global control laws using hysteresis switching.

C. Nonholonomic integrator

Consider the control system

$$\dot{\theta} = k(\mathrm{sat}_{\pi/4}(u_1) - \theta); \qquad \dot{x} = \cos(\theta)\,[u_2 + d]\,; \qquad \dot{y} = \sin(\theta)\,[u_2 + d] \ \ , \tag{87}$$

where $k > 0$, which represents a vehicle steering model where, with the function $\mathrm{sat}_{\pi/4}(\cdot)$ defined as $\mathrm{sat}_{\pi/4}(u_1) := \mathrm{sgn}(u_1)\min\left\{\frac{\pi}{4}, |u_1|\right\}$, the steering angle is limited to the range $\pm\pi/4$. The linearization of this system at the origin is uncontrollable through (u_1, u_2). The quantity d is a disturbance input.

We will develop a family of control algorithms, based on an approximate discrete-time model, that are appropriate for controlling the exact discrete-time model when the sampling period is large. In order to apply our fixed sampling period results to this case, we reparameterize time as $\tau = t/T$ where T is the sampling period in the original time scale. In the new time scale, τ, the sampling period is fixed to be one. We also define $\widetilde{u}_2 = Tu_2$, $\widetilde{d} = Td$, and a modeling parameter $h = 1/kT$. In this case, the new differential equation we consider is

$$h\theta^{'} = (\mathrm{sat}_{\pi/4}(u_1) - \theta); \qquad x^{'} = \cos(\theta)\left[\widetilde{u}_2 + \widetilde{d}\right]; \qquad y^{'} = \sin(\theta)\left[\widetilde{u}_2 + \widetilde{d}\right] \tag{88}$$

where $x^{'} = \frac{dx}{d\tau}$. In order to apply our main results, we will consider control of the differential inclusion

$$h\theta^{'} = (\mathrm{sat}_{\pi/4}(u_1) - \theta); \qquad x^{'} \in \cos(\theta)\left[\widetilde{u}_2 + \gamma\overline{\mathcal{B}}_1\right]; \qquad y^{'} \in \sin(\theta)\left[\widetilde{u}_2 + \gamma\overline{\mathcal{B}}_1\right] \ . \tag{89}$$

The approximate discrete-time model we consider is

$$\theta^+ = \mathrm{sat}_{\pi/4}(u_1); \qquad x^+ \in x + \cos(\mathrm{sat}_{\pi/4}(u_1))\widetilde{u}_2 + \gamma\overline{\mathcal{B}}_1; \qquad y^+ \in y + \sin(\mathrm{sat}_{\pi/4}(u_1))\widetilde{u}_2 + \gamma\overline{\mathcal{B}}_1 \ . \tag{90}$$

Using that $\sin(\cdot)$, $\cos(\cdot)$ and $\mathrm{sat}_{\pi/4}(\cdot)$ are Lipschitz with constant equal to one, the exact discrete-time satisfies

$$\begin{aligned} \theta^+ &= \exp(-1/h)\theta + (1 - \exp(-1/h))\mathrm{sat}_{\pi/4}(u_1) \\ x^+ &\in x + \cos(\mathrm{sat}_{\pi/4}(u_1))\widetilde{u}_2 + \left(h\left[1 - e^{-1/h}\right]|\theta - u_1||\widetilde{u}_2| + \gamma\right)\overline{\mathcal{B}}_1 \\ y^+ &\in y + \sin(\mathrm{sat}_{\pi/4}(u_1))\widetilde{u}_2 + \left(h\left[1 - e^{-1/h}\right]|\theta - u_1||\widetilde{u}_2| + \gamma\right)\overline{\mathcal{B}}_1 \ . \end{aligned} \tag{91}$$

It is immediate that for any compact set of initial conditions, the approximate discrete-time model is upper semi-consistent with the exact discrete-time model, i.e., the third condition of Theorem 1 is satisfied.

To control the approximate model (90) we first consider stabilizing the (x, y) subsystem with $\gamma = 0$ by choosing

$$\widetilde{u}_2 = \frac{v_2}{\cos(\mathrm{sat}_{\pi/4}(u_1))}; \qquad \mathrm{sat}_{\pi/4}(u_1) = \arctan(\mathrm{sat}_1(v_1)) \ , \tag{92}$$

so that

$$x^+ = x + v_2; \qquad y^+ = y + \mathrm{sat}_1(v_1)v_2 \ , \tag{93}$$

and then picking

$$v_1 = \begin{cases} -\dfrac{y}{v_2} & v_2 \neq 0 \\ 0 & \text{otherwise} \end{cases} ; \qquad v_2 = \begin{cases} -x & |x|^2 \geq |y| \\ \sqrt{|y|} & \text{otherwise} \end{cases} . \tag{94}$$

Observe that $|v_1| \leq \sqrt{|y|}$ and $v_2 = 0 \Longrightarrow (x, y) = (0, 0)$. Moreover, the control law is uniformly bounded and hence the second condition of Theorem 1 holds.

Using the (partial) Lyapunov function $V_\circ(x, y) = \frac{1}{4}|x|\mathrm{sat}_1(|x|) + |y|$, we show that the (x, y) subsystem with $\gamma = 0$ is stabilized with the above given control law:

Case 1: $|x|^2 \geq |y|$, $v_2 \neq 0$:[1]

$$\begin{aligned} V_\circ(x^+,y^+) - V_\circ(x,y) &= |y - \mathrm{sat}_1\,(y/v_2)\,v_2| - \tfrac{1}{4}|x|\mathrm{sat}_1(|x|) - |y| \\ &= -\tfrac{1}{4}|x|\mathrm{sat}_1(|x|) - |y|\frac{\mathrm{sat}_1\,(|v_1|)}{|v_1|} \\ &\leq -\tfrac{1}{4}|x|\mathrm{sat}_1(|x|) - \sqrt{|y|}\mathrm{sat}_1\left(\sqrt{|y|}\right)\ . \end{aligned} \tag{95}$$

Case 2: $|x|^2 < |y|$, $v_2 \neq 0$:[2]

$$\begin{aligned} V_\circ(x^+,y^+) - V_\circ(x,y) &\leq \tfrac{1}{2}|x|\mathrm{sat}_1(|x|) + \tfrac{1}{2}\sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|}) - \tfrac{1}{4}|x|\mathrm{sat}_1(|x|) - \sqrt{|y|}\mathrm{sat}_1\left(\sqrt{|y|}\right) \\ &= \tfrac{1}{4}|x|\mathrm{sat}_1(|x|) - \tfrac{1}{2}\sqrt{|y|}\mathrm{sat}_1\left(\sqrt{|y|}\right) \\ &\leq \tfrac{1}{4}\sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|}) - \tfrac{1}{2}\sqrt{|y|}\mathrm{sat}_1\left(\sqrt{|y|}\right) \\ &= -\tfrac{1}{4}\sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|}) \\ &\leq -\tfrac{1}{5}\left(\tfrac{1}{4}|x|\mathrm{sat}_1(|x|) + \sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|})\right)\ . \end{aligned} \tag{96}$$

Now we consider as a complete Lyapunov function for the approximate closed-loop system $V(\theta,x,y) := \frac{1}{10}|\theta|\mathrm{sat}_1(|\theta|) + V_\circ(x,y)$. We have

$$\begin{aligned} V(\theta^+,x^+,y^+) - V(\theta,x,y) &= \tfrac{1}{10}\sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|}) - \tfrac{1}{10}|\theta|\mathrm{sat}_1(|\theta|) - \tfrac{1}{20}|x|\mathrm{sat}_1(|x|) - \tfrac{1}{5}\sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|}) \\ &= -\tfrac{1}{10}\left(|\theta|\mathrm{sat}_1(|\theta|) + \tfrac{1}{2}|x|\mathrm{sat}_1(|x|) + \sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|})\right)\ . \end{aligned} \tag{97}$$

Finally, we allow $\gamma \neq 0$. Using that the function V is globally Lipschitz with Lipschitz constant equal to two, since $s \to |s|\mathrm{sat}_1(|s|)$ is globally Lipschitz with Lipschitz constant equal to two, we then have

$$V(\theta^+,x^+,y^+) - V(\theta,x,y) \leq -\frac{1}{10}\left(|\theta|\mathrm{sat}_1(|\theta|) + \frac{1}{2}|x|\mathrm{sat}_1(|x|) + \sqrt{|y|}\mathrm{sat}_1(\sqrt{|y|})\right) + 2\gamma\ . \tag{98}$$

It follows that the first condition of Theorem 1 holds with $\delta_0 = 2\gamma$ for sufficiently large set of initial conditions. Hence, we can conclude using Corollary 2 and Remark 2 that for sufficiently large compact set of initial conditions and arbitrarily small ℓ_3, the trajectories of the exact system corresponding to (88) satisfy the stability bound (62) with $T = 1$ for sufficiently small values of h. Note that because of (50) we have that trajectories converge to a ball whose radius depends on the size of disturbance (i.e. γ).

D. A partially quasi-continuous Lyapunov certificate for hysteresis switching

In this section, we show how to construct a partially quasi-continuous Lyapunov certificate for a control algorithm that combines local and global controllers through hysteresis switching. We consider two approximate models:

$$x^+ \in F^{a,glob}_{T,h}(x,u) \tag{99}$$

$$x^+ \in F^{a,loc}_{T,h}(x,u)\ , \tag{100}$$

and we assume that each model is upper semi-consistent with the corresponding exact discrete-time model. A consequence of this is that the map defined at x by arbitrarily picking either $F^{a,glob}_{T,h}(x,u)$ or $F^{a,loc}_{T,h}(x,u)$ is also upper semi-consistent with the exact discrete-time model. To simplify notation in what follows, we will use F^{glob} and F^{loc} for $F^{a,glob}_{T,h}$ and $F^{a,loc}_{T,h}$ respectively, and similarly for the functions given below.

[1] The final inequality in (95) uses the fact that $0 \leq a \leq b$ implies $a\mathrm{sat}_1(a) \geq b\mathrm{sat}_1(b)$.

[2] The first inequality in (96) uses Fact 1 given in the appendix.

Assumption 2: There exist a closed set $\mathcal{A}$, functions $u_{glob}, u_{loc} : \mathbb{R}^n \to \mathbb{R}^m$, functions $V_{glob}, V_{loc} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_{glob}$, $\underline{\alpha}_{loc}$, $\overline{\alpha}_{glob}$, $\overline{\alpha}_{loc}$, continuous, positive definite functions $\gamma_{glob}, \gamma_{loc} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and positive real numbers c_{glob}, c_{loc} such that:

1. for all $x \in \mathbb{R}^n$ we have

$$\underline{\alpha}_{glob}(|x|_{\mathcal{A}}) \leq V_{glob}(x) \leq \overline{\alpha}_{glob}(|x|_{\mathcal{A}}) \tag{101}$$

$$\underline{\alpha}_{loc}(|x|_{\mathcal{A}}) \leq V_{loc}(x) \leq \overline{\alpha}_{loc}(|x|_{\mathcal{A}}) \ ; \tag{102}$$

2. for all x such that $V_{loc}(x) \leq c_{loc}$,

$$V_{loc}(F^{loc}(x, u_{loc}(x))) - V_{loc}(x) \leq -\gamma_{loc}(|x|_{\mathcal{A}}) \ ; \tag{103}$$

3. for all x such that $V_{glob}(x) \geq c_{glob}$,

$$V_{glob}(F^{glob}(x, u_{glob}(x))) - V_{glob}(x) \leq -\gamma_{glob}(|x|_{\mathcal{A}}) \ ; \tag{104}$$

4. there exists $\varepsilon > 0$ such that

$$V_{glob}(x) \leq c_{glob} \qquad \Longrightarrow \qquad V_{loc}(x) \leq c_{loc} - \varepsilon \ . \tag{105}$$

The composite control strategy will use a switching variable $s \in \{glob, loc\}$ (these two states can be arbitrarily associated with numbers). We let η be an arbitrary mapping to the set $\{glob, loc\}$ satisfying $\eta(glob) = glob$ and $\eta(loc) = loc$, let the controller be defined as

$$s^+ = \left\{ \begin{array}{ll} glob & \text{if } V_{loc}(x) \geq c_{loc} \\ \eta(s) & \text{if } V_{glob}(x) \geq c_{glob} \ \wedge \ V_{loc}(x) < c_{loc} \\ loc & \text{if } V_{glob}(x) < c_{glob} \end{array} \right\} =: G(x, s) \tag{106}$$

$$u(x, s) = u_{G(x,s)}(x) \tag{107}$$

and let the approximate discrete-time model be defined as

$$x^+ \in F(x, s, u) = F^{G(x,s)}(x, u) \ . \tag{108}$$

We then can state the following result:

Proposition 10: Under Assumption 2, there exists a continuously differentiable function $\rho \in \mathcal{K}_\infty$ such that, defining $\widetilde{V}_{glob} := V_{glob}$ and $\widetilde{V}_{loc} := \rho \circ V_{loc}$, the function

$$(x, s) \mapsto \widetilde{V}_{\eta(s)}(x) =: W(x, s) \tag{109}$$

is a Lyapunov certificate for (106)-(108), (with $x \mapsto W(x, s)$ inheriting sufficient regularity from $x \mapsto V_{glob}(x)$ and $x \mapsto V_{loc}(x)$.)

Proof. The proof of this proposition hinges of the following Lemma:

Lemma 2: Given strictly positive real numbers ε and c_{loc} satisfying $\varepsilon < c_{loc}$ and class-$\mathcal{K}_\infty$ functions $\underline{\kappa}$, $\overline{\kappa}$, there exists a continuously differentiable function $\rho \in \mathcal{K}_\infty$, with ρ' nondecreasing, such that

$$\rho(s) \leq \underline{\kappa}(s) \qquad \forall s \in [0, c_{loc} - \varepsilon] \tag{110}$$

$$\overline{\kappa}(s) \leq \rho(s) \qquad \forall s \in [c_{loc}, \infty) \ . \tag{111}$$

Proof. See the Appendix. ■

Continuing with the proof of the proposition, we define

$$\underline{\kappa} \ := \ \frac{2}{3} \cdot \underline{\alpha}_{glob} \circ \overline{\alpha}_{loc}^{-1}; \qquad \overline{\kappa} \ := \ 2 \cdot \overline{\alpha}_{glob} \circ \underline{\alpha}_{loc}^{-1} \tag{112}$$

and we apply Lemma 2 to get $\rho \in \mathcal{K}_\infty$. Now we consider the difference in the Lyapunov function W along trajectories. We first consider the two cases:

1. $V_{loc}(x) \geq c_{loc}$, $\eta(s) \neq G(x,s) = glob$. (It follows from (105) that $V_{glob}(x) \geq c_{glob}$ and from the properties of $\eta(\cdot)$ that $\eta(s) = loc$.) Using (106)-(109), the bounds in Assumption 2, the definition of $\overline{\kappa}$ in (112), and (111), we have

$$\begin{aligned} W(F(x,s,u(x,s)),G(x,s)) - W(x,s) \ &= \ V_{glob}(F^{glob}(x,u_{glob}(x)) - \rho(V_{loc}(x)) \\ &\leq \ V_{glob}(x) - \rho(V_{loc}(x)) \\ &\leq \ \frac{1}{2}\overline{\kappa}(V_{loc}(x)) - \rho(V_{loc}(x)) \\ &\leq \ -\frac{1}{2}\rho(V_{loc}(x)) \leq -\frac{1}{2}\rho \circ \underline{\alpha}_{loc}(|x|_{\mathcal{A}}) \ . \end{aligned} \tag{113}$$

2. $V_{glob}(x) < c_{glob}$, $\eta(s) \neq G(x,s) = loc$. (It follows that $\eta(s) = glob$ and $V_{loc}(x) \leq c_{loc} - \varepsilon$.) Using (106)-(109), the bounds in Assumption 2, the definition of $\underline{\kappa}$ in (112), and (110), we have

$$\begin{aligned} W(F(x,s,u(x,s)),G(x,s)) - W(x,s) \ &= \ \rho \circ V_{loc}(F^{loc}(x,u_{loc}(x))) - V_{glob}(x) \\ &\leq \ \rho(V_{loc}(x)) - \frac{3}{2}\underline{\kappa}(V_{loc}(x)) \\ &\leq \ -\frac{1}{2}\rho(V_{loc}(x)) \leq -\frac{1}{2}\rho \circ \underline{\alpha}_{loc}(|x|_{\mathcal{A}}) \ . \end{aligned} \tag{114}$$

In all of the other cases to be considered, we have that $\eta(s) = G(x,s)$. For these cases, it follows from (106), the fact that ρ' is nondecreasing, and the bounds in Assumption 2 that there exists a continuous, positive definite function γ such that

$$W(F(x,s,u(x,s),G(x,s)) - W(x,s) \leq -\gamma(|x|_{\mathcal{A}}) \ . \tag{115}$$

This completes the proof of the Proposition. ■

VII. Conclusions

A general framework for stabilization of sampled-data nonlinear differential inclusions via their approximate discrete-time models was presented. The generality of our approach is reflected in the following: (i) plants and (dynamic) controllers that are considered are modeled, respectively, by general differential and difference inclusions; (ii) no regularity assumptions are needed for the controller dynamics; (iii) stability with respect arbitrary (not necessarily compact) sets is considered; (iv) arbitrary approximate plant models are considered; (v) both fixed and varying sampling periods are considered. All conditions that are presented are checkable (although they may be hard to check) since they use the properties of the continuous-time plant model, the controller and the approximate model, all of which are known to the designer. The results are prescriptive in nature and they can be used as a guide when designing controllers for sampled-data systems based on their approximate discrete-time models. Finding systematic procedures for controller design for classes of systems and their specific approximate models using the framework of this paper is an important area of further research.

References

[1] D. Angeli, E.D. Sontag and Y. Wang, *A characterization of integral input-to-state stability*, IEEE Trans. Automat. Contr., 45 (2000), pp. 1082-1097.

[2] D. Angeli and E. Mosca, *Command governors for constrained nonlinear systems*, IEEE Trans. Automat. Contr., 44 (1999), pp. 816-818.

[3] J-P. Aubin and H. Frankowska, *Set-valued analysis*. Birkhäuser: Boston, 1990.

[4] J.S. Baras and N.S. Patel, *Robust control of set-valued discrete-time dynamical systems*, IEEE Trans. Automat. Contr., 43 (1998), pp. 61-75.

[5] B. Castillo, S. Di Gennaro, S. Monaco and D. Normand-Cyrot, *On regulation under sampling*, IEEE Trans. Automat. Contr., 42 (1997), pp. 864-868.

[6] T. Chen and B. A. Francis, Optimal sampled-data control systems. Springer-Verlag: London, 1995.

[7] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag and A. I. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Contr., 42 (1997), pp. 1394-1407.

[8] F. H. Clarke, Y. S. Ledyaev, L. Rifford and R. J. Stern, *Feedback stabilization and Lyapunov functions*, SIAM J. Contr. Optimiz., vol. 39 (2000), pp. 25–48.

[9] G. De Nicolao, L. Magni and R. Scattolini, *Stabilizing receding horizon control of nonlinear time-varying systems*, IEEE Trans. Automat. Contr., 43 (1998), pp. 1030-1038.

[10] D. Dochain and G. Bastin, *Adaptive identification and control algorithms for nonlinear bacterial growth systems*, Automatica, 20 (1984), pp. 621-634.

[11] A. F. Filippov, *Differential equations with discontinuous right-hand side*. Kluwer Academic, Dordrecht, 1988.

[12] G. C. Goodwin, B. McInnis and R. S. Long, *Adaptive control algorithm for waste water treatment and pH neutralization*, Optimal Contr. Applic. Meth., 3 (1982), pp. 443-459.

[13] L. Grüne, *Asymptotic controllability and exponential stabilization of nonlinear control systems at singular points*, SIAM J. Contr. Optimiz., 36 (1998), pp. 1485-1503.

[14] L. Grüne, *Asymptotic behavior of dynamical and control systems under perturbation and discretization*, Habilitationsschrift, J. W. Goethe University, 2001.

[15] J. Hauser and R. Hindman, *Maneuvering regulation from trajectory tracking: feedback linearizable systems*, NOLCOS, Lake Tahoe, CA, USA, 1995, pp. 595-600.

[16] R. Hindman and J. Hauser, *Maneuvering modified trajectory tracking*, Int. Symp. Math. Theory Networks Systems (MTNS), St. Louis, MO, USA, 1996.

[17] A. Isidori, Nonlinear control systems II. Springer-Verlag: London, 1999.

[18] Z.P. Jiang, I.M.Y.Mareels and Y. Wang, *A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems*, Automatica, 32 (1966), pp. 1211-1215.

[19] Z.P. Jiang and Y. Wang, *Input-to-state stability for discrete-time nonlinear systems*, Automatica, 37 (2001), pp. 857-869.

[20] H. K. Khalil, *Nonlinear systems*. Prentice-Hall: New Jersey, 1996.

[21] P.E. Kloeden and J. Lorenz. *Stable attracting sets in dynamical systems and their one-step discretizations*, SIAM J. Num. Anal., 23 (1986), pp. 986-995.

[22] P.E. Kloeden and B. Schmalfuss, *Asymptotic behavior of non autonomous difference inclusions*, Syst. Contr. Lett., 33 (1998), pp. 275-280.

[23] G. Kreisselmeier and T. Birkhölzer, *Numerical nonlinear regulator design*, IEEE Trans. Automt. Contr., 39 (1994), pp. 33-46.

[24] M. Krstić, I. Kanellakopoulos and P. V. Kokotović, *Nonlinear and adaptive control design*. J. Wiley & Sons: New York, 1995.

[25] D.S.Laila and D. Nešić, *Changing supply rates for input-output to state stable discrete-time nonlinear systems with applications*, submitted, 2001.

[26] Y. Lin, E.D.Sontag and Y.Wang, *A smooth converse Lyapunov theorem for robust stability*, SIAM J. Contr. Opt., 34 (1996), pp. 124-160.

[27] Y. Lin, *Lapunov function techniques for stabilization*, PhD Thesis, Rutgers, The State University of New Jersey, 1992.

[28] I. M. Y. Mareels, H. B. Penfold and R. J. Evans, *Controlling nonlinear time-varying systems via Euler approximations*, Automatica, 28 (1992), pp. 681-696.

[29] D. Q. Mayne, J. B. Rawlings, C. V. Rao and P. O. M. Scokaert, *Constrained model predictive control: Stability and optimality*, Automatica, vol. 36 (2000), pp. 789-814.

[30] E. S. Meadows, M. A. Henson, J. W. Eaton and J. B. Rawlings, "Receding horizon control and discontnuous state feedback stabilization", *Int. J. Contr.*, vol. 62 (1995), pp. 1217–1229.

[31] S. Monaco and D. Normand-Cyrot, *Sampling of a linear analytic control system*, Proc. CDC'85, pp. 1457-1462.

[32] D. Nešić, A. R. Teel and E.D.Sontag, *Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems*, Syst. Contr. Lett., vol. 38 (1999), pp. 49-60.

[33] D. Nešić, A.R. Teel and P. V. Kokotović, *Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations*, Sys. Contr. Lett., 38 (1999), pp. 259-270.

[34] D. Nešić and A.R. Teel, *Set stabilization of nonlinear sampled-data differential inclusions via their approximate discrete-time models*, Proc. IEEE Conf. Decis. Contr., Sydney, pp. 2112-2117, 2000.

[35] D. Nešić and D.S.Laila, *A note on input-to-state stabilization for nonlinear sampled-data systems*, to appear in *IEEE Trans. Automat. Contr.*, 2001.

[36] D. Nešić, A.R. Teel and P. V. Kokotović, *On the design of a controller based on the discrete-time approximation of the nonlinear plant model*, Proc. American Control Conference, San Diego, CA, pp. 3474-3478, 1999.

[37] D. Nešić and A.R. Teel, *Backstepping on the Euler approximate model for stabilization of sampled-data nonlinear systems*, to appear in Proc. Conference on Decision and Control, Orlando, FL, 2001.

[38] D. Nešić and D. Angeli, *Integral versions of ISS for sampled-data nonlinear systems via their approximate discrete-time models*, submitted, 2001.

[39] D. Nešić and A. R. Teel, *Changing supply rates in input-to-state stable systems: the discrete-time case*, IEEE Trans. Automt. Contr., vol. 46 (2001), pp. 960-962.

[40] E. Skafidas, A. Fradkov, R. J. Evans and I. M. Y. Mareels, *Trajectory based adaptive control for nonlinear systems under matching conditions*, Automatica, 34 (1998), pp. 287-299.

[41] E. D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Contr., 34 (1989), pp. 435-443.

[42] E. D. Sontag, *Clocks and insensitivity to small measurement errors*, Contr. Optimiz. Calc. Vari., vol. 4 (1999), pp. 537–557.

[43] A. M. Stuart and A. R. Humphries, Dynamical systems and numerical analysis. Cambridge University Press: New York, 1996.

[44] A. R. Teel, L. Moreau and D. Nešić, *A unification of time-scale methods for systems with disturbances*, submitted, 2001.

[45] A. Donchev and F. Lempio, *Difference methods for differential inclusions: a survey*, SIAM Review, 34 (1992), pp. 263-294.

VIII. Appendix

A. *Proof of Proposition 1:*

Our first claim is that $x \in \mathcal{V}(0, \ell_2)$ and $x^+ \in F(x)$ imply

$$V(x^+) \leq \max\{V(x), \ell_1 + T\ell_3\} \ . \tag{116}$$

For the case where $V(x) \leq \ell_1$, (116) follows from (28). The only case left to consider is when $V(x^+) \geq \ell_1$ (otherwise (116) holds) and $\ell_2 \geq V(x) \geq \ell_1$, which by (28) implies $V(x^+) \leq \ell_2 + T\ell_3$. But under these conditions (27) applies and so it follows that $V(x^+) \leq V(x)$, i.e., (116) holds.

The relation (116) guarantees that the set $\mathcal{V}(0, \ell_1 + T\ell_3)$ is forward invariant. Moreover, using (116) together with the fact that $\ell_1 + T\ell_3 \leq \ell_2$, we obtain that the set $\mathcal{V}(0, \ell_2)$ is also forward invariant.

Now, suppose there exist $x_\circ \in \mathcal{V}(0, \ell_2)$, $\phi(\cdot, x_\circ)$ (a solution to (19)) and a nonnegative integer k such that (29) is violated, i.e.,

$$V(\phi(k, x_\circ)) > \max\{\beta(V(x_\circ), kT), \ell_1 + T\ell_3\} \ . \tag{117}$$

Since $\beta(s, 0) \geq s, \forall s \geq 0$, it must be the case that $k \geq 1$ and $V(\phi(j, x_\circ)) \geq \ell_1 + T\ell_3$ for all $j \in \{0, \ldots, k\}$. It follows from (27) that

$$V(\phi(j+1, x_\circ)) - V(\phi(j, x_\circ)) \leq -T\alpha(V(\phi(j, x_\circ))) \qquad \forall j \in \{0, \ldots, k-1\} \ . \tag{118}$$

Define $t_j := jT\rho_2(V(x_\circ))$, $j \in \{0, 1, \ldots, k\}$ and the function $y : [0, t_k] \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} y(t) := V(\phi(j, x_\circ)) + \left(\frac{t}{T\rho_2(V(x_\circ))} - j\right)[V(\phi(j+1, x_\circ)) - V(\phi(j, x_\circ))] \\ \forall t \in [t_j, t_{j+1}] \quad \forall j \in \{0, \ldots, k-1\} \ . \end{aligned} \tag{119}$$

The function $y(\cdot)$ is well defined since $\rho_2(V(x_\circ)) > 0$ for all $x_\circ$. Moreover, it is continuous, piece-wise linear, and satisfies

$$y(t_j) = V(\phi(j, x_\circ)) \qquad \forall j \in \{0, \ldots, k\} \ . \tag{120}$$

Using (119) and (118), for almost all $t \in [0, t_k]$ and $j \in \{0, \dots, k-1\}$ such that $t \in [t_j, t_{j+1}]$, we have

$$\begin{aligned} \dot{y} &= \tfrac{1}{T\rho_2(V(x_\circ))}\left[V(\phi(j+1, x_\circ)) - V(\phi(j, x_\circ))\right] \le -\tfrac{\alpha(V(\phi(j,x_\circ)))}{\rho_2(V(x_\circ))} \\ &\le -\tfrac{\rho_1(V(\phi(j,x_\circ)))\cdot\rho_2(V(\phi(j,x_\circ)))}{\rho_2(V(x_\circ))} \le -\rho_1(V(\phi(j, x_\circ))) \ , \end{aligned} \tag{121}$$

where the last inequality follows from the facts that $\rho_2 \in \mathcal{M}$ and $V(\phi(j, x_\circ)) \le V(x_\circ)$, which imply

$$\frac{\rho_2(V(\phi(j, x_\circ)))}{\rho_2(V(x_\circ))} \ge 1, \forall j \in \{0, \dots, k\} \ .$$

It follows from this relation that $y(\cdot)$ is non increasing and thus, using (120), we have

$$y(t) \le V(\phi(j, x_\circ)) \qquad \forall t \in [t_j, t_{j+1}] \ , \forall j \in \{0, \dots, k-1\} \ . \tag{122}$$

Using that $\rho_1 \in \mathcal{K}_\infty$, it follows from (121) and (122) that, for almost all $t \in [0, t_k]$,

$$\dot{y} \le -\rho_1(y(t)) \ . \tag{123}$$

It follows from standard comparison theorems (see Lemma 4.4 in [41]) and the definition of $\tilde{\beta} \in \mathcal{KL}$ that

$$y(t) \le \tilde{\beta}(y(0), t) = \tilde{\beta}(V(x_\circ), t) \ . \tag{124}$$

Letting $t = t_j$ and using (120) with $j = k$ we get

$$V(\phi(j, x_\circ)) \quad \le \quad \tilde{\beta}(V(x_\circ), t_j) \ = \ \tilde{\beta}(V(x_\circ), \rho_2(V(x_\circ))kT) \ =: \ \beta(V(x_\circ), kT) \ ,$$

which is a contradiction to (117). Finally, $\beta(s, t) = \tilde{\beta}(s, \rho_2(s)t)$ is class-$\mathcal{KL}$ since $\tilde{\beta} \in \mathcal{KL}$ and $\rho_2 \in \mathcal{M}$. ∎

B. Proof of Proposition 4:

Let (Δ, M) be given. Define $\Delta_1 := \max\{\Delta, M\}$. Let item 2b of the proposition generate $h_1^* > 0$ and $\rho_1 \in \mathcal{K}_\infty$. Define $\widetilde{\Delta} = \Delta_1 + 1$ and let items 1 and 2c of the proposition generate $M_1 > 0$ and $\widetilde{\rho} \in \mathcal{K}_\infty$. Define $h^* := \min\left\{h_1^*, M_1^{-1}\right\}$ and $\rho(s) := \rho_1(s) + \widetilde{\rho}(M_1 s)$. It follows from item 1 of the proposition that, for all $(x, u) \in \mathcal{H}_\mathcal{A}(0, \Delta) \times \Delta\overline{\mathcal{B}}_m$,

$$\{\phi \in \mathcal{S}(x, u) \ , \ t \in [0, h^*]\} \Longrightarrow \left\{\phi(t, x, u) \in \mathcal{H}_\mathcal{A}(0, \widetilde{\Delta}) \ , \ |\phi(t, x, u) - x| \le M_1 t\right\} . \tag{125}$$

For each $v \in \mathbb{R}^n$ and $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, define $g_{(x,u)}(v)$ to be the unique (since $\widetilde{F}(x, u)$ is closed and convex) closest point in $\widetilde{F}(x, u)$ to v. Since $\widetilde{F}(x, u)$ is closed and convex, the function $g_{(x,u)}(\cdot)$ is continuous. Let $w : \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ be a measurable function such that $w(t) \in F(\phi(t, x, u), u)$, $\forall t \ge 0$ and, for almost all $t \ge 0$, $w(t) = \dot{\overbrace{\phi(t, x, u)}} \in F(\phi(t, x, u), u)$. Then the function $g_{(x,u)}(w(\cdot))$ enjoys the following properties:

1. $g_{(x,u)}(w(\cdot))$ is measurable (since $w(\cdot)$ is measurable and $g_{(x,u)}(\cdot)$ is continuous);
2. for all $t \in [0, h^*]$, $|w(t) - g_{(x,u)}(w(t))| \le \widetilde{\rho}(|\phi(t, x, u) - x|) \le \widetilde{\rho}(M_1 t)$;
3. from convexity of $\widetilde{F}(x, u)$ and the fact that $g_{(x,u)}(w(t)) \in \widetilde{F}(x, u)$ for all t, $\int_0^h g_{(x,u)}(w(t))dt \in h\widetilde{F}(x, u)$.

It follows that, for all $h \in (0, h^*)$

$$\begin{aligned} \phi(h, x, u) &= x + \int_0^h w(t)dt \ = \ x + \int_0^h g_{(x,u)}(w(t))dt + \int_0^h \left[w(t) - g_{(x,u)}(w(t))\right] dt \\ &\in x + h\widetilde{F}(x, u) + h\widetilde{\rho}(M_1 h)\overline{\mathcal{B}}_n \ \subseteq \ f_h(x, u) + h\left(\rho_1(h) + \widetilde{\rho}(M_1 h)\right)\overline{\mathcal{B}}_n \ = \ f_h(x, u) + h\rho(h)\overline{\mathcal{B}}_n \ . \end{aligned} \tag{126}$$

It follows that, for all $(x, u) \in \mathcal{H}_\mathcal{A}(0, \Delta) \times M\overline{\mathcal{B}}_m$ and all $h \in (0, h^*)$, $F_h^e(x, u) \subseteq f_h(x, u) + h\rho(h)\overline{\mathcal{B}}_n$, i.e., f_h is $\mathcal{A}$-one-step upper semi-consistent with F_h^e. ∎

C. Proof of Proposition 9.

Let $\delta \in \mathcal{K}_\infty$ satisfy $2\delta_1(s) + \max\{3\delta_1(s), \alpha_2 \circ 3\delta_1(s)\} \leq \delta(s)$. Consider the function

$$\widetilde{V}_h(x) = \mathrm{sat}_{2\delta_1(h)}(|x|_{\mathcal{A}})[1 - \ell_{\delta_1(h)}(|x|_{\mathcal{A}} - \delta_1(h))] + \ell_{\delta_1(h)}(|x|_{\mathcal{A}})V_h(x) \tag{127}$$

where $\mathrm{sat}_{2\delta_1(h)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as $\mathrm{sat}_{2\delta_1(h)}(s) = \min\{2\delta_1(h), s\}$ and $\ell_{\delta_1(h)} : \mathbb{R} \to [0,1]$ is defined as $\ell_{\delta_1(h)}(s) = 0$ for $s \leq \delta_1(h)$, $\ell_{\delta_1(h)}(s) = 1$ for $s \geq 2\delta_1(h)$ and $\ell_{\delta_1(h)}(s) = \frac{s-\delta_1(h)}{\delta_1(h)}$ for $s \in [\delta_1(h), 2\delta_1(h)]$. From these definitions and the fact that $3\delta_1(h) \leq \delta(h)$, it immediately follows that (85) holds. To see that (86) holds, we use (82) and (83) to deduce that

$$\begin{aligned} \widetilde{V}_h(x^+) - \widetilde{V}_h(x) &\leq 2\delta_1(h) + V_h(x^+) - V_h(x) + [1 - \ell_{\delta_1(h)}(|x|_{\mathcal{A}})]V_h(x) \\ &\leq V_h(x^+) - V_h(x) + 2\delta_1(h) + \alpha_2(3\delta_1(h)) \\ &\leq V_h(x^+) - V_h(x) + \delta(h)\ . \end{aligned} \tag{128}$$

Finally, we establish (84). We have

$$\widetilde{V}_h(x) \leq |x|_{\mathcal{A}} + \ell_{\delta_1(h)}(|x|_{\mathcal{A}})\alpha_2(|x|_{\mathcal{A}} + \delta_1(h)) \leq |x|_{\mathcal{A}} + \alpha_2(2|x|_{\mathcal{A}}) \tag{129}$$

and, for all $|x|_{\mathcal{A}} \leq 2\delta_1(h)$, $\widetilde{V}_h(x) \geq |x|_{\mathcal{A}}$, while for all $|x|_{\mathcal{A}} \geq 2\delta_1(h)$ $\widetilde{V}_h(x) \geq \alpha_1(|x|_{\mathcal{A}} - \delta_1(h)) \geq \alpha_1\left(\frac{1}{2}|x|_{\mathcal{A}}\right)$. ■

D. Proof of fact used to derive the first inequality in (96)

Fact 1: The following is true for all $a, b \in \mathbb{R}$:

$$(|a| + |b|)\mathrm{sat}_1(|a| + |b|) \leq 2|a|\mathrm{sat}_1(|a|) + 2|b|\mathrm{sat}_1(|b|)\ . \tag{130}$$

■

Proof: The proof follows from considering the following cases:

1. If $\mathrm{sat}_1(|a| + |b|) = \mathrm{sat}_1(|a|) = \mathrm{sat}_1(|b|)$, then it is straightforward to show that (130) holds.
2. If $\mathrm{sat}_1(|a|) < 1$, $\mathrm{sat}_1(|b|) < 1$, then

$$(|a| + |b|)\mathrm{sat}_1(|a| + |b|) \leq (|a| + |b|)^2 \leq 2|a|^2 + 2|b|^2 = 2|a|\mathrm{sat}_1(|a|) + 2|b|\mathrm{sat}_1(|b|)\ .$$

3. If $\mathrm{sat}_1(|a|) < 1$, $\mathrm{sat}_1(|b|) = 1$, then

$$(|a| + |b|)\mathrm{sat}_1(|a| + |b|) = |a| + |b| \leq 2|b| = 2|b|\mathrm{sat}_1(|b|) \leq 2|a|\mathrm{sat}_1(|a|) + 2|b|\mathrm{sat}_1(|b|)\ .$$

4. If $\mathrm{sat}_1(|a|) = 1$, $\mathrm{sat}_1(|b|) < 1$, then using the same calculations as in Case 2 by replacing roles of a and b we obtain:

$$(|a| + |b|)\mathrm{sat}_1(|a| + |b|) \leq 2|a|\mathrm{sat}_1(|a|) + 2|b|\mathrm{sat}_1(|b|)\ ,$$

which completes the proof. ■

E. Proof of Lemma 2

To prove Lemma 2, we use the following result:

Lemma 3: [18] Let $\chi_1 \in \mathcal{K}$ and $\chi_2 \in \mathcal{K}_\infty$ satisfy $\chi_1(s) < \chi_2(s)$ for all $s > 0$. Then, there exists a function $\chi \in \mathcal{K}_\infty$ such that: (i) $\chi_1(s) < \chi(s) < \chi_2(s)$ for all $s > 0$; (ii) χ is C^1 on $(0, \infty)$ and $\frac{d\chi}{ds} > 0$ for all $s > 0$. ■

Let the functions $\underline{\kappa}, \overline{\kappa} \in \mathcal{K}_\infty$ and numbers c_{loc}, ϵ come from the conditions of Lemma 2. Using Lemma 3, let the continuously differentiable functions $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ be such that for all $s > 0$ we have

$$\frac{1}{2}\underline{\kappa}(s) \leq \sigma_1(s) \leq \underline{\kappa}(s), \quad \overline{\kappa}(s) \leq \sigma_2(s) \leq 2\overline{\kappa}(s), \quad \frac{d\sigma_1}{ds}(s) > 0, \quad \frac{d\sigma_2}{ds}(s) > 0\ . \tag{131}$$

Introduce

$$
\begin{aligned}
q_1(s) &:= \inf_{t\in(s,c_{loc}]} \frac{d\sigma_1}{ds}(t) && \forall s\in[0,c_{loc}) \\
q_2(s) &:= \max\left\{\frac{2}{\epsilon}\sigma_2(c_{loc}), q_1(c_{loc}-\epsilon), \sup_{t\in[c_{loc}-\epsilon,s)} \frac{d\sigma_2}{ds}(t)\right\} && \forall s\in(c_{loc}-\varepsilon,\infty)
\end{aligned}
$$

and define

$$
q(s) := \begin{cases} q_1(s), & s\in[0,c_{loc}-\epsilon] \\ \frac{q_2(c_{loc})-q_1(c_{loc}-\epsilon)}{\epsilon}(s-c_{loc}+\epsilon)+q_1(c_{loc}-\epsilon), & s\in[c_{loc}-\epsilon,c_{loc}] \\ q_2(s), & s\in[c_{loc},\infty)\ . \end{cases}
$$

Note, that since σ_1 and σ_2 are continuously differentiable, bounded and strictly positive for all $s>0$, then $q_1(\cdot)$ and $q_2(\cdot)$ are continuous, nonnegative and nondecreasing on their domains of definition and $q_1(s)>0$ for all $s\in(0,c_{loc})$. Moreover, since $q_1(c_{loc}-\epsilon)\le q_2(c_{loc})$, it follows that $q(\cdot)$ is continuous, nondecreasing and positive definite. Define

$$
\rho(s) := \int_0^s q(t)dt\ .
$$

The function ρ is obviously $\mathcal{K}_\infty$ and continuously differentiable with $\frac{d\rho}{ds}(s)=q(s)$ for all s. Finally, using the definitions of q and q_1 and the first inequality in (131) we can write for all $s\in[0,c_{loc}-\epsilon]$ that

$$
\rho(s) = \int_0^s q_1(t)dt \le \int_0^s \frac{d\sigma_1}{dt}dt = \sigma_1(s) \le \underline{\kappa}(s)\ .
$$

Moreover, using the definitions of q and q_2 and the second inequality in (131) we can write for all $s\ge c_{loc}$ that

$$
\begin{aligned}
\rho(s) &= \int_0^{c_{loc}-\epsilon} q_1(t)dt + \int_{c_{loc}-\epsilon}^{c_{loc}} q(t)dt + \int_{c_{loc}}^{s} q_2(t)dt \\
&\ge \int_{c_{loc}-\epsilon}^{c_{loc}} q(t)dt + \int_{c_{loc}}^{s} \frac{d\sigma_2}{dt}dt \\
&= \int_{c_{loc}-\epsilon}^{c_{loc}} q(t)dt + \sigma_2(s)-\sigma_2(c_{loc})\ .
\end{aligned} \tag{132}
$$

Integration and the definition of $q_2(\cdot)$ yield the following:

$$
\begin{aligned}
\int_{c_{loc}-\epsilon}^{c_{loc}} q(t)dt &= \int_{c_{loc}-\epsilon}^{c_{loc}} \left[\frac{q_2(c_{loc})-q_1(c_{loc}-\epsilon)}{\epsilon}(t-(c_{loc}-\epsilon))+q_1(c_{loc}-\epsilon)\right]dt \\
&= \left[\frac{q_2(c_{loc})-q_1(c_{loc}-\epsilon)}{\epsilon}\left(\frac{t^2}{2}-(c_{loc}-\epsilon)t\right)+q_1(c_{loc}-\epsilon)t\right]\Bigg|_{c_{loc}-\epsilon}^{c_{loc}} \\
&= \left[\frac{q_2(c_{loc})-q_1(c_{loc}-\epsilon)}{\epsilon}\left(\frac{2c_{loc}\epsilon-\epsilon^2}{2}-(c_{loc}-\epsilon)\epsilon\right)+q_1(c_{loc}-\epsilon)\epsilon\right] \\
&= \left[\frac{[q_2(c_{loc})-q_1(c_{loc}-\epsilon)]\cdot\epsilon}{2}+q_1(c_{loc}-\epsilon)\epsilon\right] \\
&= \frac{[q_2(c_{loc})+q_1(c_{loc}-\epsilon)]\cdot\epsilon}{2} \\
&\ge \frac{q_2(c_{loc})\epsilon}{2} \\
&\ge \sigma_2(c_{loc})\ .
\end{aligned} \tag{133}
$$

Combining (131), (132) and (133) yields:

$$
\rho(s) \ge \int_{c_{loc}-\epsilon}^{c_{loc}} q(t)dt + \sigma_2(s)-\sigma_2(c_{loc}) \ge \sigma_2(c_{loc})+\sigma_2(s)-\sigma_2(c_{loc}) = \sigma_2(s) \ge \overline{\kappa}(s)\ ,
$$

which completes the proof of the lemma. ∎