[10] A. Vannelli and M. Vidyasagar, “Maximal Lyapunov functions and domain of attraction for autonomous nonlinear systems,” *Automatica*, vol. 21, pp. 69–80, Jan. 1985.
[11] M. Vidyasagar, *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1978.
[12] V. I. Zubov, “Methods of A. M. Lyapunov and their applications,” U.S. Atomic Energy Commission, 1961.

Dead-Beat Control of Simple Hammerstein Models

D. Nešić and I. M. Y. Mareels

***Abstract*—Dead-beat controllers for simple Hammerstein systems are investigated. Several designs for nonminimum-time state dead-beat controllers are given for certain classes of simple Hammerstein systems. A general minimum-time state dead-beat controller is presented for a class of simple Hammerstein systems. A design for a family of minimum-time control laws is provided. This enables, to a certain extent, shaping of transient response via choosing an appropriate control law. Finally, the authors design an output dead-beat controller for a class of Hammerstein systems that are not necessarily state dead-beat controllable.**

***Index Terms*—Controllability, dead-beat control, discrete-time, Hammerstein systems, time-optimal control.**

I. Introduction

Dead-beat controllers for linear systems have long been investigated, and successful applications have been reported in the literature [5]; some results on dead-beat controllability for classes of nonlinear systems can be found in [1], [3], [6], [8], and references therein. In this paper, we investigate dead-beat control for a class of nonlinear systems which are sometimes referred to as simple Hammerstein systems. Identification techniques for block-oriented models often yield systems of this form [4]. The block diagram of these systems is given in Fig. 1. The system consists of a linear dynamical block W and a static nonlinearity $f(u)$.

If the image of the nonlinearity f is such that $im(f) =]-\infty, +\infty[$, the design of a dead-beat (or any other) controller can be completely based on the design of such a controller for the linear subsystem W [8]. If, on the other hand, we have that the image $im(f) =]-\infty, \delta]$ or $[\delta, +\infty[$, it is no longer possible to complete the design of dead-beat controllers for the simple Hammerstein systems using the controllers designed for the linear subsystem W.

We propose a design method for nonminimum and minimum-time dead-beat controllers for the above simple Hammerstein systems with $im(f) =]-\infty, 0]$ or $[0, +\infty[$. We emphasize that our results can be used with minor changes to simple Hammerstein systems $im(f) =]-\infty, \delta]$ or $[\delta, +\infty[$. A family of nonminimum-time and a family of minimum-time dead-beat controllers is obtained. One can change to a certain degree the transient response while keeping time-optimality by choosing one controller from the family. Finally, we present a nonminimum-time output dead-beat controller for a class of simple Hammerstein systems that are not necessarily state dead-beat controllable.

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D. Nešić was with the Department of Systems Engineering, RSISE, The Australian National University, Canberra, Australia. He is now with the Department of Electrical and Computer Engineering, The University California, Santa Barbara, CA 93106 USA (e-mail: dragan@gibren.ece.ucsb.edu).

I. M. Y. Mareels was with the Department of Engineering, FEIT, The Australian National University, Canberra, Australia. He is now with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville 3052, Victoria, Australia.

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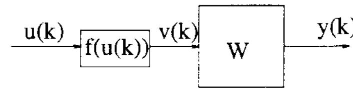

Fig. 1. Block diagram of simple Hammerstein systems.

II. Notation and Definitions

$\mathbb{R}$, $\mathbb{N}$, and $\mathbb{C}$ are, respectively, the sets of real, nonnegative integer, and complex numbers. For $\delta \in \mathbb{R}$, we write $\mathbb{R}_\delta^+ = [\delta, +\infty[$ and $\mathbb{R}_\delta^- =]-\infty, \delta]$. The class of nonlinear discrete-time systems that we consider can be written in the form

$$\begin{aligned} x(k+1) &= Ax(k) + bf(u(k)) \\ y(k) &= cx(k) + df(u(k)) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}$ are, respectively, the state and the input of the system at time $k \in \mathbb{N}$. The control sequence $\{u(0), u(1), \cdots\}$ is denoted as $\mathcal{U}$, and its truncation $\{u(0), \cdots, u(N-1)\}$ of length N is $\mathcal{U}_N$. Let $x(N, x(0), \mathcal{U}_N)$ denote the state at time N which results from the initial state $x(0)$ and the control sequence $\mathcal{U}_N$.

Assumption 1: With reference to (1), if $x(0) = 0$ then there exists $u(0) \in \mathbb{R}$ such that $x(1, x(0), \{u(0)\}) = 0$. In other words, $\delta \leq 0$ if $im(f) = \mathbb{R}_\delta^+$, or $\delta \geq 0$ if $im(f) = \mathbb{R}_\delta^-$.

Definition 1: System (1) is dead-beat controllable if $\forall\, x(0) \in \mathbb{R}^n$ there exists a control sequence which transfers $x(0)$ to the origin in finite time, i.e., $\exists N \in \mathbb{N}$ and $\mathcal{U}_N$ such that $x(N, x(0), \mathcal{U}_N) = 0$.

Definition 2: System (1) is completely controllable if $\forall\, x(0)$, $x^* \in \mathbb{R}^n$ there exist an integer $F = F(x(0), x^*)$ and a finite control sequence $\mathcal{U}_F$ such that the system is transferred from the state $x(0)$ to the state x^* under the action of the sequence $\mathcal{U}_F$, that is $x(F, x(0), \mathcal{U}_F) = x^*$.

A *minimum-time dead-beat controller* $u(k) = g(x(k))$ is such that it transfers any initial state $x(0)$ to the origin in minimum time. If a controller $u(k) = g(x(k))$ transfers any initial state to the origin but not necessarily in minimum time it is called *dead-beat*.

We use the following notation for a cone: $C = \{x\colon x = \sum_i^r c_i v_i,\ v_i \in \mathbb{R}_0^+\},\ c_i \in \mathbb{R}^{n\times 1}, \forall i = 1, \cdots, r$. A convex polyhedral cone can be also defined by r inequalities $l_i x \geq 0$, $l_i \in \mathbb{R}^{1\times n}$, $\forall i = 1, 2, \cdots, r$ [7].

Theorem 1 [2]: System (1) with $im f(u) = \mathbb{R}_0^+$ is completely controllable on $\mathbb{R}^n$ if and only if rank$[b\, Ab \cdots A^{n-1}b] = n$ (or alternatively rank$(\lambda I - A : b) = n$, $\forall \lambda \in \mathbb{C}$) and the matrix A has no real positive or zero eigenvalues.

The following theorem is an easy consequence of results in [2].

Theorem 2: System (1) with $im(f) = \mathbb{R}_0^+$ (or $\mathbb{R}_0^-$) is dead-beat controllable if and only if rank$(\lambda I - A : b) = n$, $\forall \lambda \in \mathbb{C} - \{0\}$ and A has no real strictly positive eigenvalues.

In order to appreciate the concreteness of the above theorem it suffices to reason as follows. The system can be decomposed into one without zero modes, which is completely controllable on an appropriate subspace, and another one which has only zero modes, which may be uncontrollable. Since dead-beat controllability is considered, the zero modes die out with $u(k) \equiv 0$ and the only issue is controllability of the subsystem without zero modes, which is completely controllable.

Comment 1: When the conditions of Theorem 1 are satisfied there exists a bound on the dead-beat time uniform over the state space. In other words, there exists an integer $T \in \mathbb{N}$ such that $\forall x(0) \in \mathbb{R}^n$, $\exists \mathcal{U}_k$ which yields $x(k, x(0), \mathcal{U}_k)) = 0$, $k \leq T$ (T is fixed for all initial states).

Using the results of [2], it is not difficult to see that if $im(f) = \mathbb{R}_\delta^+$, $\delta < 0$ (in Theorem 1), it is possible to have real eigenvalues of A, $\lambda \in]0, 1]$. Apparently, in this case there is no uniform bound on the dead-beat time and, in principle, on a subspace of the state space we have the situation that the further the state is from the origin, the longer the time required to zero that state.

The two types of dead-beat behavior differ considerably. We will concentrate on the class of systems for which there is a *uniform bound on the dead-beat time*. In other words we consider simple Hammerstein systems for which the conditions of Theorem 2 are satisfied. If $im(f) = \mathbb{R}_\delta^+$, $\delta < 0$ and $\exists \lambda(A) \in]0, 1]$, it is possible to modify the design that we present to obtain a minimum-time dead-beat controller, but the design can only be carried out on a subset of the state space (no uniform bound on dead-beat time).

III. State Dead-Beat Controllers

We present below two families of dead-beat controllers, assuming full state measurements for (1). In the first family (Class 1) we make use of both open- and closed-loop paradigms. In general, this leads to nonminimum-time dead-beat behavior. The second family is a class of closed-loop controllers that yield minimum-time dead-beat behavior. To simplify the presentation we consider in the sequel only the case $im(f) = \mathbb{R}_0^+$, but a generalization to $im(f) = \mathbb{R}_\delta^+$ is straightforward.

A. Class 1

We point out that there is no loss of generality if we concentrate only on completely controllable simple Hammerstein systems with a nonsingular matrix A (zero modes die out in finite time by applying $f(u(k)) = 0$, $\forall k$). Moreover, in the sequel we assume that the pair (A, b) is in controllability canonical form. If we apply the minimum-time dead-beat control law to the unconstrained linear system (1) ($v(k) = Kx(k)$, $v \in \mathbb{R}$), we obtain that the closed loop matrix is $(A + bK) = J$, where J is $n \times n$ nilpotent Jordan matrix. Consider the cone

$$C = \{x: Kx \geq 0, KJx \geq 0, \cdots, KJ^{n-1}x \geq 0\}. \tag{2}$$

Notice that if the initial state is in the cone, it can be transferred to the origin in at most n time steps by applying the control action $u(k) \in \mathbb{R}_0^\pm$ which is equal to (any) real solution to the equation $f(u(k)) = Kx(k)$. Is straightforward to show that no initial state in the complement of C can be transferred to the origin in less than $n+1$ time steps. Hence, the design of a dead-beat controller can be formulated into the design of a controller which transfers any state outside of the cone to the cone in finite time. On the cone the linear controller $Kx(k)$ with an inversion yields minimum-time behavior.

Since A is not singular, the cone C has a nonempty interior $\overset{\circ}{C}$ in $\mathbb{R}^n$. Indeed, nonsingularity of A implies that the matrix whose rows are the vectors KJ^i, $i = 0, 1, \cdots, n-1$ has a full rank and the conclusion follows [7]. Moreover, it is easy to show that the cone C given by (2) is convex.

Suppose that the integer N represents the time such that any state $x(0) \in \mathbb{R}^n$ can be transferred to any $x^* \in \mathbb{R}^n$ in at most N steps. Such a number exists since the system is completely controllable [2]. Therefore, for any $x^* \in \overset{\circ}{C}$ and for any positive ε there exists a sequence of controls $u(0), \cdots, u(N-1)$ such that

$$\varepsilon x^* = A^{N-1}bf(u(0)) + \cdots + Abf(u(N-2)) + bf(u(N-1)). \tag{3}$$

Assume that $x^* \in \overset{\circ}{C}$ has been chosen (it is one of the design parameters). Since x^* is in the interior of the cone C, εx^* is also in the interior of the cone for all positive ε.

To complete the design of Class 1 controllers we need [7, Corollary VI-D-1].

Proposition 1: Let C be a convex set in $\mathbb{R}^n$. Then $z \in \overset{\circ}{C}$ if and only if for every $y \in \mathbb{R}^n$, there exists some $\alpha > 0$ such that $z + \alpha y \in \overset{\circ}{C}$.

Using this result we can say that for any $x(0)$ in the complement of the cone C, there exists a positive ε ($\varepsilon = \alpha^{-1}$) such that $A^N x(0) + \varepsilon x^* \in \overset{\circ}{C}$.

In other words, there exists a sequence of controls satisfying (3) which yields $x(N) \in \overset{\circ}{C}$.

Given any $x(0)$ and $x^* \in \overset{\circ}{C}$, let us find the value of $\hat{\varepsilon}$ which yields $x(N) \in \overset{\circ}{C}$. Consider the set of inequalities

$$Kx(N) > 0;\ KJx(N) > 0;\ \cdots;\ KJ^{n-1}x(N) > 0. \tag{4}$$

If all of the inequalities are simultaneously satisfied, the state $x(N)$ belongs to the interior of the cone C. Therefore, the inequalities

$$KJ^iA^Nx(0) + \varepsilon KJ^ix^* > 0, \qquad \forall i = 0, 1, \cdots, n-1 \tag{5}$$

must be satisfied. Any $\hat{\varepsilon}$ satisfying

$$\hat{\varepsilon} > \max_i \left(-\frac{KJ^iA^Nx(0)}{KJ^ix^*}\right), \qquad \hat{\varepsilon} > 0$$

guarantees that $x(N)$ belongs to the interior of the cone C.

Hence, we can compute $\hat{\varepsilon}$ using

$$\hat{\varepsilon} = \max\left(\max_i \left[-\frac{KJ^iA^Nx(0)}{KJ^ix^*}\right], 0\right) + \zeta, \qquad \zeta > 0. \tag{6}$$

Using (3) we can compute controls $u(i)$, $i = 0, \cdots, N-1$ which transfer $x(0)$ to the interior of the cone C

$$\hat{\varepsilon} x^* = A^{N-1}bf(u(0)) + \cdots + Abf(u(N-2)) + bf(u(N-1)). \tag{7}$$

The design is summarized below.

Theorem 3: Consider a simple Hammerstein system for which (A, b) is a controllable pair, A is nonsingular, and Assumption 1 holds.

The following controller yields dead-beat behavior:

if $x(k) \in C$ apply any real root u to $f(u) = K(x(k))$.

Otherwise, apply a control sequence $u(0), \cdots, u(N-1)$ which satisfies

$$\hat{\varepsilon} x^* = A^{N-1}bf(u(0)) + \cdots + Abf(u(N-2)) + bf(u(N-1))$$

where C is defined by (2), $x^* \in \overset{\circ}{C}$, and $\hat{\varepsilon}$ is computed using (6).

We present below two special situations in which there exists an integer L such that $A^Lb \in \overset{\circ}{C}$. In the first case $L > 0$, and in the second $L = 0$. It is interesting that if $L = 0$, then a minimum-time dead-beat controller is obtained using this approach. Moreover, the obtained controller is closed loop (on the whole state space). This situation corresponds to the case when the characteristic polynomial of matrix A has all coefficients strictly positive.

Corollary 1: Consider the Hammerstein system (1) for which (A, b) is a controllable pair and Assumption 1 holds. If there exists an integer L such that $KJ^iA^Lb > 0, \forall i = 0, 1, \cdots, n-1$ then we have the control law, as shown at the bottom of the page, where $S = \cup_i S_i$, $S_i = \{x\colon KA^ix \geq 0, KJA^ix \geq 0, \cdots, KJ^{n-1}A^ix \geq 0\}$, $i = 1, \cdots, L$ is dead-beat, and it transfers every initial state to the origin in at most $n + L + 1$ time steps.

It is important to emphasize that ε may be a constant $\varepsilon \geq 0$ or a function $\varepsilon = \varepsilon(x(k), k) \geq 0, \forall x(k), \forall k$.

Proof: Consider the following equations:

$$\begin{gathered} KA^{L+1}x + K(A^Lbf(u(0)) + A^{L-1}bf(u(1)) \\ + \cdots + bf(u(L-1))) \geq 0 \\ \cdots \\ KJ^{n-1}A^{L+1}x + KJ^{n-1}(A^Lbf(u(0)) + A^{L-1}bf(u(1)) \\ + \cdots + bf(u(L-1))) \geq 0. \end{gathered} \tag{8}$$

Since $KJ^iA^Lb > 0$, $\forall i = 0, 1, \cdots, n-1$, it follows that $\forall x(0) \notin C$ the control law $f(u(0)) = \max(\max_{i=0,1,\cdots,n-1}(-KJ^iA^{L+1}x/KJ^iA^Lb), 0) + \text{const}$ and $u(i) = 0, \forall i = 1, 2, \cdots, L-1$ transfers $x(0)$ to C in $L + 1$ steps. Q.E.D.

We denote the characteristic polynomial of the matrix A as $\chi(A) = p^n + \sum_{i=0}^{i=n-1} a_i p^i$.

Corollary 2: If the matrix A has a characteristic polynomial with all coefficients strictly positive, that is $a_i > 0, \forall i = 0, 1, \cdots, n-1$, then the controller

apply any real solution u to

$$\begin{cases} f(u) = Kx, & \text{if } x \in C \\ f(u) = \max\left(\max\limits_{i=0,1,\cdots,n-1} \dfrac{-KJ^iAx}{a_i}, 0\right) + \varepsilon, \ \varepsilon \geq 0, & \text{if } x \notin C \end{cases}$$

is dead-beat and it transfers every initial state to the origin in at most $n + 1$ time steps.

Proof: Notice that $KJ^ib = a_i, \forall i = 0, 1, \cdots, n-1$. Since $a_i > 0, \forall i = 0, 1, \cdots, n-1$, it follows that there exists a control value $u(0)$ which transfers every $x(0) \in C^C$ to C in one step. All such control laws are given in Corollary 2, since obviously $KJ^ix(1) > 0, \forall i = 0, 1, \cdots, n-1$. Q.E.D.

B. Class 2

We show now how it is possible to design a minimum-time dead-beat controller for general simple Hammerstein systems. We denote the cone C as $C_n = \{x\colon Kx \geq 0, \cdots, KJ^{n-1}x \geq 0\}$. On this cone we can apply the closed-loop control scheme for Class 1 controllers, which yields minimum-time behavior. To complete the design of a minimum-time dead-beat controller we show how it is possible to construct the set of states that can be transferred to the cone in one, two, etc., time steps which we denote as $C_{n+1}, \cdots, C_N$. Once such sets are found, it is easy to find a controller which is such that it maps C_{j+1} to C_j, $j = 1, \cdots, N-1$.

Let us first compute the set C_{n+1} which is mapped to the cone C_n in one step. Find the compositions

$$KJ^iAx + KJ^ibf(u) \geq 0, \qquad i = 0, 1, \cdots, n-1. \tag{9}$$

We split the set of inequalities (9) into three groups according to the sign of KJ^ib. The set of i for which $KJ^ib = 0$ is relabeled as $s_1, \cdots, s_{n_0}$. The same is done for the sets of indexes i for which $KJ^ib > 0$ and $KJ^ib < 0$. They are denoted, respectively, as $t_1, \cdots, t_{n_+}$ and $p_1, \cdots, p_{n_-}$. It is obvious that the set $KJ^{s_i}Ax \geq 0$, $i = 1, \cdots, n_0$ is a part of the set of equations that define C_{n+1}.

Moreover, we have that there exists a control u which transfers a state x from C_{n+1} to C_n if and only if the following inequalities are satisfied:

$$\begin{gathered} \min_{p_i}\left(-\frac{KJ^{p_i}Ax}{KJ^{p_i}b}\right) \geq f(u) \geq \max_{t_j}\left(-\frac{KJ^{t_j}Ax}{KJ^{t_j}b}\right), \\ \forall p_i, t_j, i = 1, \cdots, n_-, \\ j = 1, \cdots, n_+ f(u) \geq 0. \end{gathered} \tag{10}$$

Using these inequalities we see that the following inequalities in x must be satisfied:

$$\begin{gathered} -\left(\frac{KJ^{p_i}A}{KJ^{p_i}b} - \frac{KJ^{t_j}A}{KJ^{t_j}b}\right) \geq 0 \, \forall p_i, t_j, \qquad i = 1, \cdots, n_-, \\ j = 1, \cdots, n_+ \\ -\frac{KJ^{p_i}A}{KJ^{p_i}b}x \geq 0, \qquad \forall p_i, i = 1, \cdots, n_-. \end{gathered} \tag{11}$$

The defining set of inequalities for C_{n+1} is

$$\begin{gathered} KJ^{s_i}Ax \geq 0, \qquad i = 1, \cdots, n_0 \\ -\left(\frac{KJ^{p_i}A}{KJ^{p_i}b} - \frac{KJ^{t_j}A}{KJ^{t_j}b}\right)x \geq 0, \qquad \forall p_i, t_j, i = 1, \cdots, n_-, \\ j = 1, \cdots, n_+ \\ -\frac{KJ^{p_i}A}{KJ^{p_i}b}x \geq 0, \qquad \forall p_i, i = 1, \cdots, n_-. \end{gathered} \tag{12}$$

If we denote the set of inequalities (12) as $l_i^{n+1}x \geq 0$, $i = 1, 2, \cdots, n_{n+1}$, we can write that $C_{n+1} = \{x\colon l_i^{n+1}x \geq 0, i = 1, 2, \cdots, n_{n+1}\}$. The set C_{n+2} is computed in a similar way where we start the same procedure from the following set of inequalities:

$$l_i^{n+1}Ax + l_i^{n+1}bf(u) \geq 0, \qquad i = 1, 2, \cdots, n_{n+1}.$$

It is important to notice that there exists a uniform bound on the minimum number of steps necessary to transfer any initial state to the origin. This can be seen from the proof given in [2]. Consequently, there exists an integer L which is such that $\cup_{i=1}^{i=L+n+1} C_i = \mathbb{R}^n$. It only remains to compute the controls that transfer any state in C_{i+1} to C_i, $\forall i = 1, 2, \cdots, L + n + 1$.

It is obvious that the control law $u = Kx$ maps C_{i+1} to C_i, $\forall i = 1, 2, \cdots, n-1$. We use the notation $C_i = \{x\colon l_j^ix \geq 0, j = 1, 2, \cdots, n_i\}$, $i = 1, \cdots, n, n+1, \cdots, L+n+1$. We also use the indexes s_m^i, p_s^i, and t_j^i to denote the indexes f for which l_f^ib is, respectively, equal, less than and greater than zero. Then the control law $f(u) = v_i(x)$, if $x \in C_{i+1}$, $i = n, \cdots, L-1$ and $f(u) \geq 0$ where $v_i(x)$ can take values from the following interval:

$$v_i(x) \in \left[\max\left(0, \max_{t_j^k}\left(-\frac{l_{t_j^k}^kAx}{l_{t_j^k}^kb}\right)\right), \min_{p_s^k}\left(-\frac{l_{p_s^k}^kAx}{l_{p_s^k}^kb}\right)\right], \ x \in C_{i+1}$$

which transfers any state in C_{i+1} to C_i in one step.

apply any real solution u to

$$\begin{cases} f(u) = Kx, & \text{if } x \in C \\ f(u) = 0, & \text{if } x \in S \\ f(u) = \max\left(\max\limits_{i=0,1,\cdots,n-1} \dfrac{-KJ^iA^{L+1}x}{KJ^iA^Lb}, 0\right) + \varepsilon, \ \varepsilon \geq 0, & \text{if } x \notin (C \cup S) \end{cases}$$

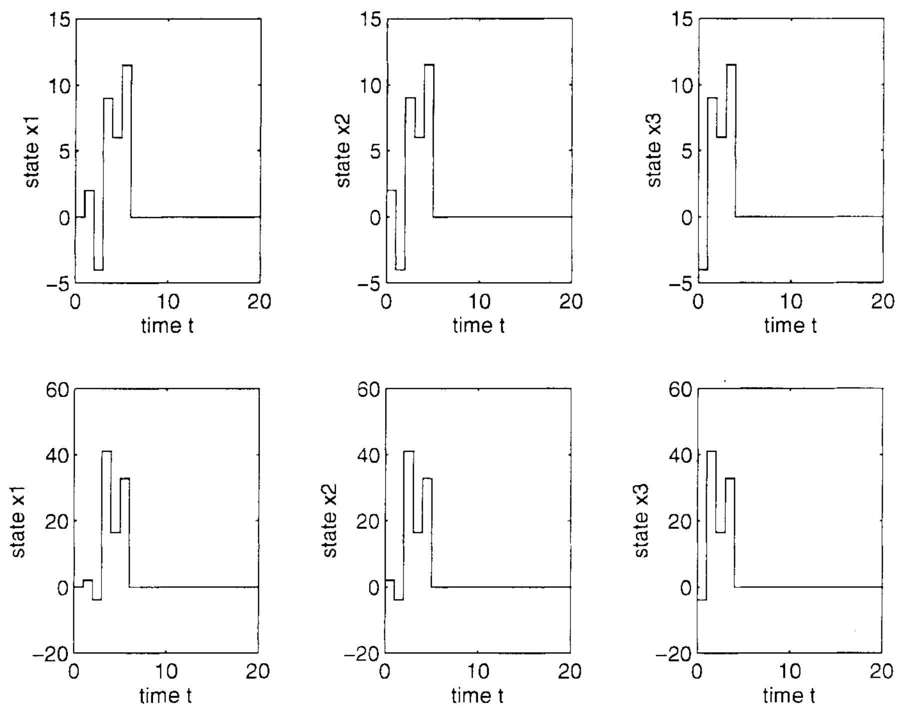

Fig. 2. Controller 2 (top) and Controller 1 (bottom) for initial state $(0, 2, -4)$.

Hence, we designed a family of minimum-time dead-beat controllers. By specifying the law according to which we chose $v_i(x)$, we can shape to a certain degree the response of the system.

IV. An Output Dead-Beat Controller

If instead of zeroing the state of system (1) we wish to zero its output in finite time, we need an output dead-beat controller. Necessary and sufficient conditions for output dead-beat controllability of simple Hammerstein systems are not known. It is obvious though [see (1)] that output dead-beat controllability is an easy consequence of state dead-beat controllability.

We present below sufficient conditions under which output dead-beat control can be achieved and design an output dead-beat controller. Assumption 1 is still used.

Theorem 4: Consider system (1) under Assumption 1. Assuming that $d \neq 0$, let $H = A - bcd^{-1}$. Define

$$C_O = \{x \colon d^{-1}cH^i x \leq 0,\ i = 0, 1, \cdots, L-1\}.$$

If the following conditions are satisfied:

1) the matrix H satisfies a polynomial equation

$$H^L - \sum_{i=0}^{i=L-1} c_i H^i = 0, \qquad \text{where } c_i \geq 0,\ \forall i = 0, \cdots, L-1;$$

2) there exists a number N such that $d^{-1}cH^iA^Nb < 0,\ \forall i = 0, 1, \cdots, L-1$;

system (1) is output dead-beat controllable. If H is a stable matrix (with all eigenvalues inside the closed unit disk), the system is output dead-beat controllable with stable zero dynamics.

Proof: Because of Condition 1 in Theorem 4, it is not difficult to see that the cone C_O is positively invariant. In other words, if an initial state is in the cone, it stays inside the cone when the control $f(u) = -d^{-1}cx$ is applied to the system.

Consider the following inequalities:

$$\begin{aligned}
&d^{-1}(cA^{N+1}x(0) + cA^Nbf(u(0)) + cA^{N-1}bf(u(1))\\
&\quad + \cdots + cbf(u(N-1))) \leq 0\\
&d^{-1}(cHA^{N+1}x(0) + cHA^Nbf(u(0)) + cHA^{N-1}bf(u(1))\\
&\quad + \cdots + cHbf(u(N-1))) \leq 0\\
&\qquad\qquad \cdots\\
&d^{-1}(cH^{L-1}A^{N+1}x(0) + cH^{L-1}A^Nbf(u(0)) + \cdots\\
&\quad + cH^{L-1}bf(u(N-1))) \leq 0.
\end{aligned}$$

If Condition 2 of Theorem 4 is satisfied, we can transfer any state outside the cone C_O to the cone C_O by applying in the first step

$$f(u(0)) = \max\left(\max_i \frac{-cH^iA^{N+1}x(0)}{cH^iA^Nb},\ 0\right)$$

and $f(u(k)) = 0,\ \forall k = 1, 2, \cdots, N-1$. We have that $x(N) \in C_O$ and then we can apply $f(u(k)) = -d^{-1}cx(k)$. Q.E.D.

Notice that under the conditions of Theorem 4 system (1) does not have to be state dead-beat controllable. Observe that $0 \in C_O$ is always satisfied and that if $\{0\} = C_O$, (1) must necessarily

any real root u to

$$\begin{cases}
f(u) = -d^{-1}cx, & \text{if } x \in C_O\\
f(u) = 0, & \text{if } x \in S\\
f(u) = \max\left(\max_i \dfrac{-cH^iA^{N+1}x}{cH^iA^Nb},\ 0\right) + \delta, & \text{if } x \in \mathbb{R}^n - (C_O \cup S),\ \delta \geq 0
\end{cases}$$

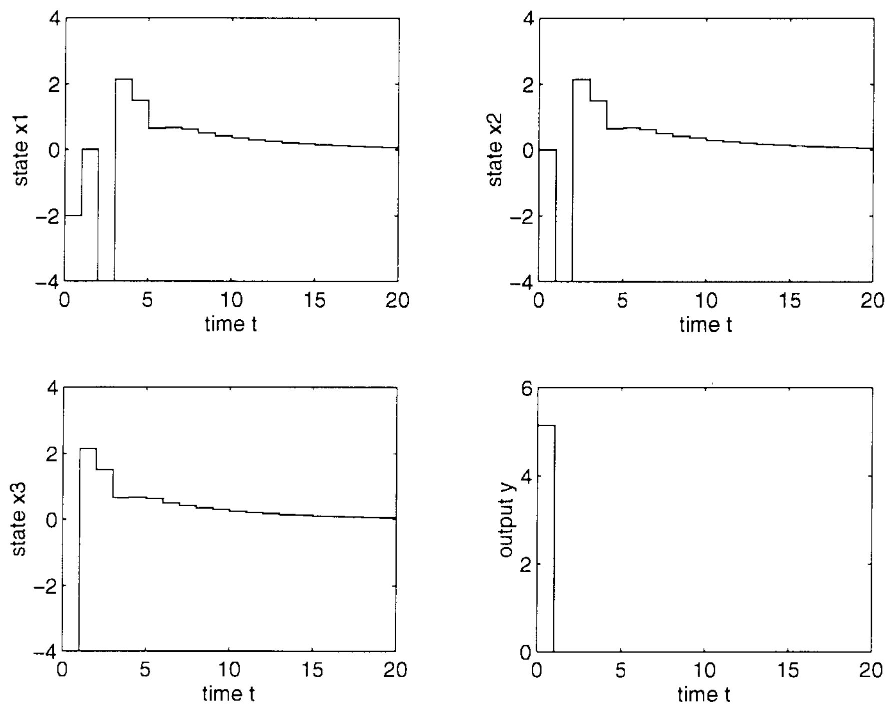

Fig. 3. Simulation results for the output dead-beat controller.

be state dead-beat controllable in order to have output dead-beat controllability.

From the proof of Theorem 4 it follows that the output dead-beat control law is as shown in the equation at the bottom of the previous page, where

$$S = \bigcup_{i=1}^{i=N-1} \{x: d^{-1}cA^i x \le 0, \cdots, d^{-1}cH^{L-1}A^i x \le 0\}.$$

Example 1: The system is given by

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0.5 & 0.5 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u^2(k).$$

Due to space limitations we do not present analytic expressions for Controllers 1 and 2. The simulations of the two controllers for the initial conditions $(0, 4, -2)$ are given in Fig. 2. It is clearly possible to change the transient response while maintaining minimum-time dead-beat behavior.

Example 2: Consider the system

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.2 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u^2(k)$$
$$y(k) = (-0.1\,0\ -0.5)x(k) + u^2(k).$$

It is straightforward to check that the conditions of Theorem 4 are satisfied, with $L = 3$ and $N = 0$. Simulation of the output dead-beat controller is given in Fig. 3. Notice that the system is not state dead-beat controllable.

References

[1] G. Bastin, F. Jarachi, and I. M. Y. Mareels, "Dead beat control of recursive nonlinear systems," in *Proc. 32nd IEEE Conf. Decision and Control,* San Antonio, TX, 1993, pp. 2965–2971.

[2] M. E. Evans and D. N. P. Murthy, "Controllability of discrete-time systems with positive controls," *IEEE Trans. Automat. Contr.,* vol. AC-22, pp. 942–945, 1977.

[3] S. T. Glad, "Dead beat control for nonlinear systems," *Analysis and Control of Nonlinear Systems,* C. I. Byrnes, C. F. Martin, and R. E. Saeks, Eds. Amsterdam, The Netherlands: North-Holland, 1988, pp. 437–442.

[4] R. Haber and H. Unbehauen, "Structure identification of nonlinear dynamic systems—A survey of input–output approaches," *Automatica,* vol. 26, pp. 651–677, 1990.

[5] J. O'Reilly, "The discrete linear time invariant time-optimal control problem—An overview," *Automatica,* vol. 17, pp. 363–370, 1981.

[6] D. Nešić and I. M. Y. Mareels, "Invariant sets and output dead-beat controllability for odd polynomial systems: The Gröbner basis method," in *Proc. 13th IFAC World Congr.,* San Francisco, CA, 1996, vol. E, pp. 221–226.

[7] R. T. Rockafellar, *Convex Analysis.* Princeton, NJ: Princeton Univ. Press, 1970.

[8] Q.-M. Zhu and K. Warwick, "Deadbeat controller for a class of nonlinear systems," *Contr. Theory and Advanced Technol.,* vol. 7, pp. 101–116, 1991.