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Controllability of Structured Polynomial Systems

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***Abstract*—Two algorithms, based on the Gröbner basis method, which facilitate the controllability analysis for a class of polynomial systems are presented. The authors combine these algorithms with some recent results on output dead-beat controllability in order to obtain sufficient, as well as necessary, conditions for complete and state dead-beat controllability for a surprisingly large class of polynomial systems. Our results are generically applicable to the class of polynomial systems in strict feedback form.**

***Index Terms*—Controllability, discrete-time, polynomial systems.**

I. Introduction

Controllability is one of the fundamental concepts in control theory, which can be used to uncover fundamental limitations to the system's control performance. Recently, we considered state and output dead-beat controllability for polynomial discrete-time systems [5], [6]. In particular, we proposed several dead-beat controllability tests for classes of polynomial systems that can be implemented using the symbolic computation packages: quantifier elimination by partial cylindrical algebraic decomposition (QEPCAD) and the Gröbner basis method.

In general, the controllability problem for polynomial systems requires the use of tools from semi-algebraic geometry [5], [6], such as QEPCAD. Semi-algebraic algorithms are, however, computationally very expensive and only modest size problems can be tackled in this way. In order to overcome the computational complexity curse, one needs to consider systems exhibiting special structure. One possible approach is to consider systems for which tools from algebraic geometry, such as the Gröbner basis method, can be used to decide controllability. The Gröbner basis method is less suited for the controllability problem than QEPCAD (it works over algebraically closed fields) but it behaves *much better* in terms of computations for the problem considered in this paper (based on the authors' experience with the current versions of the two algorithms). Hence, it is highly desirable to investigate situations when the Gröbner basis method can be used to test controllability. One such approach was pursued for output dead-beat controllability in [5] and [6] for the class of "odd" polynomial systems.

The main results of this paper are two algorithms based on the Gröbner basis method which (when combined with results from [5] and [6]) can be used to test state dead-beat and complete controllability for a large class of polynomial systems, such as strict feedback polynomial systems. The algorithms construct an algebraic variety in the state space in finite time, which contains all states due to which we may loose controllability. This variety is called "critical" and it has lower dimension than the state space. For systems for which the critical variety can be constructed we can state sufficient, as well as necessary, conditions for state dead-beat and complete controllability that are computationally less expensive than the "brute force" approach based on QEPCAD [6]. Existence of the critical variety is an important structural property for a large class of systems and we believe that our approach would play an important role in further simplifications of controllability tests for structured polynomial systems since it allows us to use tools from computational algebraic geometry rather than QEPCAD.

In Section II, we present our notation and briefly explain the Gröbner basis method. Section III contains the main results of the paper. In the last section we summarize our results.

II. Preliminaries

We use the standard definitions of rings and fields [1]. The sets of real, natural, and rational numbers are, respectively, denoted as $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Q}$. $\mathbb{R}^n$ is a set of all n-tuples of elements of $\mathbb{R}$, where $n \in \mathbb{N}$. The ring of polynomials in n variables over a field k is denoted as $k[x_1, x_2, \cdots, x_n]$. Let $f_1, f_2, \cdots, f_s$ be polynomials in $\mathbb{R}[x_1, x_2, \cdots, x_n]$. Then we define

$$V(f_1, f_2, \cdots, f_s) = \{(a_1, a_2, \cdots, a_n) \in \mathbb{R}^n\colon f_i(a_1, a_2, \cdots, a_n) = 0 \text{ for all } 1 \le i \le s\}.$$

We call $V(f_1, f_2, \cdots, f_s)$ the real algebraic set or real variety defined by the polynomials $f_1, f_2, \cdots, f_s$. Since the defining polynomials of a real variety are often clear from the context, we may denote it simply as V. If $V, W \subset \mathbb{R}^n$ are real varieties, then so are $V \cup W$ and $V \cap W$. A subset $I \subset \mathbb{R}[x_1, x_2, \cdots, x_n]$ is an ideal if: $0 \in I$; if $f, g \in I$, then $f + g \in I$; and if $f \in I$ and $h \in \mathbb{R}[x_1, \cdots, x_n]$, then $hf \in I$. Let $f_1, f_2, \cdots, f_s$ be polynomials in $\mathbb{R}[x_1, x_2, \cdots, x_n]$. Then the set $\langle f_1, \cdots, f_s\rangle$ defined as

$$\langle f_1, \cdots, f_s\rangle = \left\{\sum_{i=1}^{s} h_i f_i\colon h_1, \cdots, h_s \in \mathbb{R}[x_1, \cdots, x_n]\right\}$$

is called the ideal generated by $f_1, f_2, \cdots, f_s$. We use the notation $V(J)$ to denote the variety $V(f_1, \cdots, f_n)$, where f_i are the generators of the ideal J. Given two ideals $J_1, J_2 \in k[x_1, \cdots, x_m]$, their product $J_1 \cdot J_2 \in k[x_1, \cdots, x_m]$ is the ideal generated by all polynomials $f \cdot g$ where $f \in J_1$ and $g \in J_2$. Notice that $V(J_1) \cup V(J_2) = V(J_1 \cdot J_2)$.

All the systems that are considered in the sequel are subclasses of the following class of polynomial systems:

$$x(k+1) = f(x(k), u(k)), \qquad f(0, 0) = 0 \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}$ are, respectively, the state and the input of the system (1) at time k. The vector $f(x, u) = (f_1(x, u) \cdots f_n(x, u))^T$ is such that $f_i(x, u) \in \mathbb{Q}[x, u] = \mathbb{Q}[x_1, x_2, \cdots, x_n, u]$. The assumption that the polynomials f_i have rational coefficients is needed for computational purposes. A sequence of controls is denoted as $U = \{u(0), u(1), \cdots\}$. The truncation of U to a sequence of length $p+1$ is denoted as $U_p = \{u(0), u(1), \cdots, u(p)\}$. We use the following notation: $f_{u(k)} \circ \cdots \circ f_{u(1)} \circ f_{u(0)}(x(0)) = f(\cdots f(f(x(0), u(0)), u(1)), \cdots, u(k))$. The state of system (1) that is reached from the initial state $x(0)$ at

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time step $p+1$ under the action of a control sequence U_p is denoted as $x(p+1, x(0), U_p)$. The following sets are introduced:

$$\begin{aligned} S_0 &= \{x: \exists u \in \mathbb{R} \text{ such that } f(x, u) = 0\} \\ S_k &= \{x: \exists u(0), \cdots, u(k-2) \in \mathbb{R} \\ &\qquad \text{such that } f_{u(k-2)} \circ \cdots \circ f_{u(0)}(x) = 0\}. \end{aligned} \tag{2}$$

The set S_k consists of all states in the state space with the following property: the minimum time necessary to transfer any $x(0) \in S_k$ to the origin is *at most* $k+1$ time steps. We now give a list of definitions that are used in the sequel.

Definition 1: System (1) is completely controllable if for any initial state $x(0) \in \mathbb{R}^n$ and any terminal state x^* there exists an integer $N \in \mathbb{N}$ and a control sequence U_N such that $x^* = x(N, x(0), U_N)$.

Definition 2: System (1) is state dead-beat controllable if for any initial state $x(0) \in \mathbb{R}^n$ there exists a control sequence $\mathcal{U}$ and $\nu \in \mathbb{N}$ such that $x(p+1, x(0), \mathcal{U}_p) = 0, \forall p \geq \nu$.

Given a set of polynomials, the Gröbner basis algorithm produces a set of "simpler" polynomials (its Gröbner basis) that has the same solutions as the original set [1]. Packages for computing Gröbner bases can be found in most symbolic computation packages, such as Maple and Mathematica. Gröbner bases are not unique. However, given a monomial ordering, there exists a well-defined *reduced Gröbner basis* which is unique. One can then compare whether two ideals are the same by checking whether the reduced Gröbner bases of the ideals are the same. We denote the reduced Gröbner basis of a set of polynomials $f_1, \cdots, f_s$ for a given ordering as Gbasis$[f_1, \cdots, f_s]$. Due to space limitations it is impossible to present all the theory on Gröbner bases that we need and we refer to [1] for more details on the subject.

III. Main Results

In this section we present a methodology which shows how one can use the Gröbner basis method, together with some assumptions on the system's structure in order to obtain a state dead-beat controllability test.

Assumption 1: Consider a polynomial system (1). We assume that $S_{n-1} = \mathbb{R}^n - C$, and the smallest variety containing the set C, denoted as V_C^*, has the dimension at most $n-1$ ($\dim(V_C^*) \leq n-1$).

In other words, the set S_{n-1} is the whole state space except perhaps for the states that belong to the "critical variety" V_C^*. Since the variety V_C^* is a lower dimensional subset of the state space, it is defined by an ideal J_C^*, which is not trivial. In other words, $V_C^* = V(J_C^*)$. Without loss of generality we can assume that the variety V_C^* is generated by a single polynomial $f_C^* \in \mathbb{Q}[x]$, $f_C^* \not\equiv 0$.

Definition 3: $V_C^* = V(f_C^*)$ in Assumption 1 is called the critical variety.

We emphasize that Assumption 1 may not be satisfied for the set S_{n-1} but for some other set S_N, $N \neq n-1$. However, for the class of strict feedback systems that we consider Assumption 1 is generically satisfied for the set S_{n-1}. Moreover, our results can be applied also if we are not working with the smallest variety V_C^* but with any other variety V_C containing V_C^*, such that $\dim(V_C) = n-1$. The main result for this paper is an algorithm for computation of *a critical variety* V_C, which contains *the (smallest) critical variety* V_C^*. It is not difficult to show that a critical variety may contain invariant subsets in the following sense.

Definition 4: A set $V_{I_j} \subseteq V_C$ is invariant if

$$\forall x \in V_{I_j}, \qquad \forall u \in \mathbb{R}, \qquad f(x, u) \subseteq V_{I_j}. \tag{3}$$

The union of all invariant subsets of V_C is denoted as V_I and is called the *maximal invariant set.*

The following propositions follow directly from [5] and [6].

Proposition 1: The maximal invariant set $V_I \subseteq V_C$ is a variety and it can be computed using a finite algorithm, presented in [5] and [6].

Proposition 2: A polynomial system of the form (1) with Assumption 1 is state dead-beat controllable if $V_I = \emptyset$.

Proposition 3: Suppose $V_I \neq \emptyset$. A polynomial system of the form (1) with Assumption 1 is state dead-beat controllable only if $0 \in V_I$.

In summary, if we can identify a critical variety V_C which has a lower dimension than the state dimension, it is possible to use Propositions 1–3 to decide on state dead-beat controllability.

Notice that the main issue in the above approach is the existence of a variety with the property that all states outside of it can be transferred to the origin. This implies that we may work with "much larger" critical variety which contains many "good" states as well. For instance, suppose that at some step K the set $S_K = \mathbb{R}^n - \{x: x_1 = 0, x_2 = 0, \cdots, x_{n-1} = 0, x_n > 0\}$. So the critical set is in this case a half line in $\mathbb{R}^n$. However, nothing stops us from defining a critical variety $V_C = \{x: x_1 = 0, x_2 = 0, \cdots, x_{n-1} = 0\}$, which obviously contains all "critical states" but also some "good" states. Then we can apply the same methodology to compute the maximal invariant set of a critical variety. In certain situations it may be straightforward and easier to compute such a larger critical variety. We present below such an approach based on the Gröbner basis method.

Consider (1). Let us compute the composition

$$f_{u(n-1)} \circ f_{u(n-2)} \circ \cdots \circ f_{u(0)}(x) = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} \tag{4}$$

where obviously $F_i \in \mathbb{Q}[x_1, \cdots, x_n, u(0), \cdots, u(n-1)]$. Using the lexicographic ordering $u(n-1) \succ u(n-2) \succ \cdots \succ u(0) \succ x_1 \succ \cdots \succ x_n$, compute the Gröbner basis

$$\text{Gbasis}[F_1, F_2, \cdots, F_n] = \{g_1, g_2, \cdots, g_N\}. \tag{5}$$

Now we can give sufficient conditions for systems (1) to satisfy Assumption 1.

Theorem 1: If the following conditions are satisfied:

1) *Triangular Structure Condition*: In (5) we have that $N = n$ and $g_k \in \mathbb{Q}[x_1, \cdots, x_n, u(0), \cdots, u(n-k)]$, $\forall k = 1, 2, \cdots, n$;

2) *Odd Polynomials Condition*: g_k in (5) are of the following form:

$$\begin{aligned} g_n &= \sum_{i=0}^{m_n} p_i^n u(0)^i, \qquad p_i^n \in \mathbb{Q}[x_1, \cdots, x_n] \\ g_k &= \sum_{i=0}^{m_k} p_i^k u(n-k)^i, \\ &\qquad p_i^k \in \mathbb{Q}[x_1, \cdots, x_n, u(0), \cdots, \qquad u(n-k-1)], \end{aligned} \tag{6}$$

$k = 1, \cdots, n-1$, where $m_k = 2t_k + 1$, $t_k \in \mathbb{N}$;

3) *Nontriviality Condition*: All the ideals:

$$\begin{aligned} J_1^x &= \langle p_{m_n}^n \rangle \\ J_k^x &= \text{Gbasis}[g_n, \cdots, g_{n-k+2}, p_{m_{n-k+1}}^{n-k+1}] \cap \mathbb{Q}[x_1, \cdots, x_n], \end{aligned} \tag{7}$$

$k = 2, \cdots, n$ are nontrivial. All Gröbner bases in (7) are computed using the lexicographic ordering $u(n-k-1) \succ \cdots \succ u(0) \succ x_n \succ \cdots \succ x_1$;

then the following holds.

1) Assumption 1 holds.
2) A critical variety is given by
$$V_C = V\left(\prod_{i=1}^{n} J_i^x\right).$$
3) The algorithm used to obtain the variety V_C (all ideals J_k^x) stops in a finite number of steps.

Remark: From conditions 1) and 2) of the theorem, we see that the Gröbner basis of polynomials F_i in (4) has a very special triangular structure. Indeed, there are exactly n polynomials in the Gröbner basis and each polynomial g_k has odd highest degree in control variable $u(n-k)$.

Proof of Theorem 1: Notice that we need to prove that if the conditions of Theorem 1 are satisfied, there exists a real solution $u(0), \cdots, u(n-1)$ to the equation:

$$f_{u(n-1)} \circ f_{u(n-2)} \circ \cdots \circ f_{u(0)}(x) = 0. \tag{8}$$

Now we prove that if $x \notin V\left(\prod_{i=1}^{n} J_i^x\right)$, there exists a real solution $u(0), \cdots, u(n-1)$ to the equations

$$g_1 = 0, \qquad g_2 = 0, \cdots, g_n = 0 \tag{9}$$

and moreover, the same values of $u(0), \cdots, u(n-1)$ solve the system of (8).

Suppose that $x \notin V(J_1^x)$. Then it is guaranteed that there is a real value of $u(0)$ which renders $g_n = 0$, since the highest degree of $u(0)$ in g_n is an odd integer. Suppose now that $x \notin V(J_1^x \cdot J_2^x)$. In this case, we can guarantee that there exists real roots $u(0), u(1)$ to the equations $g_n = 0, g_{n-1} = 0$. Indeed, since $x \notin V(J_1^x)$ there is a real root to the equation $g_n = 0$. Since $x \notin V(J_2^x)$, for the values of $u(0)$ which zero g_n, the highest degree of $u(1)$ in g_{n-1} is odd since $p_{m_{n-1}}^{n-1}$ when evaluated at x and $u(0)$ is not zero. By continuing the same argument we verify the existence of a solution $u(0), \cdots, u(n-1)$ to the systems of (9).

Notice, that for all $x \notin V\left(\prod_{i=1}^{n} J_i^x\right)$ the real solution to the set of (9) also solves (8) and this proves the first claim.

The second claim is obvious from the above given argument. Indeed, the geometric interpretation of the ideal J_k^x is that for all states in $V(J_k^x)$ there are real controls $u(0), u(1), \cdots, u(n-k-1)$ which solve equations $g_n = 0, \cdots, g_{n-k-1} = 0$ but not necessarily the equation $g_k = 0$. The union of all such varieties yields the variety V_C.

The third claim of the theorem follows from the fact that a Gröbner basis of a set of polynomials can be computed using an algorithm which stops in finite time. In our algorithm for the computation of a critical variety, we need to compute $n+1$ Gröbner bases. Q.E.D.

It may seem that Assumption 1 is strong. Surprisingly, however, several important classes of polynomial systems fall into this category. It is generically satisfied for the large class of polynomial systems in strict feedback form (a discrete-time version of systems considered in [4])

$$\begin{aligned}
x_1(k+1) &= F_1(x_1(k)) + G_1(x_1(k))x_2(k)\\
x_2(k+1) &= F_2(x_1(k), x_2(k)) + G_2(x_1(k), x_2(k))x_3(k)\\
\cdots & \qquad \cdots\\
x_n(k+1) &= F_n(x_1(k), \cdots, x_n(k)) + G_n(x_1(k), \cdots, x_n(k))u(k)
\end{aligned} \tag{10}$$

with $x_i \in \mathbb{R}, \forall i = 1, \cdots, n$ and $u \in \mathbb{R}$. We also have that $G_i, F_i \in \mathbb{Q}[x_1, \cdots, x_i]$. Notice that the functions $F_i(x_1, \cdots, x_i) + G_i(x_1, \cdots, x_i)x_{i+1}$ *may not be surjective in* $x_{i+1}, \forall x \in \mathbb{R}^n$. In other words, we allow for the possibility that the real varieties $V(G_i), i = 1, 2, \cdots, n$ are not empty. We denote $x(k) = (x_1(k)\ x_2(k) \cdots x_n(k))^T$. If we take n compositions of this map, starting from $x(0) \in \mathbb{R}^n$, we obtain for $i = 1, \cdots, n$

$$\begin{aligned}
x_i(n) = {} & c_i(x(0), u(0), \cdots, u(i-2))\\
& + d_i(x(0), u(0), \cdots, u(i-2))u(i-1)
\end{aligned} \tag{11}$$

where c_i, d_i are polynomials obtained by straightforward computations. Observe the "triangular structure" of polynomials on the right hand side of (11) with respect to controls $u(i), i = 0, 1, \cdots, n-1$, which is required in Condition 1 of Theorem 1. Hence, for this class of systems *we do not have to compute the Gröbner basis of the polynomials on the right-hand side of (11) and we can work with the polynomials themselves.* Using (11), it is easy to show that the systems (10) generically satisfy the conditions of Theorem 1.

Comment 1: If $V_C = \mathbb{R}^n$ (Assumption 1 not satisfied) for a strict feedback polynomial system, it often indicates that the system is not controllable. For instance, this is always the case for linear systems.

We show below how it is possible to modify the presented methods in order to use them for complete controllability testing. Consider (1). Let us compute the composition (4) and consider the system of polynomial equations

$$t_1 - F_1 = 0; \quad t_2 - F_2 = 0; \quad \cdots; \quad t_n - F_n = 0 \tag{12}$$

where $F_i \in \mathbb{Q}[x_1, \cdots, x_n, u(0), \cdots, u(n-1)]$. The variables that are introduced $t_i, i = 1, \cdots, n$ can be viewed as the state of the system at time step $n+k$ and the variables $x_i, i = 1, 2, \cdots, n$ as states at time step k. If for any real values of t_i, x_i there is a real solution $u(k), \cdots, u(k+n-1)$ of (12), (1) is completely controllable. Using the lexicographic ordering $u(n-1) \succ u(n-2) \succ \cdots \succ u(0)$ and regarding x_i, t_i as parameters, compute the Gröbner basis:

$$\text{Gbasis}[t_1 - F_1, t_2 - F_2, \cdots, t_n - F_n] = \{\hat{g}_1, \hat{g}_2, \cdots, \hat{g}_N\}. \tag{13}$$

We can state the following result.

Theorem 2: If the following conditions are satisfied:

1) *Triangular Structure Condition*: In (13) we have that $N = n$ and $\hat{g}_k \in \mathbb{Q}[x_1, \cdots, x_n, t_1, \cdots, t_n, u(0), \cdots, u(n-k)], \forall k = 1, 2, \cdots, n$;
2) *Odd Polynomials Condition*: $\hat{g}_k$ in (5) are of the following form:
$$\begin{aligned}
\hat{g}_n &= \sum_{i=0}^{m_n} \hat{p}_i^n u(0)^i, \qquad \hat{p}_i^n \in \mathbb{Q}[x_1, \cdots, x_n, t_1, \cdots, t_n]\\
\hat{g}_k &= \sum_{i=0}^{m_k} \hat{p}_i^k u(n-k)^i, \qquad \hat{p}_i^k \in \mathbb{Q}[x_1, \cdots, x_n, t_1, \cdots, t_n,\\
&\qquad u(0), \cdots, u(n-k-1)],
\end{aligned} \tag{14}$$
$k = 1, \cdots, n-1$, where $m_k = 2t_k + 1, t_k \in \mathbb{N}$;
3) *Nontriviality Condition*: All of the following ideals:
$$\begin{aligned}
J_1^{x,t} &= \langle \hat{p}_{m_n}^n \rangle\\
J_k^{x,t} &= \text{Gbasis}[\hat{g}_n, \cdots, \hat{g}_{n-k+2}, \hat{p}_{m_{n-k+1}}^{n-k+1}]\\
&\quad \cap \mathbb{Q}[x_1, \cdots, x_n, t_1, \cdots, t_n],
\end{aligned} \tag{15}$$

$k = 2, \cdots, n$ are nontrivial. All Gröbner bases are computed using the lexicographic ordering $u(n-k-1) \succ \cdots \succ u(0) \succ t_n \succ \cdots \succ t_1 \succ x_n \succ \cdots \succ x_1$.

then:

1) the dimension of the variety $V\left(\prod_{i=1}^n J_i^{t,x}\right) \subset \mathbb{R}^{2n}$ is at most $2n-1$;
2) it is possible to transfer any initial state x to the terminal state t, if the $(t, x) \notin V\left(\prod_{i=1}^n J_i^{t,x}\right) \subset \mathbb{R}^{2n}$.

For obvious reasons, the variety $\hat{V}_C = V(\prod_{i=1}^n J_i^{t,x})$ is termed critical. The variety $\hat{V}_C$ contains all terminal t and initial states x in the space $\mathbb{R}^{2n}$ which are such that it may not be possible to transfer x to t in n time steps. There is no loss of generality if it is assumed that the variety $\hat{V}_C$ is defined by a single polynomial $\hat{f}_C \in \mathbb{Q}[x_1, \cdots, x_n, t_1, \cdots, t_n]$. Hence, given $x(0)$ and $x(n)$, there exists a control sequence U_{n-1} such that $x(n) = x(n, x(0), U_{n-1})$ if $\hat{f}_C(x(n), x(0)) \neq 0$.

Consider the polynomial $\hat{f}_C(t, x)$, which defines a critical variety $\hat{V}_C$. We use the following notation:

$$\hat{f}_C \circ f_u(x) = \hat{f}_C(t, f(x, u)).$$

As before, we can compute the set of initial $x \in \mathbb{R}^n$ and terminal $t \in \mathbb{R}^n$ states which are such that if $\hat{f}_C(t, x(0)) = 0$, then $\forall k, \forall U_k$ we have that $\hat{f}_C(t, x(k, x(0), U_k))) = 0$. In other words, we can find the maximal invariant set $\hat{V}_I$ of the variety $\hat{V}_C$. To do this, we introduce notation

$$\begin{aligned} \hat{f}_C \circ f_{u(0)} &= \sum_{i=0}^{m_1} B_{i_1}^1(t, x)u(0)^i \\ \hat{f}_C \circ f_{u(1)} \circ f_{u(0)} &= \sum_{i_1=0,\, i_2=0}^{m_2,\, p_2} B_{i_1, i_2}^2(t, x)u(0)^{i_1}u(1)^{i_2} \\ \cdots & \quad \cdots \\ \hat{f}_C \circ f_{u(k)} \circ \cdots \circ f_{u(0)} &= \sum_{i_1=0, \cdots, i_{k+1}=0}^{m_{k+1}, p_{k+1}, \cdots, l_{k+1}} \\ & \quad \cdot B_{i_1, \cdots, i_{k+1}}^{k+1}(t, x)\, u(0)^{i_1} \cdots u(k)^{i_{k+1}}. \end{aligned} \tag{16}$$

Theorem 3: The maximal invariant set $\hat{V}_I \subseteq \hat{V}_C$ can be computed by the following finite algorithm.

1) Initialize: $\hat{f}_C(t, x), f(x, u)$; $G_0 = \{f_C\}$; $k = 0$; Fix a monomial ordering.
2) Iterate: $k = k+1$.
3) Compute $\hat{f}_C \circ f_{u(k-1)} \circ \cdots \circ f_{u(0)}(t, x)$.
4) Compute the reduced Gröbner basis G_k

$$G_k = \text{Gbasis}[\hat{f}_C, B_0^1, \cdots, B_{m_1}^1, B_{00}^2, \cdots, B_{m_2, p_2}^2, \cdots, B_{m_k, p_k, \cdots, l_k}^k]$$

where the polynomials $B_{i_1, \cdots, i_s}^s \in \mathbb{Q}[t, x]$, $s = 1, \cdots, k$ are defined in (16).

5) If $G_k = G_{k-1}$, stop. $\langle G_k \rangle$ defines the maximal invariant set $\hat{V}_I$. If $G_k \neq G_{k-1}$ go to 2).

The following theorems are easily established.

Theorem 4: Suppose the variety $\hat{V}_C$ in Theorem 2 is of dimension $\dim \hat{V}_C \leq 2n-1$. A polynomial system of the form (1) is completely controllable if $\hat{V}_I = \emptyset$.

Theorem 5: Suppose that the conditions of Theorem 2 are satisfied with $\hat{f}_C \in \mathbb{Q}[x]$. Then a polynomial system of the form (1) is completely controllable if and only if $\hat{V}_I = \emptyset$.

In the example below we use subscripts to denote time steps for controls and omit the time index for states in the Gröbner bases, that is we write u_k instead of $u(k)$ and x_1 instead of $x_1(0)$.

Example 1: Consider the strict feedback system:

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= x_1^2(k) + (x_1(k) - x_3(k))u(k). \end{aligned} \tag{17}$$

Consider $x(3)$

$$\begin{aligned} x_1(3) &= x_1^2(0) + (x_1(0) - x_3(0))u(0) \\ x_2(3) &= x_2^2(0) + [x_2(0) - x_1^2(0) - (x_1(0) - x_3(0))u(0)]u(1) \\ x_3(3) &= x_3^2(0) + [x_3(0) - [x_2(0) - (x_1^2(0) + (x_1(0) \\ &\quad - x_3(0))u(0))]u(1)]u(2). \end{aligned} \tag{18}$$

We may not be able to attain $x_1(3) = 0$ if $x_1(0) = x_3(0)$, that is $V(x_1 - x_3)$ may contain critical states. Consider now those states for which we can zero the first equation, whereas the second equation may not be possible to zero. We compute the Gröbner basis

$$\text{Gbasis}[x_1^2 + (x_1 - x_3)u_0,\, x_2 - x_1^2 - (x_1 - x_3)u_0]$$

with lexicographic ordering $u_0 \succ x_1 \succ x_2 \succ x_3$. The obtained basis consists of only one polynomial which does not depend on u_0, namely the polynomial x_2. Hence, the states that belong to the variety $V(x_2)$ are also critical.

Finally, we compute the following basis:

$$\begin{aligned} \text{Gbasis}[&x_1^2 + (x_1 - x_3)u_0,\, x_2^2 + (x_2 - x_1^2 - (x_1 - x_3)u_0)u_1, \\ &x_3 - (x_2 - (x_1^2 + (x_1 - x_3)u_0))u_1] \end{aligned}$$

with lexicographic ordering $u_1 \succ u_0 \succ x_1 \succ x_2 \succ x_3$. The only polynomial in the computed basis that does not depend on u_1 and u_0 is $x_3 + x_2^2$. Thus, for the states in the variety $V(x_3 + x_2^2)$ we may zero the first two equations but not necessarily the third one.

As a result, we obtain that a critical variety is given by $V_C = V(x_2(x_1 - x_3)(x_3 + x_2^2))$. By using the Gröbner basis method in Theorem 1, we obtain that the maximal invariant set is $V_I = \{(0, 0, 0), (1, 1, 1)\}$. By simple calculations one can verify that both of these states are invariant sets themselves. Moreover, the state $(1, 1, 1)$ cannot be transferred to the origin. We conclude that the system is not state dead-beat controllable.

IV. Concluson

The results of this paper characterize a large class of discrete-time polynomial systems for which the Gröbner basis method can be used to facilitate the state dead-beat and complete controllability analysis. We have shown that an object called a critical variety, which we use in controllability tests, can be generically constructed for a large class of polynomial systems. For instance, the class of strict feedback systems was shown to generically satisfy our conditions. The computational complexity of the proposed algorithms is large but this is an intrinsic feature of the problem and the class of systems we consider.

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