Stabilization of sampled-data nonlinear systems via backstepping on their Euler approximate model ⋆

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Abstract

Two integrator backstepping designs are presented for digitally controlled continuous-time plants in special form. The controller designs are based on the Euler approximate discrete-time model of the plant and the obtained control algorithms are novel. The two control laws yield, respectively, semiglobal-practical stabilization and global asymptotic stabilization of the Euler model. Both designs achieve semiglobal-practical stabilization (in the sampling period that is regarded as a design parameter) of the closed loop sampled-data system. A simulation example illustrates that the obtained controllers may sometimes be superior to backstepping controllers based on the continuous-time plant model that are implemented digitally.

Key words: Disturbances, Networked Control Systems, Nonlinear, Stability.

1 Introduction

Backstepping control techniques have attracted much attention in the last ten years [1]. While backstepping is well understood for continuous-time systems, the case when a continuous-time plant needs to be controlled via a digital controller is not understood as well. Two different approaches have been suggested in the literature for this situation:

Approach 1: A continuous-time controller is designed for the plant ignoring sampling (e.g. using [1]) and then the controller is discretized and implemented using sample and hold devices (e.g. using [8]). Since this approach ignores sampling at the controller design stage, it is reasonable to expect that other approaches that take sampling into account would yield much better results.

Approach 2: A discrete-time controller is designed for the exact discrete-time model of the plant, which is in strict feedback form (see [2–4,14–16]). In this case, *it is assumed* that the exact discrete-time model of the plant is known and it has a feedback structure that is amenable to backstepping. However, both of these assumptions are unrealistic in the case when a continuous-time plant that has strict feedback structure needs to be controlled via a digital controller. Indeed, in this case the exact discrete-time plant model is typically unknown since we need to solve explicitly a nonlinear differential equation over one sampling interval. Moreover, even if the exact discrete-time model of the plant could be found, the model would usually not have the needed strict feedback structure (sampling destroys the strict feedback structure).

Recently the authors proposed a framework for digital controller design based on approximate discrete-time models of the plant [6]. We note that controller design was not addressed in [6], which we do in this paper for a class of strict feedback systems using their Euler approximate model.

In this paper we present several backstepping designs based on the Euler approximate discrete-time model of a continuous-time plant that is in strict feedback form. Motivation for doing this comes from the following: (i) The Euler approximate discrete-time model preserves the strict feedback structure of the continuous-time plant. Hence, the strict feedback assumption of the Euler approximate model is justified. (ii) We obtain completely new control algorithms in this way. (iii) The backstepping controllers based on the Euler ap-

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proximate plant model may sometimes outperform discretized continuous-time backstepping controllers (see the example in the last section). While we could not prove this fact in general, we observed this in simulations for several examples we considered. (iv) Not every backstepping controller based on the Euler approximate plant model stabilizes the sampled-data plant (for a example see [9]). Note that if backstepping is based on the exact discrete-time model, then this can not happen (dead-beat controllers are often used to illustrate discrete-time backstepping designs [14]).

Our first design achieves semiglobal-practical asymptotic (SPA) stabilization of the Euler model whereas the second design achieves global stabilization of the Euler model. Both designs achieve SPA stabilization of the closed-loop sampled-data system via the sampling period which is assumed to be a design parameter. In our second design we use results from [7] on change of supply rates for discrete-time input-to-state stable (ISS) systems.

The paper is organized as follows. In Section 2 we present preliminaries. Main results are stated in Section 3 and an example is given in Section 4. Several auxiliary results are given in the appendix.

2 Preliminaries

The sets of real and natural numbers are denoted $\mathbb{R}$ and $\mathbb{N}$, respectively. We use the standard definitions of class $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ functions [1], and we use $|x| := \max_i |x_i|$. We consider systems of the form

$$\dot{\eta} = f(\eta) + g(\eta)\xi; \qquad \dot{\xi} \;=\; u \; . \tag{1}$$

For simplicity, we assume $\eta \in \mathbb{R}^n, \xi \in \mathbb{R}, u \in \mathbb{R}$, $f(0) = 0$ and f, g are differentiable sufficiently many times. The control will be a piecewise constant signal $u(t) = u(kT) =: u(k), \forall t \in [kT, (k+1)T[, k \in \mathbb{N}$ where $T > 0$ is a sampling period and the state measurements $\eta(k) := \eta(kT)$ and $\xi(k) := \xi(kT)$ are available at sampling instants $kT, k \in \mathbb{N}$. The sampling period is assumed to be a design parameter that can be assigned arbitrarily. The difference equations corresponding to the exact plant model and its Euler approximation respectively are denoted:

$$x(k+1) = F_T^e(x(k), u(k)) \tag{2}$$
$$x(k+1) = F_T^{Euler}(x(k), u(k)) \; , \tag{3}$$

where we used the notation $x := \begin{pmatrix}\eta^T \; \xi^T\end{pmatrix}^T$ and $F_T^{Euler}(x,u) := \begin{pmatrix} \eta + T(f(\eta) + g(\eta)\xi) \\ \xi + Tu \end{pmatrix}$. Both models (2) and (3) are parameterized by T. We emphasize that F_T^e is not known in most cases. Moreover, even when F_T^e is known, it usually does not preserve the strict feedback structure of (1). On the other hand, F_T^{Euler} preserves the strict feedback structure like (1). Note that F_T^{Euler} is defined globally if the functions f and g in (1) are defined globally. In general, F_T^e is defined semi-globally in T since the initial value problem (1) may exhibit finite escape times. We will use the following definitions:

Definition 1 *We say that the family of controllers u_T semiglobally-practically asymptotically (SPA) stabilizes F_T if there exists $\beta \in \mathcal{KL}$ such that for any pair of strictly positive real numbers (D, ν) there exists $T^* > 0$ such that for each $T \in (0, T^*)$ the solutions of $x(k+1) = F_T(x(k), u_T(x(k)))$ satisfy: $|x(k, x(0))| \leq \beta(|x(0)|, kT) + \nu$, for all $k \geq 0$, whenever $|x(0)| \leq D$.*

Definition 2 *Let $\hat{T} > 0$ be given and for each $T \in (0, \hat{T})$ let functions $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $u_T : \mathbb{R}^n \to \mathbb{R}$ be defined. We say that the pair of families (u_T, V_T) is a semiglobally-practically asymptotically (SPA) stabilizing pair for F_T if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for any pair of strictly positive real numbers (Δ, δ) there exists a triple of strictly positive real numbers (T^*, L, M), with $T^* \leq \hat{T}$, such that for all $x, z \in \mathbb{R}^n$ with $\max\{|x|, |z|\} \leq \Delta$, and $T \in (0, T^*)$:*

$$\alpha_1(|x|) \;\leq\; V_T(x) \leq \alpha_2(|x|) \tag{4}$$
$$V_T(F_T(x, u_T(x))) - V_T(x) \leq -T\alpha_3(|x|) + T\delta \tag{5}$$
$$|V_T(x) - V_T(z)| \leq L|x - z| \tag{6}$$
$$|u_T(x)| \leq M \; . \tag{7}$$

Moreover, if there exists $T^{**} > 0$ such that (4), (5), (6), (7), with $\delta = 0$, hold for all $x \in \mathbb{R}^n$, $T \in (0, T^{**})$, then we say that the pair (u_T, V_T) is a globally asymptotically (GA) stabilizing pair for F_T.

A direct consequence of Definition 2 is that if (u_T, V_T) is a GA stabilizing pair for F_T, then (u_T, V_T) is a SPA stabilizing pair for F_T. The proofs of the following two results come directly from [6]:

Theorem 1 *If the pair (u_T, V_T) is SPA stabilizing for F_T^{Euler}, then it is u_T is SPA stabilizing for F_T^e.*

Hence, if we can find a family of pairs of (u_T, V_T) that is a GA or SPA stabilizing pair for F_T^{Euler}, then the controller u_T will SPA stabilize the exact model F_T^e. This is done in the next section.

3 Integrator backstepping

Design of SPA and GA control laws for F_T^{Euler} is hard in general. However, these control laws can be derived systematically when the system has the form (1) so that the Euler model[1] is

$$\eta(k+1) = r_T(\eta(k), \xi(k)) \tag{8}$$
$$\xi(k+1) = \xi(k) + Tu(k)\ , \tag{9}$$

where $r_T(\eta,\xi) := \eta + T(f(\eta)+g(\eta)\xi)$. First we design a SPA stabilizing family of pairs (u_T, V_T) (Theorem 2) and then a GA stabilizing pair (Theorem 3).

Theorem 2 *Suppose that there exists $\hat{T} > 0$ and a pair (ϕ_T, W_T) that is defined for each $T \in (0,\hat{T})$ and that is a SPA stabilizing pair for the subsystem (8), with $\xi \in \mathbb{R}$ regarded as its control. Suppose also that:*

1. ϕ_T and W_T are continuously differentiable for any $T \in (0,\hat{T})$;

2. there exists $\tilde{\varphi} \in \mathcal{K}_\infty$ such that $|\phi_T(\eta)| \le \tilde{\varphi}(|\eta|)$ for all $\eta \in \mathbb{R}^n$, $T \in (0,\hat{T})$.

3. for any $\tilde{\Delta} > 0$ there exists a pair of strictly positive numbers $(\tilde{T}, \tilde{M})$ such that for each $T \in (0,\tilde{T})$ and $|\eta| \le \tilde{\Delta}$ we have $\max\left\{\left|\frac{\partial W_T}{\partial \eta}\right|, \left|\frac{\partial \phi_T}{\partial \eta}\right|\right\} \le \tilde{M}$.

Then there exists a SPA stabilizing family of pairs (u_T, V_T) for the Euler model (8), (9). In particular, we can take (with arbitrary $c > 0$ and $r_T^\phi := \eta + T(f(\eta) + g(\eta)\phi_T(\eta))$):

$$u_T(x) = -c(\xi - \phi_T(\eta)) - \frac{\widetilde{\Delta W}_T}{T} + \frac{\Delta\phi_T}{T} \tag{10}$$
$$\Delta\phi_T := \phi_T(r_T) - \phi_T(\eta) \tag{11}$$
$$\widetilde{\Delta W}_T = \begin{cases} \frac{\overline{\Delta W}_T}{(\xi - \phi_T(\eta))}, & \xi \ne \phi_T(\eta) \\ T\frac{\partial W_T}{\partial \eta}(r_T)g, & \xi = \phi_T(\eta) \end{cases} \tag{12}$$
$$\overline{\Delta W}_T := W_T(r_T) - W_T(r_T^\phi) \tag{13}$$
$$V_T(x) = W_T(\eta) + \frac{1}{2}(\xi - \phi_T(\eta))^2\ . \tag{14}$$

Remark 1 *Since W_T is continuously differentiable and ϕ_T is continuous, the control law u_T is continuous. Also, $(\xi - \phi_T(\eta)) \cdot \widetilde{\Delta W}_T = \overline{\Delta W}_T$ for all x.*

Remark 2 *A continuous-time counterpart of the control law (10) can be found, for instance, in Lemma 2.8 in [1]. It takes the following form $u(x) = -c(\xi - \phi(\eta)) + \frac{\partial \phi}{\partial \eta}(\eta)[f(\eta) + g(\eta)\xi] - \frac{\partial W}{\partial \eta}(\eta)g(\eta)$, where ϕ is the control law that stabilizes the η subsystem and W is the Lyapunov function for the η subsystem with the control law ϕ. Although (10) and the above control law are similar, they are in general different. More importantly, we will show by simulations in the last section that (10) may outperform the emulated continuous time control law.*

[1] Since we only need to stabilize the Euler approximation, there is no loss of generality in considering $\dot{\xi} = u$ in (1) instead of $\dot{\xi} = h(\eta,\xi) + k(\eta,\xi)v$, where $k(\eta,\xi) \ne 0, \forall \eta, \xi$ since pre-compensation can be used to transform the Euler approximation into (8), (9).

Proof of Theorem 2: Since (ϕ_T, W_T) is a SPA stabilizing pair for $\eta + T(f(\eta) + g(\eta)u)$ and using conditions in Theorem 2, the following holds:

Property P: There exist $\hat{T} > 0$ and $\tilde{\varphi} \in \mathcal{K}_\infty$ such that the pair (ϕ_T, W_T) is defined for each $T \in (0,\hat{T})$ and

$$\eta \in \mathbb{R}^n,\ T \in (0,\hat{T})\ \Rightarrow\ |\phi_T(\eta)| \le \tilde{\varphi}(|\eta|)\ . \tag{15}$$

Moreover, there exist functions $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3 \in \mathcal{K}_\infty$ such that for any pair of strictly positive real numbers $(\tilde{\Delta}, \tilde{\delta})$ there exists a pair of strictly positive real numbers $(\tilde{T}, \tilde{M})$ such that for all $|\eta| \le \tilde{\Delta}$, $T \in (0,\tilde{T})$:

$$\tilde{\alpha}_1(|\eta|) \le W_T(\eta)\ \le\ \tilde{\alpha}_2(|\eta|) \tag{16}$$
$$\Delta W_T := W_T\left(r_T^\phi\right) - W_T(\eta) \le -T[\tilde{\alpha}_3(|\eta|) + \tilde{\delta}] \tag{17}$$
$$\max\left\{\left|\frac{\partial W_T}{\partial \eta}\right|, \left|\frac{\partial \phi_T}{\partial \eta}\right|\right\} \le \tilde{M}\ . \tag{18}$$

Let $c > 0$ come from the control law. We introduce $\alpha_1(s) := \min\left\{\tilde{\alpha}_1(s), \tilde{\alpha}_1 \circ \tilde{\varphi}_1^{-1}\left(\frac{1}{2}s\right), \frac{1}{8}s^2\right\}$, $\alpha_2(s) := \tilde{\alpha}_2(s) + s^2 + \tilde{\varphi}^2(s)$, $\alpha_3(s) := \min\left\{\tilde{\alpha}_3(s), \tilde{\alpha}_3 \circ \tilde{\varphi}_1^{-1}\left(\frac{1}{2}s\right), \frac{c}{4}s^2\right\}$ and $\tilde{\varphi}_1(s) := \max\{s, \tilde{\varphi}(s)\}$. Let (Δ, δ) be a pair of strictly positive real numbers. Let $\hat{T} > 0$ come from Property P. Let $\tilde{\delta} := \frac{\delta}{2}$. Define $\Delta_1 := \sup_{|x| \le \Delta, T \in (0,\hat{T})} \max\{|r_T|, \left|r_T^\phi\right|\}$. The number Δ_1 is finite for any $\Delta > 0$ since f and g are continuous (and hence bounded on compact sets), $\hat{T}$ is bounded and (15) holds. Let $\tilde{\Delta} := \max\{\Delta, \Delta_1\}$. Let the pair $(\tilde{\Delta}, \tilde{\delta})$ generate $(\tilde{T}, \tilde{M})$ using Property P. Let $\tilde{M}_1 := \sup_{|x| \le \Delta, T \in (0,\hat{T})} \max\{|\xi - \phi_T(\eta)|, |g(\eta)|, |f(\eta) + g(\eta)\xi|\}$. The number $\tilde{M}_1$ is finite for any $\Delta > 0$ since f and g are continuous and (15) holds. We define $T^* := \min\left\{\tilde{T}, \hat{T}, \frac{\delta}{\tilde{M}_1^2(c+\tilde{M})^2}\right\}$, $M := (c + 2\tilde{M})\tilde{M}_1$, $L := \tilde{M} + 2\tilde{M}_1(1 + \tilde{M})$. Now we consider arbitrary $x, z \in \mathbb{R}^{n+1}$, with $\max\{|x|, |z|\} \le \Delta$ and $T \in (0, T^*)$ and show that inequalities (4),(5),(6) and (7) hold for F_T^{Euler} with the pair (u_T, V_T) defined by (10), (14).

The lower bound in (4) follows directly from Proposition 1 in the appendix by letting $\theta_1(s) = s$ and $\theta_2(s) = \frac{1}{2}s^2$. The upper bound in (4) follows from:

$$\begin{aligned} V_T(x) &= W_T(\eta) + \frac{1}{2}(\xi - \phi_T(\eta))^2 \\ &\le \tilde{\alpha}_2(|x|) + |x|^2 + \tilde{\varphi}^2(|x|) = \alpha_2(|x|)\ . \end{aligned} \tag{19}$$

We now prove (5). Using (10), (14), (13), (11), Remark 1 and the Mean Value Theorem, we obtain:

$$\Delta V_T = W_T(r_T) - W_T(\eta) - 0.5(\xi - \phi_T(\eta))^2$$
$$+ 0.5\,(\xi + Tu_T - \phi_T(\eta + T(f + g\xi)))^2$$
$$= \underbrace{W_T(r_T^{\phi}) - W_T(\eta)}_{\Delta W_T} + \underbrace{W_T(r_T) - W_T(r_T^{\phi})}_{(\xi-\phi_T)\cdot\widetilde{\Delta W}_T} - 0.5(\xi - \phi_T)^2$$
$$+ \frac{1}{2}\left(\xi - \phi_T - \Delta\phi_T + T\left[c(\phi_T - \xi) - \frac{\widetilde{\Delta W}_T}{T} + \frac{\Delta\phi_T}{T}\right]\right)^2$$
$$= \Delta W_T + (\xi - \phi_T)\cdot\widetilde{\Delta W}_T - 0.5(\xi - \phi_T(\eta))^2$$
$$+ 0.5\left((1 - cT)(\xi - \phi_T(\eta)) - \widetilde{\Delta W}_T\right)^2$$
$$= \Delta W_T + (\xi - \phi_T)\cdot\widetilde{\Delta W}_T - 0.5(\xi - \phi_T)^2 + 0.5\widetilde{\Delta W}_T^2$$
$$+ 0.5(1 - 2cT + c^2T^2)(\xi - \phi_T)^2 - (1 - cT)(\xi - \phi_T)\cdot\widetilde{\Delta W}_T$$
$$= \Delta W_T - cT(\xi - \phi_T(\eta))^2$$
$$+ T^2\left\{\frac{c^2(\xi - \phi_T)^2}{2} + \frac{c(\xi - \phi_T)\cdot\widetilde{\Delta W}_T}{T} + \frac{\widetilde{\Delta W}_T^2}{2T^2}\right\}$$
$$\leq -T\tilde{\alpha}_3(|\eta|) + T\tilde{\delta} - Tc(\xi - \phi_T(\eta))^2$$
$$+ \frac{T^2}{2}\left\{c(\xi - \phi_T(\eta)) + \frac{\widetilde{\Delta W}_T}{T}\right\}^2$$
$$\leq -T\tilde{\alpha}_3(|\eta|) - Tc(\xi - \phi_T(\eta))^2 + T\tilde{\delta}$$
$$+ \frac{T^2}{2}\left\{c\,|\xi - \phi_T(\eta)| + \left|\frac{\partial W_T}{\partial\eta}(\eta^*)\right|\cdot|g(\eta)|\right\}^2$$

$$\leq -T\tilde{\alpha}_3(|\eta|) - Tc(\xi - \phi_T(\eta))^2 + T\tilde{\delta} + \frac{T^2}{2}\tilde{M}_1^2(c + \tilde{M})^2$$
$$\leq -T\alpha_3(|x|) + T\left(\frac{\delta}{2} + \frac{\delta}{2}\right)\ ,$$

where $\eta^* = \eta + T[f + g(\theta\xi + (1-\theta)\phi_T)], \theta \in (0,1)$ and the last step follows from the fact that: $\tilde{\alpha}_3(|\eta|) + c(\xi - \phi_T(\eta))^2 \geq \alpha_3(|x|)$, which follows from Proposition 1 in the appendix by letting $W_T(\eta) = \tilde{\alpha}_3(|\eta|)$, $\theta_1(s) = s$ and $\theta_2 = \frac{1}{2}s^2$. This proves condition (5).

Denoting $x = (\eta_1^T\ \xi_1)^T$, $z = (\eta_2^T\ \xi_2)^T$, using the Mean Value Theorem and the definition L, we have

$$|V_T(x) - V_T(z)| \leq |W_T(\eta_1) - W_T(\eta_2)|$$
$$+ \frac{1}{2}\left|(\xi_1 - \phi_T(\eta_1))^2 - (\xi_2 - \phi_T(\eta_2))^2\right|$$
$$\leq \left|\frac{\partial W_T}{\partial\eta}(\eta_1^*)\right|\cdot|\eta_1 - \eta_2| + |\xi_1 + \xi_2 - \phi_T(\eta_1) - \phi_T(\eta_2)|\cdot$$
$$\{|\xi_1 - \xi_2| + |\phi_T(\eta_1) - \phi_T(\eta_2)|\}$$
$$\leq (\tilde{M} + 2\tilde{M}_1(1 + \tilde{M}))\,|x - z| = L\,|x - z|\ ,$$

where $\eta_1^* = \theta_1\eta_1 + (1-\theta_1)\eta_2$ and $\eta_2^* = \theta_2\eta_1 + (1-\theta_2)\eta_2$ for some $\theta_1, \theta_2 \in (0,1)$. Thus condition (6) is satisfied.

Finally, using the Mean Value Theorem,

$$|u_T(x)| \leq c\,|\xi - \phi_T(\eta)| + \left|\frac{\partial W_T}{\partial\eta}(\eta_3^*)\right|\cdot|g(\eta)|$$
$$+ \left|\frac{\partial\phi_T}{\partial\eta}(\eta_4^*)\right|\cdot|f(\eta) + g(\eta)\xi|$$
$$\leq c\tilde{M}_1 + 2\tilde{M}\tilde{M}_1 = M\ ,$$

where $\eta_3^* = \eta + T[f + g(\theta_3\xi + (1-\theta_3)\phi_T)]$, $\eta_4^* = \eta + \theta_4 T(f + g\xi)$ for some $\theta_3, \theta_4 \in (0,1)$. Thus (7) holds.

Before stating Theorem 3 on GA stability we give Lemma 1 which is instrumental in proving Theorem 3 and that is inspired by the results in [11].

Lemma 1 *Consider the system* $\eta(k+1) = \eta(k) + T(f(\eta(k)) + g(\eta(k))u(k))$*, where* $\eta \in \mathbb{R}^n, u \in \mathbb{R}$*,* $f(0) = 0$ *and* f, g *are continuous. Suppose that there exist functions* $\alpha_1, \alpha_2, \alpha_3, \tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathcal{K}_\infty$*,* $c_1 \geq 0$*,* $T^* > 0$ *and for each* $T \in (0, T^*)$ *there exist a function* $\phi_T : \mathbb{R}^n \to \mathbb{R}$ *and a continuously differentiable function* $W_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that for all* $\eta \in \mathbb{R}^n$ *and all* $T \in (0, T^*)$ *we have: (i)* $W_T(r_T^{\phi}) - W_T(\eta) \leq -T\alpha_3(|\eta|)$*; (ii)* $|g(\eta)\phi_T(\eta)| \leq \tilde{\varphi}_1(|\eta|)$*; (iii)* $\left|\frac{\partial W_T}{\partial\eta}(\eta)\right| \leq c_1 + \tilde{\varphi}_2(|\eta|)$*. Then there exist* $\gamma \in \mathcal{K}_\infty$*, a (smooth) function* $\rho : \mathbb{R}_{\geq 0} \to (0,1]$ *and a family of control laws:* $\tilde{u}_T(\eta, \zeta) = \phi_T(\eta) + \rho(\eta)\zeta$ *such that for all* $\eta \in \mathbb{R}^n, \zeta \in \mathbb{R}$ *and* $T \in (0, T^*)$ *we have:*

$$W_T(r_T^{\tilde{u}}) - W_T(\eta) \leq -\frac{T}{2}\alpha_3(|\eta|) + T\gamma(|\zeta|)\ , \tag{20}$$

where $r_T^{\tilde{u}} := \eta + T(f(\eta) + g(\eta)\tilde{u}_T(\eta, \zeta))$*.*

Proof of Lemma 1: Suppose conditions of Lemma 1 hold. Since f and g are continuous and $f(0) = 0$, there exist functions $\gamma_f, \gamma_g \in \mathcal{K}_\infty$ and $c_2 \geq 0$ such that $|f(\eta)| \leq \gamma_f(|\eta|)$ and $|g(\eta)| \leq c_2 + \gamma_g(|\eta|)$ for all $\eta \in \mathbb{R}^n$. Define the functions $\chi(s) := c_1 + \tilde{\varphi}_2\,(s + T^*(\gamma_f(s) + \tilde{\varphi}_1(s) + (c_2 + \gamma_g(s))\cdot\alpha_3(s)))$, $\tilde{\varphi}_3(s) := \gamma_f \circ \alpha_3^{-1}(s) + \tilde{\varphi}_1 \circ \alpha_3^{-1}(s) + [c_2 + \gamma_g \circ \alpha_3^{-1}(s)]\cdot s$, $\gamma(s) := \frac{s}{2} + [c_1 + \tilde{\varphi}_2\left(\alpha_3^{-1}(s) + T^*\tilde{\varphi}_3(s)\right)]\cdot\tilde{\varphi}_3(s)$ Note that $\gamma \in \mathcal{K}_\infty$. Let $\rho : \mathbb{R}_{\geq 0} \to (0,1]$ be any non-increasing smooth function that satisfies $\rho(s) \leq \min\left\{\frac{1}{2\chi(s)}, 1\right\}$ for all $s \geq 0$. Such ρ exists since the function χ is non-negative and bounded for any bounded η. We show now that (20) holds by considering two cases: $\alpha_3(|\eta|) \leq |\zeta|$ and $\alpha_3(|\eta|) \geq |\zeta|$.

Consider first the case when $\alpha_3(|\eta|) \geq |\zeta|$.

$$\begin{aligned}\Delta W_T &= W_T\left(r_T^{\tilde{u}}\right) - W_T(\eta)\\ &= W_T\left(r_T^{\phi}\right) - W_T(\eta) + W_T\left(r_T^{\tilde{u}}\right) - W_T\left(r_T^{\phi}\right)\\ &\leq -T\alpha_3(|\eta|) + \underbrace{W_T\left(r_T^{\tilde{u}}\right) - W_T\left(r_T^{\phi}\right)}_{\overline{\Delta W}_T}.\end{aligned}$$

Using the Mean Value Theorem, we can write $\overline{\Delta W}_T = T\rho\frac{\partial W_T}{\partial\eta}(\eta^*)\zeta$ and since $\eta^* = \eta + T\left(f + g(\phi_T + \theta\rho\zeta)\right)$, $\theta \in (0,1)$, it follows from definitions of χ and ρ that

$$\begin{aligned}\overline{\Delta W}_T &\leq T\rho(|\eta|)\left|\frac{\partial W_T}{\partial\eta}(\eta^*)\right| \cdot |\zeta| \;\leq\; T\rho(|\eta|)\chi(|\eta|)\,|\zeta|\\ &\leq T\frac{1}{2\chi(|\eta|)}\chi(|\eta|)\alpha_3(|\eta|) \;\leq\; \frac{T}{2}\alpha_3(|\eta|) \qquad (21)\end{aligned}$$

Using (21) and (21), $\alpha_3(|\eta|) \geq |\zeta|$ implies

$$\begin{aligned}\Delta W_T &\leq -T\alpha_3(|\eta|) + \frac{T}{2}\alpha_3(|\eta|) \;=\; -\frac{T}{2}\alpha_3(|\eta|)\\ &\leq -\frac{T}{2}\alpha_3(|\eta|) + T\gamma(|\zeta|) \qquad (22)\end{aligned}$$

Consider now $|\zeta| \geq \alpha_3(|\eta|)$. By definition of γ and using the notation $\eta_1^* = \eta + T\theta_1(f + g(\phi_T + \rho\zeta)), \theta_1 \in (0,1)$, we have that $\alpha_3(|\eta|) \leq |\zeta|$ implies

$$\begin{aligned}&\Delta W_T + \frac{T}{2}\alpha_3(|\eta|) \leq T\left|\frac{\partial W_T}{\partial\eta}(\eta_1^*)[f + g(\phi_T + \rho\zeta)]\right| + \frac{T}{2}|\zeta|\\ &\leq [c_1 + \tilde{\varphi}_2(|\eta^*|)]\tilde{\varphi}_3(|\zeta|) + \frac{T}{2}|\zeta|\\ &\leq [c_1 + \tilde{\varphi}_2(\alpha_3^{-1}(|\zeta|) + T^*\tilde{\varphi}_3(|\zeta|))]\tilde{\varphi}_3(|\zeta|) + \frac{T}{2}|\zeta|\\ &= T\gamma(|\zeta|) .\end{aligned}$$

The proof follows from (22) and (23).

Theorem 3 *Consider the Euler approximate model (8), (9). Suppose that there exists $\hat{T} > 0$ and a pair of families (ϕ_T, W_T) that is defined for each $T \in (0,\hat{T})$ and that is GA stabilizing for the subsystem (8), with $\xi \in \mathbb{R}$ regarded as its control. Let ρ come from Lemma 1. Moreover, suppose that the pair of families (ϕ_T, W_T) has the following properties:*

(1) ϕ_T and W_T are continuously differentiable for any $T \in (0,\hat{T})$;

(2) There exists $\tilde{\varphi} \in \mathcal{K}_\infty$ such that for all $\eta \in \mathbb{R}^n$ and all $T \in (0,\hat{T})$ we have $|\phi_T(\eta)| \leq \tilde{\varphi}(|\eta|)$;

(3) For any $\tilde{\Delta} > 0$ there exists a pair of strictly positive numbers $(\tilde{T}, \tilde{M})$ such that for all $T \in (0,\tilde{T})$ and $|\eta| \leq \tilde{\Delta}$ we have $\max\left\{\left|\frac{\partial W_T}{\partial\eta}\right|, \left|\frac{\partial\phi_T}{\partial\eta}\right|\right\} \leq \tilde{M}$.

Then, there exists a GA stabilizing pair (u_T, V_T) for the Euler model (8), (9). In particular, the family of control laws can be taken to be:

$$u_T(x) = \frac{\Delta\phi_T}{T} + \frac{\Delta\rho}{T}\zeta - c\rho(|\eta + T(f + g\xi)|)\zeta\ , \qquad (23)$$

where $c > 0$ is arbitrary and $\Delta\phi_T := \phi_T(\eta + T(f + g\xi)) - \phi_T(\eta)$, $\Delta\rho := \rho(|\eta + T(f + g\xi)|) - \rho(|\eta|)$, $\zeta := \frac{\xi - \phi_T(\eta)}{\rho(|\eta|)}$ and there exist two smooth functions $\tilde{\theta}_1, \tilde{\theta}_2 \in \mathcal{K}_\infty$ such that we can take $V_T(x) = \tilde{\theta}_1(W_T(\eta)) + \tilde{\theta}_2\left(\frac{1}{2}\frac{(\xi - \phi_T(\eta))^2}{\rho^2(|\eta|)}\right)$.

Proof of Theorem 3: Let conditions of Theorem 3 be satisfied. Then the following property P1 holds: *Property P1: There exist $\hat{T} > 0$ and $\tilde{\varphi} \in \mathcal{K}_\infty$ such that the pair of families (ϕ_T, W_T) is defined for all $T \in (0,\hat{T})$ and $\eta \in \mathbb{R}^n$, $T \in (0,\hat{T})$ imply $|\phi_T(\eta)| \leq \tilde{\varphi}(|\eta|)$. Moreover, there exist $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3 \in \mathcal{K}_\infty$ and for each $T \in (0,\hat{T})$ and functions $W_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\phi_T : \mathbb{R}^n \to \mathbb{R}$ such that for all $\eta \in \mathbb{R}^n$ and all $T \in (0,\hat{T})$ the following holds:*

$$\tilde{\alpha}_1(|\eta|) \;\leq W_T(\eta) \;\leq \tilde{\alpha}_2(|\eta|) \qquad (24)$$

$$W_T(r_T^{\phi}) - W_T(\eta) \quad\leq\quad -T\tilde{\alpha}_3(|\eta|) \qquad (25)$$

Using the change of coordinates ζ, the control law (23) and definitions of $\Delta\phi_T$ and $\Delta\rho$ we can rewrite the system (8) as follows:

$$\begin{aligned}\eta(k+1) &= \eta(k) + T\left(f(\eta(k)) + g(\eta(k))\xi(k)\right)\\ &= r_T^{\tilde{u}}(\eta(k), \zeta(k)) \qquad (26)\\ \zeta(k+1) &= \frac{\xi(k+1) - \phi_T(\eta(k+1))}{\rho(|\eta(k+1)|)} = (1 - cT)\zeta(k)\ .\end{aligned}$$

From Lemma 1 it follows that the function ρ has the property that there exists $\tilde{\gamma} \in \mathcal{K}_\infty$ such that for all $\eta \in \mathbb{R}^n$, $\zeta \in \mathbb{R}$ and $T \in (0,\hat{T})$ we have:

$$W_T(r_T^{\tilde{u}}) - W_T(\eta) \leq -\frac{T}{2}\tilde{\alpha}_3(|\eta|) + T\tilde{\gamma}(|\zeta|) \qquad (27)$$

Moreover, by denoting $U(\zeta) := \frac{1}{2}\zeta^2$, we can see that:

$$\Delta U = (1 - cT^2)\zeta^2 - \zeta^2 \leq -T\frac{c}{2}\zeta^2\ , \qquad (28)$$

for all $\zeta \in \mathbb{R}$ and all $T \in (0, \frac{1}{c})$. Hence, for all $\eta \in \mathbb{R}^n$, $\zeta \in \mathbb{R}$ and all $T \in (0, T^*)$ with $T^* = \min\{\hat{T}, \frac{1}{c}\}$ (27) and (28) hold. By Corollary 1 of the appendix there exist $\tilde{\theta}_1, \tilde{\theta}_2, \hat{\alpha}_1, \hat{\alpha}_2$ such that for all η, ζ and $T \in (0, T^*)$ we have with $V_T(x) = \tilde{\theta}_1(W_T) + \tilde{\theta}_2(U)$ that the following holds $V_T(F_T^{Euler}) - V_T \leq -T\hat{\alpha}_1(|\eta|) - \frac{T}{2}\hat{\alpha}_2(|\zeta|)$. Note that $\rho(0) > 0$ and $\rho(|\eta|)$ is non increasing in $|\eta|$. Now we prove that there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the inequalities (4) and (5) hold in original coordinates $x = (\eta^T\ \xi)^T$. The first inequality in (4) follows directly from:

$$V_T(x) = \tilde{\theta}_1(W_T(\eta)) + \tilde{\theta}_2\left(\frac{|\xi - \phi_T(\eta)|^2}{2\rho^2(|\eta|)}\right)$$
$$\geq \tilde{\theta}_1(W_T(\eta)) + \tilde{\theta}_2\left(\frac{1}{2\rho^2(0)}|\xi - \phi_T(\eta)|^2\right) \geq \alpha_1(|x|)$$

by applying Proposition 1 in the appendix (we let in the proposition $\theta_1(s) = \tilde{\theta}_1(s)$, $\theta_2(s) = \tilde{\theta}_2(\frac{1}{2\rho^2(0)}s^2)$), where $\alpha_1(s) = \min\left\{\tilde{\theta}_1 \circ \tilde{\alpha}_1(s), \tilde{\theta}_1 \circ \tilde{\alpha}_1 \circ \tilde{\varphi}_1^{-1}\left(\frac{1}{2}s\right), \tilde{\theta}_2\left(\frac{1}{8\rho^2(0)}s^2\right)\right\}$ and $\tilde{\varphi}_1$ is given in Proposition 1. The second inequality in (4) follows from:

$$\begin{aligned} V_T(x) &= \tilde{\theta}_1(W_T(\eta)) + \tilde{\theta}_2\left(\frac{(\xi - \phi_T(\eta))^2}{2\rho^2(|\eta|)}\right) \\ &\leq \tilde{\theta}_1 \circ \tilde{\alpha}_2(|\eta|) + \tilde{\theta}_2\left(\frac{2\xi^2}{\rho^2(|\eta|)}\right) + \tilde{\theta}_2\left(\frac{2\tilde{\varphi}^2(|\eta|)}{\rho^2(|\eta|)}\right) \\ &\leq \alpha_2(|x|)\ , \end{aligned} \tag{29}$$

where $\alpha_2(s) = \tilde{\theta}_1 \circ \tilde{\alpha}_2(s) + \tilde{\theta}_2\left(\frac{2s^2}{\rho^2(s)}\right) + \tilde{\theta}_2\left(\frac{2\tilde{\varphi}^2(s)}{\rho^2(s)}\right)$. Finally, the bound (5) follows from $\hat{\alpha}_1(|\eta|)+\frac{1}{2}\hat{\alpha}_2\left(\frac{|\xi-\phi_T(\eta)|}{\rho(|\eta|)}\right) \geq \hat{\alpha}_1(|\eta|)+\frac{1}{2}\hat{\alpha}_2\left(\frac{|\xi-\phi_T(\eta)|}{\rho(0)}\right) \geq \alpha_3(|x|)$ by using Proposition 1 of the appendix with $\theta_1(s) = s$, $\theta_2(s) = \frac{1}{2}\hat{\alpha}_2(\frac{1}{\rho(0)}s)$, $W_T(\eta) = \hat{\alpha}_1(|\eta|)$, where $\alpha_3(s) := \min\left\{\hat{\alpha}_1(s), \hat{\alpha}_1 \circ \tilde{\varphi}_1^{-1}(s), \frac{1}{2}\hat{\alpha}_2\left(\frac{s}{2\rho(0)}\right)\right\}$ and $\tilde{\varphi}_1$ is defined in Proposition 1. Now we need to show that the inequalities (6) and (7) hold in a semiglobal sense. Let $\Delta > 0$ be given. Let $\hat{T} > 0$ come from Property P1. We define $\Delta_1 := \sup_{|x|\leq\Delta, T\in(0,\hat{T})} \max\{|r_T|, \left|r_T^{\phi}\right|\}$ and let $\tilde{\Delta} = \max\{\Delta, \Delta_1\}$. Let $\tilde{\Delta}$ generate (using condition 3 of Theorem) a pair of strictly positive real numbers $\tilde{T}, \tilde{M}$. Define $T^* = \min\{\hat{T}, \tilde{T}\}$ and $\tilde{M}_1 := \sup_{|x|\leq\tilde{\Delta}, T\in(0,T^*)} \max\left\{|\xi - \phi_T|, |f + g\xi|, \left|\frac{d\rho}{ds}\right|, \frac{1}{\rho}, |\xi|, |\phi_T|\right\}$, $\Delta_2 := \sup_{|x|\leq\tilde{\Delta}, T\in(0,T^*)} \max\left\{\tilde{\alpha}_2(\tilde{\Delta}), \frac{1}{2}\tilde{M}_1^4\right\}$, $\tilde{M}_2 := \sup_{s\leq\Delta_2, T\in(0,T^*)}\left\{\left|\frac{d\tilde{\theta}_1}{ds}(s)\right|, \left|\frac{d\tilde{\theta}_2}{ds}(s)\right|\right\}$. Finally, we define $M := \tilde{M}\tilde{M}_1 + \tilde{M}_1^4 + c\tilde{M}_1^2$, $L := \tilde{M}\tilde{M}_2 + \tilde{M}_1^3\tilde{M}_2(1 + \tilde{M} + 2\tilde{M}_1^3)$.

Consider arbitrary $x = (\eta_1^T\ \xi_1)^T$, $z = (\eta_2^T\ \xi_2)^T$ with $\max\{|x|, |z|\} \leq \Delta$ and $T \in (0, T^*)$. Then we can write $|V_T(x) - V_T(z)| \leq \left|\tilde{\theta}_1(W_T(\eta_1)) - \tilde{\theta}_1(W_T(\eta_2))\right| + \left|\tilde{\theta}_2\left(\frac{1}{2}\zeta_1^2\right) - \tilde{\theta}_2\left(\frac{1}{2}\zeta_2^2\right)\right|$, where $\zeta_i := (\xi_i - \phi_T(\eta_i))/\rho(|\eta_i|)$, $i = 1, 2$. Hence, using the Mean Value Theorem and triangle inequality we can write:

$$\begin{aligned} |V_T(x) - V_T(z)| &\leq \left|\frac{d\tilde{\theta}_1}{ds}(w_1^*)\right| \cdot \left|\frac{\partial W_T}{\partial \eta}(\eta_1^*)\right| \cdot |\eta_1 - \eta_2| \\ &+ \frac{1}{2}\left|\frac{d\tilde{\theta}_2}{ds}(w_2^*)\right| \cdot |\zeta_1 + \zeta_2| \cdot |\zeta_1 - \zeta_2| \\ &\leq \tilde{M}\tilde{M}_2|\eta_1 - \eta_2| + \tilde{M}_1^2\tilde{M}_2|\zeta_1 - \zeta_2| \end{aligned} \tag{30}$$

where $w_1^* = \ell_1 W_T(\eta_1) + (1 - \ell_1)W_T(\eta_2)$, $w_2^* = \frac{1}{2}[\ell_2\zeta_1^2 + (1 - \ell_2)\zeta_2^2]$, $\eta_1^* = \ell_3\eta_1 + (1 - \ell_3)\eta_2$ and $\ell_1, \ell_2, \ell_3 \in (0, 1)$. Also, we can write:

$$\begin{aligned} |\zeta_1 - \zeta_2| &\leq \left|\frac{\xi_1}{\rho(|\eta_1|)} - \frac{\xi_2}{\rho(|\eta_2|)}\right| + \left|\frac{\phi_T(\eta_1)}{\rho(|\eta_1|)} - \frac{\phi_T(\eta_2)}{\rho(|\eta_2|)}\right| \\ &\leq \left|\frac{\xi_1}{\rho(|\eta_1|)} - \frac{\xi_2}{\rho(|\eta_1|)}\right| + \left|\frac{\xi_2}{\rho(|\eta_1|)} - \frac{\xi_2}{\rho(|\eta_2|)}\right| \\ &+ \left|\frac{\phi_T(\eta_1)}{\rho(|\eta_1|)} - \frac{\phi_T(\eta_2)}{\rho(|\eta_1|)}\right| + \left|\frac{\phi_T(\eta_2)}{\rho(|\eta_1|)} - \frac{\phi_T(\eta_2)}{\rho(|\eta_2|)}\right| \\ &\leq \frac{1}{\rho(\eta_1)}\left\{|\xi_1 - \xi_2| + \left|\frac{\partial\phi_T}{\partial\eta}(\eta_2^*)\right| \cdot |\eta_1 - \eta_2|\right\} \\ &+ \{|\xi_2| + \phi_T(\eta_2)\} \cdot \left\{\left|\frac{d\rho}{ds}(s_1^*)\right| \frac{1}{\rho^2(s_1^*)} \cdot |\eta_1 - \eta_2|\right\} \\ &\leq (1 + \tilde{M} + 2\tilde{M}_1^3)\tilde{M}_1|x - z|\ , \end{aligned} \tag{31}$$

where $\eta_2^* = \ell_4\eta_1 + (1 - \ell_4)\eta_2$, $s_1^* = \ell_5|\eta_1| + (1 - \ell_5)|\eta_2|$ and $\ell_4, \ell_5 \in (0, 1)$. Combining definition of M, (30), (31) we obtain that (6) holds.

Consider an arbitrary x with $|x| \leq \Delta$ and $T \in (0, T^*)$. Using the Mean Value Theorem, the triangle inequality and definition of M we can write:

$$\begin{aligned} |u_T(x)| &\leq \left|\frac{\Delta\phi_T}{T}\right| + \left|\frac{\Delta\rho}{T}\right| \cdot |\zeta| + c|\zeta| \\ &\leq \left|\frac{\partial\phi_T}{\partial\eta}(\eta_3^*)\right| \cdot |f + g\xi| + \left|\frac{d\rho}{ds}(s_2^*)\right| |f + g\xi| \left|\frac{\xi - \phi_T}{\rho}\right| \\ &+ c\left|\frac{\xi - \phi_T}{\rho}\right| \leq \tilde{M}\tilde{M}_1 + \tilde{M}_1^4 + c\tilde{M}_1^2 \leq M\ , \end{aligned} \tag{32}$$

where $\eta_3^* = \eta + \ell_6(f + g\xi)$, $s_2^* = \ell_7|\eta + T(f + g\xi)| + (1 - \ell_7)|\eta|$ and $\ell_6, \ell_7 \in (0, 1)$. Thus (7) holds.

Example 1 *Consider the continuous-time plant:*

$$\dot{\eta} = \eta^2 + \xi; \qquad \dot{\xi} = u\ . \tag{33}$$

First we design the continuous-time backstepping controller based on (33). The first subsystem can be stabilized with the "control" $\phi(\eta) = -\eta^2 - \eta$. This is verified using the Lyapunov function $W(\eta) = \frac{1}{2}\eta^2$. Using this information and applying controller from Remark 2 with $c = 1$, we obtain $u^{ct}(\eta, \xi) = -2\eta - \eta^2 - \xi - (2\eta + 1)(\xi + \eta^2)$. Consider now the Euler approximate model of (33). Again, the control law $\phi(\eta) = -\eta^2 - \eta$ and the Lyapunov function $W(\eta) = \frac{1}{2}\eta^2$ are a GA stabilizing pair for the η-subsystem of the Euler approximate model. Using (10) with $c = 1$ in Theorem 2, we obtained the controller: $u_T^{Euler}(\eta, \xi) = u^{ct}(\eta, \xi) - T[0.5\eta^2 + 0.5\xi - 0.5\eta + (\xi + \eta^2)^2]$.

The term $-T[0.5\eta^2 + 0.5\xi - 0.5\eta + (\xi + \eta^2)^2]$ can be regarded as a modification of the controller u_T^{ct}. Moreover, for $T = 0$ we have that $u_0^{Euler}(\eta, \xi) = u^{ct}(\eta, \xi)$. We have compared the performance of the sampled-data systems with the two different controllers and have observed that u_T^{Euler} consistently yielded at least 4 times larger domain of attraction than u^{ct} for all tested sampling periods ($T \in \{0.1, 0.2, 0.5, 1\}$). In particular, Figures 1 and 2 show respectively trajectories with the $u^{ct}(\eta, \xi)$ and $u_T^{Euler}(\eta, \xi)$ starting from the same initial condition and with the same sampling period. While the trajectory with $u^{ct}(\eta, \xi)$ escapes in finite time, the trajectory with $u_T^{Euler}(\eta, \xi)$ is bounded and it converges to the origin. Domain of attraction (DOA) estimates with the two controllers for the sampling period $T = 0.5$ s are given in Figure 3. Hence, DOA for the system with u_T^{Euler} may be much larger than the estimate given in Figure 3.

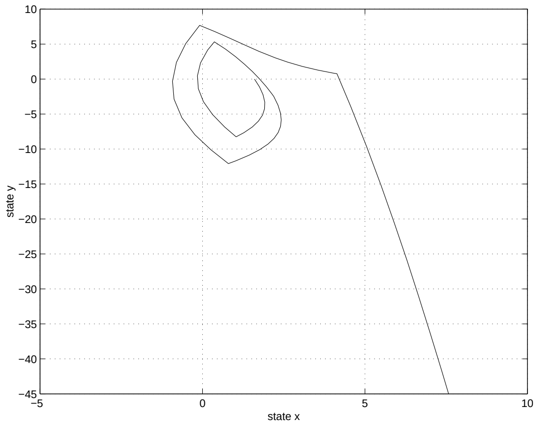

Fig. 1. Using $u^{ct}(\eta, \xi)$ from $x_\circ = [1.6\ 0]^T$ with $T = 0.5$ s.

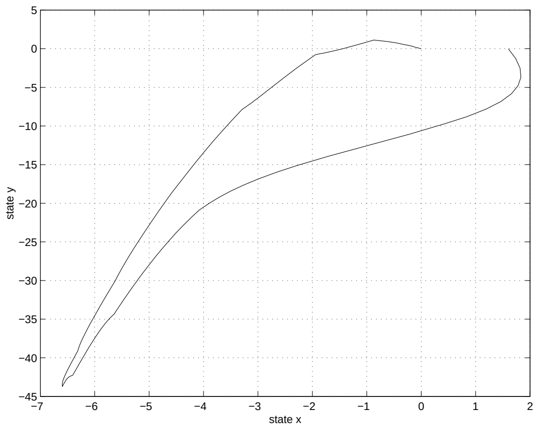

Fig. 2. Using $u_T^{Euler}(\eta, \xi)$ from $x_\circ$=[1.6 0]^T with T=0.5 s.

4 Conclusions

We have presented two control algorithms designed via Euler approximate models for sampled-data systems whose continuous-time plant is in strict feedback form. Advantages of our approach are illustrated via an example where a larger domain of attraction is achieved using

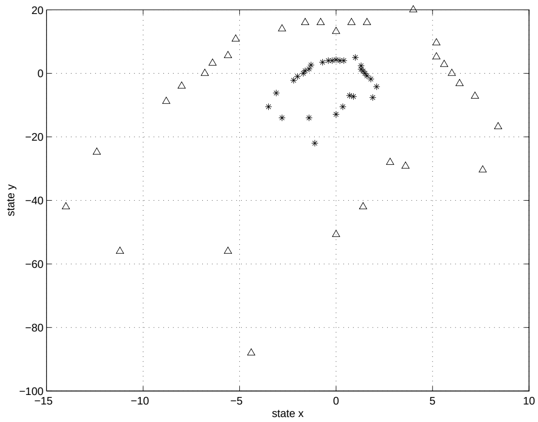

Fig. 3. Domain of attraction estimates, $T = 0.5$ s.

our controller when compared with the emulated classical backstepping controller. Our method is amenable to further extensions, such as robust backstepping.

References

[1] M. Krstić, I. Kanellakopoulos and P. V. Kokotović, *Nonlinear and adaptive control design*. J.Wiley & Sons: New York, 1995.

[2] W. Lin and C. I. Byrnes, *Design of discrete-time nonlinear control systems via smooth feedback*, IEEE Trans. Automat. Contr., 39 (1994), pp. 2340–2346.

[3] W. Lin, *Further results on global stabilization of discrete nonlinear systems*, Syst. Contr. Lett., 29 (1996), pp. 51–59.

[4] A. Madani, S. Monaco and D. Normand-Cyrot, "Adaptive stabilization of discrete-time strict feedback dynamics", Proc. Europ. Contr. Conf., Karlsruhe, Germany, 1999.

[5] D. Nešić, A. R. Teel and E.D.Sontag, *Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems*, *Syst. Contr. Lett.*, 38 (1999), pp. 49–60.

[6] D. Nešić, A. R. Teel and P. V. Kokotović, *Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations*, *Syst. Contr. Lett.*, 38 (1999), pp. 259–270.

[7] D. Nešić and A. R. Teel, *Changing supply functions in input-to-state stable systems: the discrete-time case*, IEEE Trans. Automat. Contr., 46 (2001), pp. 960–962.

[8] D. Nešić, D. S. Laila and A. R. Teel, *Open and closed loop dissipation inequalities under sampling and controller discretization*, *Europ. J. Contr.*, 8 (2002), pp. 109–125.

[9] D. Nešić, A. R. Teel and P. V. Kokotović, *On the design of a controller based on the discrete-time approximation of the nonlinear plant model*, Proc. Amer. Contr. Conf., San Diego, California, pp. 3474–3478, 1999.

[10] E. D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Contr., 34 (1989), pp. 435–443.

[11] E. D. Sontag, *Further facts about input to state stabilization*, IEEE Trans. Automat. Contr., 35 (1990), pp. 473–476.

[12] E.D. Sontag and A.R. Teel, *Changing supply functions in input/state stable systems*, IEEE Trans. Automat. Contr., 40 (1995), pp. 1476–1478.

[13] A. R. Teel, D. Nešić and P. V. Kokotović, *A note on input-to-state stability of sampled-data nonlinear systems*, In Proc. IEEE Conf. on Dec. and Control, Tampa, Florida, 1998, pp. 2473–2478.

[14] C. Simoes, H. Nijmeijer and J. Tsinias, "Nonsmooth stabilizability and feedback linearization of discrete-time nonlinear systems", Int. J. Robust Nonlin. Contr., vol. 6, pp. 171–188, 1996.

[15] P-C. Yeh and P. V. Kokotović, "Adaptive control of a class of nonlinear discrete-time systems", Int. J. Contr., vol. 62, pp. 303–324, 1995.

[16] J. Zhao and I. Kanellakopoulos, "Adaptive control of discrete-time strict feedback nonlinear systems", Proc. Amer. Contr. Conf., Albouquerque, New Mexico, 1997.

5 Appendix

The following result is stated without a proof.

Proposition 1 *Let $\theta_1, \theta_2, \tilde{\alpha}_1, \tilde{\varphi} \in \mathcal{K}_\infty$. If $W_T(\eta) \geq \tilde{\alpha}_1(|\eta|)$ and $|\phi_T(\eta)| \leq \tilde{\varphi}(|\eta|)$, for all $\eta \in \mathbb{R}^n$ and all $T \in (0, T^*)$, then $\theta_1(W_T(\eta)) + \theta_2(|\xi - \phi_T(\eta)|) \geq \alpha_1(|x|)$, for all $x \in \mathbb{R}^{n+1}$, $T \in (0, T^*)$, where $\alpha_1(s) := \min\left\{\theta_1 \circ \tilde{\alpha}_1(s), \theta_1 \circ \tilde{\alpha}_1 \circ \tilde{\varphi}_1^{-1}\left(\frac{1}{2}s\right), \theta_2\left(\frac{1}{2}s\right)\right\}$ and $\tilde{\varphi}_1(s) := \max\{\tilde{\varphi}(s), s\}$.*

Consider the parameterized family of systems:

$$x(k+1) = F_T(x(k), u(k)) \ . \tag{34}$$

Definition 3 *If there is some $\alpha_1, \alpha_2, \gamma, \alpha \in \mathcal{K}_\infty$, $T^* > 0$ and for all $T \in (0, T^*)$ a smooth function V_T so that $\alpha_1(|x|) \leq V_T(x) \leq \alpha_2(|x|)$ and $V_T(F_T(x,u)) - V_T(x) \leq T\gamma(|u|) - T\alpha(|x|)$, for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, then the triple of functions (V_T, γ, α) is called a Lyapunov ISS triple for system (34).*

Corollary 1 *[7] Let two ISS systems be given with their corresponding Lyapunov ISS triples $(W_T, \gamma_1, \alpha_1)$ and $(U_T, \gamma_2, \alpha_2)$. Then there are $\mathcal{K}_\infty$ smooth functions $\theta_1, \theta_2, \tilde{\gamma}, \tilde{\alpha}_1$ and $\tilde{\alpha}_2$ such that $[\theta_1(W_T), \frac{1}{2}\tilde{\alpha}_2, \tilde{\alpha}_1]$ is a Lyapunov ISS triple for the first system and $[\theta_2(U_T), \tilde{\gamma}, \tilde{\alpha}_2]$ is a Lyapunov ISS triple for the second system.*

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