Stabilizability and stability for explicit and implicit polynomial systems: a symbolic computation approach*

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Abstract

Stabilizability and stability for a large class of discrete-time polynomial systems can be decided using symbolic computation packages for quantifier elimination in the first order theory of real closed fields. A large class of constraints on states of the system and control inputs can be treated in the same way. Stability of a system can be checked by constructing a Lyapunov function, which is assumed to belong to a class of polynomial positive definite functions. Moreover, we show that stability/stabilizability is possible to decide in a rather unexpected way, namely directly from the $\varepsilon - \delta$ definition.

1 Introduction

The stability and stabilizability problems for dynamical and controlled systems respectively, represent most important topics in control systems theory. This is due to the fact that stability is a minimum requirement which most control systems need to satisfy. A vast literature on these problems exists and we mention just a few important results and refer to [9, 24] for a more complete list of references. The second method of Lyapunov is ubiquitous in analysis of stability of nonlinear systems [9] although the construction of a Lyapunov function for a general nonlinear system is to this date an important open problem in control theory. The well known Brockett's necessary condition for smooth stabilizability indicated that many important nonlinear plants can not be stabilized using smooth feedback and discontinuous or time-varying feedback laws are required [24]. More recently, the notions of the control Lyapunov functions and the fundamental Artstein's theorem paved the way to the use of the Lyapunov's second method for the stabilizability problem of nonlinear systems [24]. For some classes of discrete-time nonlinear systems these problems were investigated in [23, 12, 4] and references therein.

In this paper, we concentrate on the problems of stability/stabilizability for a class of discrete-time polynomial systems. Some applications of this class of models can be found in [3, 8, 13, 20].

The main contribution of our paper is that we show how some of the recently developed symbolic computation methods can be used to check whether a polynomial system is stable/stabilizable. With this set of tools we can approach the stability/stabilizability problems for polynomial systems from *a completely new direction*. We propose an algorithm for quantifier elimination (QE) in the first order theory of real closed fields as a tool to tackle the stability/stabilizability problems for polynomial systems. The method is implemented in the QEPCAD (Quantifier Elimination by Partial Cylindrical Algebraic Decomposition) software package, which was developed by Collins, Hong and a number of their coworkers [5, 7]. Other QE methods have been proposed in the literature which can tackle special classes of problems and are more efficient than QEPCAD [14, 26].

We emphasize several facts which follow from our results. First, QEPCAD is a natural tool to use when dealing with stability/stabilizability of polynomial systems. Indeed, it is possible to exploit this

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symbolic software in two ways when tackling the stability problem. In the first approach we chose a class of positive definite polynomial (Lyapunov) functions, e.g., quadratic forms $x^T Hx$, where H is an $n \times n$ matrix whose coefficients are to be determined. QEPCAD is used to compute whether there exists a function from the family, which is a Lyapunov function for the system. The second approach is based on the $\varepsilon - \delta$ stability definition. Other approaches are possible but we believe that QEPCAD is the key tool in a computational approach to stability analysis of polynomial systems and can be very useful if combined with any other method from [9, 24].

Second, as it was already indicated, we demonstrate how checking asymptotic stability using QEPCAD is possible via the ε-δ definition of stability. We are not aware of any results in the literature which make this observation. This approach is restricted to discrete-time systems. Indeed, it is *impossible* to do this for general continuous time nonlinear systems since this would require an analytic solution of a set of nonlinear differential equations. More surprisingly, it turns out that for some classes of discrete-time systems checking stability via the ε-δ definition may be less computationally demanding than the Lyapunov approach described above.

Our work motivates the following question: If a polynomial system is asymptotically stable, does there exist a *polynomial Lyapunov function* for the system. Notice the difference between this statement and other converse Lyapunov theorems - we require that the Lyapunov function has a special polynomial form. The class of quadratic forms is a good candidate, which generically works, so we believe that this statement is not far from being true in general. With this result and our first approach we could obtain an algorithmic approach for checking in finite time whether a polynomial system is stable or not. In summary, QEPCAD may pave the way for obtaining necessary and sufficient conditions based on Lyapunov theory [9], checkable in finite time, for the question of whether a polynomial systems is stable or not.

Third, we illustrate how QEPCAD can be used to test stabilizability of implicit recursive polynomial equations which arise in the investigation of minimum phase properties of input-output polynomial (also called NARMAX [8]) models. The approach seems to be natural in this setting since the non-uniqueness of solutions requires a quantifier elimination to be performed in testing stabilizability.

Finally, we would like to point out that irrespective of the large computational requirements of the QEPCAD based algorithms, which hamper their practicality, each of the above given observations is theoretically very important. Moreover, the symbolic computation approach to quantifier elimination is an area of ongoing active research and it is very possible that efficiency of QEPCAD is improved to the point where it can be effectively used for solving relevant engineering problems in the very near future.

In the first part of the paper we show how QEPCAD can be used to tackle the stability problem of polynomial dynamical (not controlled) systems. This methodology is generalized in the second part of the paper where we consider the problems of stabilizability and stability of zero output constrained dynamics (or zero dynamics), which can be viewed as a constrained stabilizability problem. We illustrate our approach and limitations of the current versions of QEPCAD via an example.

2 Preliminaries

The sets of real, complex and non-negative integer numbers are respectively denoted as $\mathbb{R}, \mathbb{C}$ and $\mathbb{N}$. The ring of polynomials in variables $x_1, \ldots, x_n$ with coefficients in a field $\mathcal{F}$ is denoted as $\mathcal{F}[x_1, \ldots, x_n]$.

A set $S \subset \mathbb{R}^n$ is called semi-algebraic if it can be constructed by finitely many applications of union, intersection and complementation operations on sets of the form $\{x \in \mathbb{R}^n : f_i(x) \geq 0\}$, where f_i are polynomials in x with real coefficients. Given a semi-algebraic set S we denote its defining expression as $S(x)$. For example, if $S = \{x : x_1^2 + x_2^2 - 1 < 0 \wedge x_1 > 0.5\}$ then we write $S(x) = (x_1^2 + x_2^2 - 1 < 0) \wedge (x_1 > 0.5)$.

The Euclidean norm of a vector x is denoted as $||x||$. The distance between points $x, y \in \mathbb{R}^n$ and the distance between a set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ are respectively denoted as $d(x, y) = ||x - y||$ and $d(x, A) = \inf_{y \in A} ||x - y||$. The open hyper-ball centered at a point z^* with a diameter $p > 0$ is denoted as

$$\mathcal{B}_p(z^*) = \{z \in \mathbb{R}^n : d(z, z^*) < p\}$$

and the open hyper-cube centered at a point z^* with sides $2r > 0$ is denoted as

$$\mathcal{C}_r(z^*) = \{z \in \mathbb{R}^t : (|z_1^* - z_1| < r) \wedge \ldots \wedge (|z_t^* - z_t| < r)\}$$

where $|a|$ is the absolute value of the scalar a. The composition of two mappings $f(x)$ and $h(x)$ is denoted as $f \circ h(x) = f(h(x))$. p compositions of the mapping f by itself is denoted as $f^p(x)$.

Dynamic polynomial systems are represented by:

$$x(k+1) = f(x(k)) \tag{1}$$

where $x = (x_1 \; x_2 \; \ldots x_n)^T$ and $k \in \mathbb{N}$. Moreover, without loss of generality[1] it is assumed that $f(0) = 0$ and $f = (f_1 \; f_2 \; \ldots \; f_n)^T, \; f_i \in \mathbb{Q}[x_1, \ldots, x_n], \forall i$. The assumption that the polynomials f_i have rational coefficients is needed for computational purposes. We denote the state at time step k starting from the initial state $x(0)$ as $x(k, x(0))$, that is $x(k, x(0)) = f^k(x(0))$.

A set Z is invariant if given any $x(0) \in Z$ we have that $f(x(0)) \in Z$.

Definition 1 Given an invariant set Z, we say that the origin of the system (1) is:

1. stable (conditionally to Z) if for any $\epsilon > 0$ there exists $\delta > 0, \; \delta = \delta(\epsilon)$ such that if $||x(0)|| < \delta$ then $||x(k, x(0))|| < \epsilon, \forall k \in \mathbb{N}$ ($x(0) \in Z$ and $||x(0)|| < \delta$ then $||x(k, x(0))|| < \epsilon, \forall k \in \mathbb{N}$).
2. attractive (conditionally to Z) if there exists $\Delta > 0$ such that if $||x(0)|| < \Delta$ then $\lim_{k\to\infty} ||x(k, x(0))|| = 0$ (if $||x(0)|| < \Delta$ and $x(0) \in Z$ then $\lim_{k\to\infty} ||x(k, x(0))|| = 0$).
3. asymptotically stable (conditionally to Z) if it is both stable and attractive (conditionally to an invariant set Z).
4. If in 2 and 3 we have $\Delta = +\infty$, then the corresponding properties are global.

If the origin of the system (1) is asymptotically stable, we refer to the system as a stable system.

Definition 2 A function $V : \mathbb{R}^n \to \mathbb{R}$ is (locally) positive definite if:

1. $V(0) = 0$
2. $V(x) > 0, \forall x \in \mathbb{R}^n - \{0\}$ ($\exists d > 0$ such that $V(x) > 0, \forall x \in \mathcal{B}_d(0) - \{0\}$).

Suppose that we have a parameterized family of positive definite functions, depending on the (matrix) parameter H. We denote this family of functions as $V(H, x)$. The class of quadratic positive definite functions is given by $V(H, x) = x^T H x$ and $H \in \mathbb{R}^{n\times n}$ is a positive definite matrix. For example, in the case of second order systems the class of quadratic functions has the following form:

$$V(H, x) = h_{11}x_1^2 + 2h_{12}x_1x_2 + h_{22}x_2^2$$

with

$$h_{11} > 0 \wedge h_{11}h_{22} - h_{12}^2 > 0$$

We use a shorthand notation $S_1(H)$ for the set of conditions on coefficients H that need to be satisfied in order to have a positive definite function $V(H, x)$.

Using Lyapunov theory [25] we have:

[1]We can also consider classes of systems with non-polynomial nonlinearities, such as rational functions or even trigonometric functions. For instance, if the nonlinearities $\sin^2 x$ and $\cos^2 x$ appear in the model, we substitute $\sin^2 x = p^2, \cos^2 x = 1 - p^2$ and obtain a polynomial system with the new variable p and perhaps a polynomial constraint.

Theorem 1 *Consider the dynamical polynomial system (1). If there exists a locally positive definite function V, such that*

$$(\exists s > 0)(\ V(f(x)) - V(x) < 0, \quad \forall x \in \mathcal{B}_s(0) - \{0\}\ ) \tag{2}$$

then the system (1) is stable.

In the sequel, we also consider controlled polynomial systems:

$$x(k+1) = f(x(k), u(k)) \tag{3}$$

where $x = (x_1\ x_2\ \ldots x_n)^T, u \in \mathbb{R}$ are respectively the state and control signal and $k \in \mathbb{N}$ represents time. It is assumed that $f(0,0) = 0$ and $f = (f_1\ f_2\ \ldots\ f_n)^T,\ f_i \in \mathbb{Q}[x_1, \ldots, x_n, u], \forall i$. An infinite control sequence $\{u(0), u(1) \ldots\}$ is denoted as U and its truncation of length k, $\{u(0), \ldots, u(k-1)\}$, is denoted as U_k. We denote the state at time step k starting from the initial state $x(0)$ under the sequence of controls U_k as $x(k, x(0), U_k)$.

Definition 3 *The system (3) is stabilizable if there exists a state feedback $u = u(x)$ such that the system $x(k+1) = f(x(k), u(x(k)))$ is stable.*

We emphasize that we do not assume that $u(x)$ in the above definition is continuous. Also, more general situations, such as dynamic and time varying (periodic) control laws can be analyzed in the same way but we consider only static control laws for simplicity of presentation.

QEPCAD is a symbolic computation package for quantifier elimination (QE) in the first order theory of real closed fields. It is based on the cylindrical algebraic decomposition (CAD) algorithm [5, 7]. The input to the algorithm is an expression which consists of polynomial equations and inequalities, Boolean operators and ($\wedge$), or ($\vee$), implies ($\rightarrow$) and not ($\neg$), as well as quantifiers $\exists$ and $\forall$. The solution (output formula) is a quantifier free expression. We note that all variables are assumed to be real. For more on QEPCAD see [5, 7, 11]. The simple examples we present illustrate the operation of the algorithm.

Example 1 If the input expression is $(\exists t)(\ a_2t^2 + a_1t + a_0 = 0\ )$, the output expression $(a_1^2 - 4a_2a_0 \geq 0)$ is obtained.

Example 2 [11] If the input expression is$(\forall x)(\forall y)(\ (x^2+y^2 < 1) \rightarrow (y > x^4 - 2)\ )$, the output expression $TRUE$ is obtained.

3 Stability of polynomial systems

In this section we consider stability of dynamical polynomial systems (1). We show that stability can be decided using QEPCAD in two completely different ways: by constructing Lyapunov functions or by direct verification of Definition 1.

3.1 Lyapunov functions construction (Procedure 1)

In this subsection we illustrate how QEPCAD can be used to construct Lyapunov functions for the systems (1).

From the converse stability theorems [9, 25], it follows that if the system is asymptotically stable, there exists a positive definite Lyapunov function satisfying the conditions of Theorem 2. This function does not have to be polynomial in general, but we will assume that it belongs to a class of positive definite polynomial functions $V(H, x)$, whose coefficients are free (to be determined later). Although there is no way we can say *a priori* how large the class of functions $V(H, x)$ should be, we can often have a good guess. Indeed, all systems whose linearization does not have critical modes (poles on the unit circle) allow for the use of quadratic polynomial functions to check stability/instability. Hence, we can say that the class of quadratic polynomial functions generically contains a desired Lyapunov function with which we can check whether the system is stable. Observe that using quadratic polynomial functions is already more general than checking stability via the linearization of (1).

We know that $S_1(H)$ needs to be satisfied in order to have the class of positively definite functions $V(H, x)$. Consider now the following quantifier elimination problem:

$$(\exists D)(\forall x)(\ D > 0 \wedge 0 < ||x|| < D \wedge V(H, f(x)) - V(H, x) < 0\ ).$$

The solution to the above problem can be found (in principle) using QEPCAD. The solution is a set of polynomial constraints on the coefficients of H which guarantee that the condition $V(H, f(x)) - V(H, x) < 0$ is satisfied for $x \in \mathcal{B}_D(0) - \{0\}$. These constrains are denoted as $S_2(H)$. Hence, if the decision problem:

$$(\exists H)(\ S_1(H) \wedge S_2(H)\ )$$

is TRUE, there exists a quadratic Lyapunov function for the system (1). Moreover, the conditions $S_1(H) \wedge S_2(H)$ on the coefficients H describe *all positive definite quadratic functions* from the given class $V(H, x)$, which are Lyapunov functions for the system (1).

Constructing a Lyapunov function in our approach means computing a polynomial positive definite function (or a set of functions) from a class of polynomial positive definite functions, which satisfies the conditions of the Lyapunov Theorem [9]. Notice that if we obtain that there is no Lyapunov function of the form $V(H, x)$, there still may exist a Lyapunov function which belongs to a perhaps larger class of positive definite functions $\hat{V}(H, x)$.

3.2 Verifying stability by definition (Procedure 2)

We show below that QEPCAD can be used to check the stability properties of systems (1) in a maybe surprising way, namely directly by verification of the definition. Our arguments are based on the following observations:

Observation 1 If the equilibrium of the system (1) is exponentially stable, that is there exist constants $K > 0$ and $\lambda \in]0, 1[$ such that $||x(k, x(0))|| < K||x(0)||\lambda^k$, then there exist $\Delta > 0$ and $p \in]0, 1[$ and a positive integer $N \in \mathbb{N}$ such that the following is true:

$$||x(N, x(0))|| < p||x(0)||,\ \forall x(0) \in \mathcal{B}_\Delta(0) - \{0\} \tag{4}$$

We note that the number N may be very large for poorly damped systems. This condition must necessarily be satisfied if the system (1) is to be exponentially stable. Moreover, it is easy to show that the condition is also sufficient for (exponential) attractivity of (1). Note, however, that it is not sufficient for stability. We can write:

$$||x(kN, x(0))|| < p^k||x(0)||, \forall x(0) \in \mathcal{B}_\Delta(0),$$

and hence $\lim_{k\to\infty} ||x(kN, x(0))|| = 0$.

The formula (4) can be cast into a decision problem that can be solved using QEPCAD. If we take N compositions of $f(x)$, the formula:

$$(\exists\Delta)(\exists p)(\forall x_1)(\forall x_2)\dots(\forall x_n)(0 < p < 1 \wedge 0 < ||x|| < \Delta \to ||f^N(x)|| < p||x||\ )$$

is a valid input formula to the QEPCAD algorithm. In other words, QEPCAD can produce an answer (TRUE or FALSE) for the above given decision problem and hence attractivity of the origin for the systems (1) can be tested in this manner[2]. If we want to check global attractivity, the test formula becomes

$$(\exists p)(\forall x)(0 < p < 1 \wedge ||x|| > 0 \to ||f^N(x)|| < p||x||).$$

[2]We use a shorter notation $\forall x$ to denote $\forall x_1 \forall x_2 \dots \forall x_n$.

Indeed, if the above condition was violated $\forall N \in \mathbb{N}$, the system is not globally (exponentially) attractive by definition.

Observation 2 Assume that the system (1) is globally exponentially attractive and that we have found the integer N for which condition (4) holds. It is then a simple consequence of attractivity that

$$||f^i(x)|| < \max_{j=0,1,\ldots,N}\{||f^j(x)||\}, \quad \forall i = N+1, \ldots$$

This implies that if the following condition is satisfied:

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } ||x|| < \delta \to ||f^j(x)|| < \epsilon,\ \forall j = 0, 1, 2, \ldots, N \tag{5}$$

then it is also true that

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } ||x|| < \delta \to ||f^j(x)|| < \epsilon,\ \forall j = N+1, \ldots \tag{6}$$

In other words, for attractive systems (1) if the number N is known, stability requires checking whether the conditions given by:

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } ||x|| < \delta \text{ implies } ||x|| < \epsilon, ||f(x)|| < \epsilon, \ldots, ||f^N(x)|| < \epsilon$$

are satisfied. This observation is crucial since we replace an infinite number of conditions $||x(k, x(0))|| < \epsilon, \forall k \in \mathbb{N}$ by a finite number of conditions $||x(k, x(0))|| < \epsilon, k = 0, 1, 2, \ldots, N$. Moreover, the above given conditions also qualify as an input formula to QEPCAD and hence can be checked using the algorithm.

Observations 1 and 2, together with the QEPCAD as a tool, provide us with *a constructive approach* to testing asymptotic stability of an equilibrium of a polynomial system (1) by definition. QEPCAD can be used in principle for this purpose for a very large class of polynomial dynamical systems (1). Notice also that a similar approach in continuous time framework is bound to be futile since we would need an analytic solution to a set of nonlinear differential equations. In discrete time we can find compositions of the polynomial map f more easily.

3.3 Computational complexity

The main drawback of the method based on QEPCAD is its computational complexity. Denote as F the input formula to the algorithm. The computation time of the CAD sub-algorithm [5] is over bounded by the following expression:

$$P_r(d, m, l) = (2d)^{2^{2r+8}} m^{2^{r+6}} l^3, \tag{7}$$

and of the QE sub-algorithm

$$P_r(d, m, l) = (2d)^{2^{2r+8}} m^{2^{r+6}} l^3 a, \tag{8}$$

where r is the number of variables in the input polynomials to the algorithm, m number of input polynomials F , d the maximum degree of any polynomial in F in any of the input variables, l the maximum norm length and a the number of atomic formulas in F. It is important to observe that for fixed m, a, l, d, the "computation time" is doubly exponential in the number of variables in the input polynomials F.

We note that the number of variables in the input polynomials influences the computation time the most. Consequently, it is practically impossible to tackle "large-scale" problems using this approach. It is very difficult to tackle more than four variables using the current versions of the algorithm. However, there are several variants of the algorithm and also other QE algorithms which can deal with certain classes of QE problems (input polynomials) more efficiently. For example, linear QE was discussed in [14] and the case when all quantified variables have at most degree 2 in [26]. In these cases, one can easily cope with 20 variables and hence this procedure becomes practically feasible.

Observation 3 We argue that when using QEPCAD, it may be in some cases computationally more efficient to use Procedure 2 than Procedure 1. Indeed, consider a second order polynomial system (1). Assume that $||f(x)|| < ||x||, \forall x$. Notice that this guarantees *asymptotic stability* using Procedure 2. The number of variables in the input polynomials is equal to the number of states, that is $r = 2$. If we want to check stability using Procedure 1 and if we decide to work with quadratic positive definite polynomials, we end up with 5 variables in the input polynomials (two states plus three coefficients in the quadratic function). Hence, deciding stability by definition in this case can be done in a much shorter time than if we try to construct a quadratic Lyapunov function.

Notice that a similar line of reasoning applies always: if we opt for Procedure 2, we always work with a fixed number of variables: n for attractivity (the number of states) and for stability $n+2$ (states plus δ and ϵ. If the number N for which $||f^N(x)|| < ||x||$ is large, the degrees and coefficients are very large (we use N compositions of $f(x)$) but the number of variables is at most $n+2$. On the other hand, if we use Procedure 1, even if we work with the simplest class of polynomial positive definite functions, such as quadratic, the number of variables is much larger than in the first case and is equal to the sum of the dimension of the system and the number of free coefficients H in the function class $V(H, x)$.

In summary, it seems that very often Procedure 2 is computationally more feasible than Procedure 1. Hence, in the sequel we concentrate only on different applications of Procedure 2 to stabilizability problems. The explicit bounds on computation time (7) and (8), together with Procedures 1 and 2, tell us something new about the problem of stability of a discrete-time nonlinear system. The example presented in the last section illustrates the capabilities and limitations of the current versions of the QEPCAD algorithm.

4 Stabilizability of controlled polynomial systems

In this section we show how Procedure 2 can be modified to address the problem of stabilizability of systems (3). If the system (3) can be made exponentially attractive by means of control, there exists a positive integer N, positive number $p \in]0,1[$ and control sequence U_N such that

$$||x(N, x(0), U_N)|| < p||x(0)||, \ \forall x \in \mathcal{B}_d(0) - \{0\}, \ 0 < p < 1$$

The above formula can be checked using QEPCAD. However, notice that we have N controls and n states and d as variables in the input polynomials. This is highly undesirable since the computations are not feasible for high N. It is straightforward to write the corresponding formulas for testing stability if the above attractivity condition is satisfied.

We present below, however, a recursive procedure which uses at each step $n+2$ variables and which can be used to test stabilizability in certain cases. The main assumption for the application of the method is given below:

Assumption 1 There exists a controller $u = u^*(x)$ and a proper subset of state space St_0 such that

1. The set St_0 is invariant for the system
$$x(k+1) = f(x(k), u^*(x(k))). \tag{9}$$
2. The system (9) is stable conditionally to St_0.

With Assumption 1, we introduce the following sets:

$$\begin{aligned} St_1 &= \{x : \exists u \in \mathbb{R}, f(x,u) \in St_0\} \\ St_2 &= \{x : \exists u \in \mathbb{R}, f(x,u) \in St_1\} \\ \dots & \quad \dots \\ St_k &= \{x : \exists u \in \mathbb{R}, f(x,u) \in St_{k-1}\} \end{aligned} \tag{10}$$

and denote the defining expressions for the sets as $St_k(x)$.

Suppose that a set St_N is a neighborhood of the origin, that is:

$$(\exists d)(\ (d > 0 \land ||x|| < d) \to St_N(x)\ ).$$

Then the origin of the system (3) can be made attractive by means of control u. Notice that one can easily construct a *feedback control law* which renders the system (3) exponentially attractive:

$$u(x) = \begin{array}{c} u^*(x) \\ \text{any real solution to} \end{array} \begin{cases} & , x \in St_0, \\ St_0(f(x,u)) & , x \in St_1 - St_0 \\ \dots & \dots \\ St_{N-1}(f(x,u)) & , x \in St_N - St_{N-1} \end{cases} \tag{11}$$

The above feedback yields actually a family of feedback laws that render the system (3) attractive with the property that $||x(N, x(0), U_N)|| < p||x(0)||$. The question arises whether such feedback laws may yield stability as well. This question can be again formulated as a decision problem:

$$(\forall x)(\exists u)(\forall \epsilon)(\exists \delta)[St_k(f(x,u)) \land St_{k+1}(x) \land (\epsilon > 0) \land (\delta > 0) \land ||x|| < \delta \to ||f(x,u)|| < \epsilon], k = 0, 1, \dots, N.$$

Notice that in this way we reduced the number of variables in input polynomials to $n + 3$. The above condition tests for the existence of a controller with a "small control property" which is sufficient to show stability. If it happens that the stronger condition holds:

$$(\forall x)(\forall u)(\forall \epsilon)(\exists \delta)[St_k(f(x,u)) \land St_{k+1}(x) \land (\epsilon > 0) \land (\delta > 0) \land ||x|| < \delta \to ||f(x,u)|| < \epsilon], k = 0, 1, \dots, N.$$

then *any* controller in the family (11) is stabilizing (this situation is illustrated in the example).

Assume that the above given test is satisfied with $St_N = \mathbb{R}^n$. Then the test guarantees that there exists a control law $u(x)$ (it may be discontinuous) which achieves finite time attractivity to the set St_0 and moreover which is such that $\forall \epsilon_k > 0, \exists \delta_k > 0$ such that $\forall x \in St_{k+1}, ||x|| < \delta_k$ we have that $f(x, u(x)) \in St_k, ||f(x, u(x))|| < \epsilon_k, k = 0, 1, \dots, N$. By using induction arguments, and letting $\epsilon_{k+1} = \delta_k, k = 0, 1, \dots, N-1$, it is not difficult to show stability of the closed loop system using Lyapunov stability definition.

To motivate Assumption 1, we single out several important cases:

1. $St_0 = \{0\}$, then the method given above would test if there are any stabilizing state dead-beat controllers (which are also time-optimal). Design of time-optimal dead-beat controllers was carried out in [19] but those controllers render the origin of the closed loop system globally attractive, whereas their stability must be tested using the above given formulas.

2. $St_0 \subseteq \{x : h(x) = 0\}$, then Assumption 1 asserts that the system is minimum phase and the method tests if there are any stabilizing output dead-beat controllers. In the next section we present a method for testing if a system is minimum phase or not (testing if Assumption 1 is satisfied).

3. St_0 is an arbitrary lower dimensional subset of the state space. The stabilization techniques for feed-forward and strict feedback systems presented in [22] can be interpreted as finding a "dummy" output $y = H(x)$ with respect to which the system is minimum phase and relative degree one, which imply passivity. Assumption 1 asserts that such a dummy output can be found for the system (3).

5 Stability of zero output constrained dynamics

In this section we address the problem of stability of zero output constrained dynamics of a class of input-output (IO) polynomial models. The problem can be regarded as a constrained stabilizability problem (see [18]), since the IO linearizing control law is normally not unique for this class of systems. We consider a class of polynomial IO models of the form [8]:

$$y(k+1) = F(y(k), y(k-1), \dots, y(k-s), u(k), u(k-1), \dots, u(k-t)) \tag{12}$$

where $y(k) \in \mathbb{R}$ and $u(k) \in \mathbb{R}$ are respectively output and input of the system at time instant k. F is a polynomial function in all its arguments. Notice that if we want to control the output of the system (12) to a desired value y^* and keep it for all future time steps at y^*, the system evolves according to:

$$F(y^*, y^*, \ldots, y^*, u(k), u(k-1), \ldots, u(k-t)) - y^* = 0 \tag{13}$$

The equation (13) defines the final regime in output dead beat control, which we investigate here. We assume that the system is output dead beat controllable and that after finitely many steps the system evolves according to (13). By denoting $F(y^*, \ldots, y^*, u(k), u(k-1), \ldots, u(k-t)) = G(u(k), u(k-1), \ldots, u(k-t))$, we need to consider the equation:

$$G(u(k), u(k-1), \ldots, u(k-t)) = 0 \tag{14}$$

We say that (14) defines implicit "zero" output constrained dynamics, or simply - implicit zero dynamics. Explicit zero dynamics take the form

$$u(k) = G(u(k-1), \ldots, u(k-t)) \tag{15}$$

If we introduce state variables $u(k-t) = x_1(k), u(k-t+1) = x_2(k), \ldots, u(k-1) = x_t(k)$, we obtain the linear system:

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ \ldots & \quad \ldots \\ x_t(k+1) &= u(k) \end{aligned} \tag{16}$$

which is defined on the real variety defined in $\mathbb{R}^t$ by

$$G(u(k), x_1(k), x_2(k), \ldots, x_t(k)) = 0 \tag{17}$$

Obviously, for any initial state $x(0)$ we can apply to the linear system (16) only controls $u = u(x(0))$ which are obtained as solutions of the equation $G(u, x(0)) = 0$. Notice that since $G(u, x)$ is a polynomial in u and x, for almost all x we will have finitely many roots u. We use the following assumption:

Assumption 2 $\forall x_1, \ldots, x_t \in \mathbb{R},\ \exists u$ *such that* $G(u, x_1, \ldots, x_t) = 0$.

The equilibria of the system (14) are found using $G(\zeta, \zeta, \ldots, \zeta) = 0$. We denote the equilibria as $\zeta = (\zeta\ \zeta\ \ldots \zeta)^T \in \mathbb{R}^t$.

Definition 4 A criterion of choice is a single valued function $c : \mathbb{R} \to \mathbb{R}$ (denoted also as $u_k = c(u(k-1), \ldots, u(k-t))$) such that

$$G(c(u(k-1), \ldots, u(k-t)), u(k-1), \ldots, u(k-t)) = 0, \quad \forall u(k-1), \ldots, u(k-t) \in \mathbb{R}. \tag{18}$$

Definition 5 Consider a criterion of choice applied to the system (16):

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ \ldots & \quad \ldots \\ x_t(k+1) &= c(x_t(k), \ldots, x_1(k)) \end{aligned} \tag{19}$$

We call the system (19) the "c"-resulting system.

We can now talk about stability of the origin of a "c"-resulting system in the sense of Definition 1. Hereafter, we assume that we are working around a known equilibrium ζ.

Below we propose definitions of minimum phase systems, which incorporate the criterion of choice. Motivation for these definition can be found in [18].

Definition 6 The system (12) is:

1. minimum phase if *there exists* a criterion of choice c such that the equilibrium ζ of the "c"-resulting system is asymptotically stable in the sense of Definition 1
2. uniformly minimum phase if *for any* criterion of choice c the equilibrium ζ of the "c"-resulting system is asymptotically stable in the sense of Definition 1
3. non minimum phase if *for any* criterion of choice c the equilibrium ζ of the "c"-resulting systems is unstable

It is immediately clear that if the IO linearizing control law is unique, the notions of minimum phase and uniform minimum phase systems coincide. This corresponds to the situation when the system has a bijective relative degree [23].

Comment 1 The above given definitions of minimum phase systems may be generalized in two directions. First, one may rephrase the definitions to include a possible non uniqueness of the equilibria ζ and define minimum phase system: "there exists an equilibrium ζ and a criterion of choice c which renders the equilibrium asymptotically stable". Another generalization was proposed in [1] where the stability of an invariant attractor, which is not necessarily a point, is incorporated in the definition of minimum phase systems. We note here that the methods that we propose in the next section to check different minimum phase properties of polynomial systems can be used (with appropriate modifications) to check these more general notions.

5.1 Deciding stability of zero dynamics using QEPCAD

The main results of this section are summarized in Theorems 2 and 3. The theorems give sufficient conditions for the existence of a criterion of choice c such that the "c"-resulting system is locally or globally asymptotically stable. For i-o polynomial systems we propose the use of QEPCAD symbolic computation package [5, 6, 7] to check the conditions of Theorems 2 and 3. Computational complexity of the problems may be prohibitive and this is the main hindrance to the proposed method. Nevertheless, for i-o systems of small total degrees of the defining polynomial map, the method may yield satisfactory results.

Fix a number $p \in]0,1[$. It is easily seen that we can compute the following sets using QEPCAD:

$$\begin{array}{lcl} S_1^{x_j} & = & \{x \in \mathbb{R}^t : \exists \zeta \in \mathbb{R}, |\zeta| < p|x_j|, G(\zeta, x_t, \ldots, x_1) = 0\} \\ S_2^{x_j} & = & \{x \in \mathbb{R}^t : \exists \zeta \in \mathbb{R}, |\zeta_1| < p|x_j|, G(\zeta_0, x_t, \ldots, x_1) = 0, G(\zeta_1, \zeta_0, x_t, \ldots, x_2) = 0\} \\ \ldots & & \ldots \end{array} \tag{20}$$

where $j = 1, 2, \ldots, t$. Hence, sets $S_k^{x_j}$ represent states in $\mathbb{R}^t$ for which there is a sequence of controls (criterion of choice) yielding $|x_t(k)| < p|x_j(0)|$. The above given sets can be used to check whether the system (12) is locally (globally) minimum phase.

Theorem 2 *The origin of the zero dynamics (14) is locally stable if the set $N = \cup_j S_1^{x_j}$ is a neighborhood of the origin.*

Proof of Theorem 2: Notice first that if the set N is the neighborhood of the origin this guarantees that zero dynamics are defined on this neighborhood since $\forall x \in \mathbb{R}^t, \exists u \in \mathbb{R}$ such that $G(u, x) = 0$ (see definition of sets $S_1^{x_j}$ in (20)).

Given a positive number $s > 0$, we define the hypercube

$$\mathcal{C}_s = \{x \in \mathbb{R}^t : |x_1| < s \wedge \ldots \wedge |x_t| < s\}$$

Notice that if the conditions of Theorem 2 are satisfied, there exists $s^* > 0$ such that $\mathcal{C}_{s^*} \subset N$. Then, there exists a criterion of choice c such that any hypercube $\mathcal{C}_s, s \in]0, s^*[$ satisfies that if $x(0) \in \mathcal{C}_s$ then $x(k) \in \mathcal{C}_s, \forall k = 1, 2, \ldots$. Indeed, if

$$|x_1(0)| < s \wedge \ldots \wedge |x_t(0)| < s$$

then we have from the structure of the system that

$$|x_1(1)| = |x_2(0)| < s \wedge \ldots |x_{t-2}(1)| = |x_t(0)| < s$$

Moreover, by definition of sets (20) we have that there exists a criterion of choice c such that

$$|x_t(1)| < p|x_j(0)| < s, j \in \{1, 2, \ldots, t\}$$

and hence we conclude that $x(1) \in \mathcal{C}_s$. Notice that this holds for arbitrary $x(0) \in \mathcal{C}_s$ and hence we have that $x(k) \in \mathcal{C}_s, \forall k$.

Consider now any hyper ball $\mathcal{B}_\varepsilon = \{x : \sum_{i=1}^t x_i^2 < \varepsilon\}$ and define $\delta = \delta(\varepsilon) = \min(\varepsilon/2, s^*/2)$. Then if $x(0) \in \mathcal{B}_\delta$ we have that $x(k) \in \mathcal{C}_\delta, \forall k$ since $\delta \in]0, s^*[$. Moreover, we have that $\mathcal{C}_\delta \subset \mathcal{B}_\varepsilon, \forall \varepsilon > 0$ and hence $x(k) \in \mathcal{B}_\varepsilon, \forall k$. Therefore there exists a criterion of choice c such that the "c"-resulting system is stable by definition. Q.E.D.

Notice that the criterion of choice is in general a discontinuous map. Moreover, the sets $S_1^{x_j}$ can be given in certain cases a nice interpretation based on Lyapunov functions. Indeed, assume that the set $S_1^{x_1}$ is a neighborhood of the origin. Assume that we consider the explicit zero dynamics (15) with the function G continuous and define the Lyapunov function:

$$V(x(k)) = \sum_{i=1}^{t} |x_i(k)|$$

which is positive definite. By considering the difference:

$$V(x(k+1)) - V(x(k)) = \sum_{i=2}^{t} |x_i(k)| + |G(x(k))| - \sum_{i=1}^{t} |x_i(k)|$$

we obtain $V(x(k+1)) - V(x(k)) = |G(x(k))| - |x_1(k)|$. By definition of the set $S_1^{x_1}$ we have that $|G(x(k))| < p|x_1(k)|$ on the set. Hence, we obtain

$$V(x(k+1)) - V(x(k)) < 0, \forall x \in S_1^{x_1}$$

and since $S_1^{x_1}$ is a neighborhood of the origin, the zero dynamics are stable.

This result can be generalized to implicitly defined zero dynamics (14) and even when the criterion of choice is a discontinuous mapping.

Theorem 3 *Suppose that $\exists j \in \{1, \ldots, t\}$ such that the set $S_1^{x_j}$ is a neighborhood of the origin. Then the origin of the zero dynamics is locally asymptotically stable.*

Proof of Theorem 3: Stability follows from Theorem 2. We show now that the system is also locally attractive. We know that there exists a number s^* such that any hypercube $\mathcal{C}_s, s \in]0, s^*[$ is invariant with respect to the solutions $x(k), \forall k$. Hence, we have that if $x(0) \in \mathcal{C}_s$

$$|x_t(k+t-j)| < p|x_j(k)|, \forall k$$

If $k = 0$ we have that $|x_t(t-j)| = p_0|x_j(0)|, p_0 \in [0, p[$. For $k = 1$ we have that $|x_t(1+t-j)| = p_1|x_j(1)| = p_1 p_0 |x_j(0)|, p_1 \in [0, p[$. In general we obtain that

$$|x_t(N+t-j)| = \prod_{i=0}^{N} p_i |x_j(0)|, p_i \in [0, p[$$

and by taking the limit we obtain $\lim_{N\to\infty} |x_t(N+t-j)| \to 0$. Since $x_l(k+1) = x_{l+1}(k), l = 1, \dots, t-2$ we conclude that $\lim_{N\to\infty} |x_l(N)| \to 0, \forall l = 1, \dots, t-1$. In other words we have that $\lim_{N\to\infty} ||x(N)|| \to 0$. We can therefore take $\Delta = s^*/2$ and the attractivity of the zero dynamics follows by definition. Q.E.D.

An obvious consequence of the above results is:

Corollary 1 *Suppose there exists $j \in \{1, 2, \dots, t\}$ such that $S_1^{x_j} = \mathbb{R}^t$. Then the zero dynamics (14) are globally stable.*

We have considered so far only how the sets $S_1^{x_j}$ can be used to decide on stability of zero dynamics. We show below that for a class of polynomial i-o systems we also may make use of the sets $S_k^{x_j}$ when dealing with this problem. The following assumptions defined the class of systems that we consider.

Assumption 3 $\forall x(0) \in \mathbb{R}^t, \forall \zeta \in \mathbb{R}$ *such that* $G(\zeta, x(0)) = 0$ *we have that* $|\zeta| < \infty$.

Assumption 4 Consider $G(x_1, x_2, x_3, \dots, x_{t+1}) = 0$, then

$$\forall x_3, \dots, x_{t+1} \in \mathbb{R}, \exists x_1 : \lim_{|x_2|\to 0} |x_1| \to 0$$

and in particular we have that

$$\forall x_3, \dots, x_{t+1} \in \mathbb{R}, G(0, 0, x_3, \dots, x_{t+1}) = 0$$

Assumption 3 guarantees that the domain of existence of zero dynamics for the set point $y^* = 0$ is the whole state space $\mathbb{R}^t$. Moreover, it is assumed that there are no finite escape times (for all bounded initial states, all allowable controls that satisfy the constraint (17) are bounded. The assumption is satisfied for all explicit zero dynamics (15) where the function G is a polynomial in all its variables or a rational function with the denominator not having zero values for all values of its arguments. In general, we can write the implicit zero dynamics (14) in the following form

$$G(x_1, \dots, x_t) = g_n(x_2, \dots, x_t) x_1^n + \dots + g_0(x_2, \dots, x_t) = 0.$$

Assumption 3 is satisfied if $g_n(x_2, \dots, x_t) \neq 0, \forall x_2, \dots, x_t \in \mathbb{R}$ since we have the bound on the roots [2]:

$$|x_1| < 1 + \sup_i |\frac{g_i(x_2, \dots, x_t)}{g_n(x_2, \dots, x_t)}|$$

which guarantees that $|x_1|$ is bounded on the whole state space. We use the notation $H = \{x \in \mathbb{R}^t : x_t = 0\}$.

Theorem 4 *Suppose that Assumptions 3 and 4 are satisfied for the implicit polynomial dynamics (14). The zero dynamics are globally attractive if there is an integer N such that $\cup_{i=1}^N S_i^{x_j} = \mathbb{R}^t - H$ for some $j \in \{1, \dots, t\}$. Moreover, $x(k), \forall k$ is bounded.*

Proof of Theorem 4: Suppose that conditions of Theorem 4 are satisfied. Consider any initial state $x(0) \in \mathbb{R}^t$. If $x(0) \in H$ then by simply applying $u(k) = 0, \forall k$ we have that $x(k) = 0, \forall k \geq t$. If $x(0) \in \mathbb{R}^t - H$, then we have that $x(0) \in S_{k_1}^{x_j}, k_1 \in \{1, \dots, N\}$. By definition of the set $S_{k_1}^{x_j}$ we have that

$$|x_t(k_1)| = p_{k_1} |x_j(0)|, p_{k_1} < p < 1$$

If $x(k_1) \in H$ we trivially have the attraction to the origin. Suppose that $x(k_1) \notin H$. Then, we have that $x(k_1) \in S_{k_2}^{x_j}, k_2 \in \{1, \ldots, N\}$ and by definition

$$|x_t(k_2)| = p_{k_2}|x_t(k_1)|, p_{k_2} \in [0, p[$$

Therefore, if we suppose that $x(k_i) \notin H, \forall i = 1, 2, \ldots$ we have that

$$|x_t(k_N)| = \prod_{i=1}^{N} p_{k_i}|x_j(0)|, p_{k_i} < p < 1, \forall i$$

and by taking the limit of the above expression we obtain that

$$\lim_{N\to\infty} |x_t(k_N)| \to 0$$

Because of the Assumption 4 and since $x_t(k_N) \to 0$ we have that $x_{t-j}(k_N+j) \to 0, j = 1, 2, \ldots, t-1$ and therefore $||x(k)|| \to 0$. The boundedness of $x(k), \forall k$ follows trivially from the boundedness of solutions (Assumption 3). Q.E.D.

We also have

Corollary 2 *If the conditions of Theorems 2 and 4 are satisfied, the implicit zero dynamics (14) are globally stable.*

Notice that Assumption 3 is not essential for the global attractivity result and is only used to guarantee that there are no finite escape times.

Comment 2 The computational complexity of the decision rules used to define the sets $S_k^{x_j}$ may be prohibitive and hence it is of utmost importance to investigate ways in which the complexity can be reduced. The required computations may be drastically reduced by first decomposing the polynomial G which defines the implicit zero dynamics (14)

$$G(\zeta_1, \zeta_2, \ldots, \zeta_t) = \prod_{i=1}^{M} f_i(\zeta_1, \zeta_2, \ldots, \zeta_t)$$

where f_i are all irreducible polynomials. Notice that $G = 0$ if $f_i = 0$ for some i and if any of the newly defined implicit zero dynamics

$$f_i(\zeta_1, \zeta_2, \ldots, \zeta_t) = 0$$

satisfies conditions of some of Theorems 2, 3 or 4, the zero dynamics (14) have at least the same properties as newly defined zero dynamics. The idea of factorizing an implicit system into several sub-systems can be found in [23] but we here presented tools for this factorization and tests of stability of zero dynamics for IO polynomial systems.

6 Example

We use the procedure described in Section 4 to illustrate on an example the capabilities and limitations of a current version of QEPCAD. Consider the polynomial system[3]:

$$\begin{aligned} x_1(k+1) &= x_1(k) + u(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= -x_2(k) - 2x_3(k) + u^2(k) \end{aligned} \tag{21}$$

[3]The example was solved by prof. G.E.Collins and C. Brown. The attractivity part was presented in [19, 20].

We Assume that $St_0 = \{0\}$, which means that we want to check if there are any stabilzing state dead-beat controllers. Using the procedure described in Section 4 we obtain:

$$\begin{array}{rcl} St_1 & = & \{x : x_2 - x_1^2 = 0 \ \wedge \ x_3 = 0\} \\ St_2 & = & \{x : 2x_3 + x_2 \geq 0 \ \wedge \ 2x_2x_3 + x_3^2 - 6x_1^2x_3 + x_2^2 - 2x_2x_1^2 + x_1^4 = 0\} \\ St_3 & = & \mathbb{R}^3 \end{array}$$

Using our notation, we have that $St_1(x) \equiv (x_2 - x_1^2 = 0) \wedge (x_3 = 0)$, $St_2(x) \equiv (2x_3 + x_2 \geq 0) \wedge (2x_2x_3 + x_3^2 - 6x_1^2x_3 + x_2^2 - 2x_2x_1^2 + x_1^4 = 0)$ and $St_3(x) \equiv (0 = 0)$. We also introduce $\hat{St}_i = St_i - St_{i-1}, i = 1, 2, 3$. The family of controllers that achieve finite time global attractivity of the origin is given below:

$$u(x) = \text{any real root } u \text{ to } \left\{ \begin{array}{ll} \quad 0 & \text{,if } x \in St_0 \\ (x_1 + u = 0) \wedge (-x_2 - 2x_3 + u^2 = 0) & \text{,if } x \in \hat{S}_1 \\ (x_3 - (x_1 + u)^2) = 0) \wedge (-x_2 - 2x_3 + u^2 = 0) & \text{,if } x \in \hat{S}_2 \\ (2(-x_2 - 2x_3 + u^2) + x_3 \geq 0) \wedge & \text{,if } x \in \hat{S}_3 \\ (2x_3(-x_2 - 2x_3 + u^2) + (-x_2 - 2x_3 + u^2)^2 & \\ -6(x_1 + u)^2(-x_2 - 2x_3 + u^2) + x_3^2 - & \\ 2x_3(x_1 + u)^2 + (x_1 + u)^4 = 0) & \end{array} \right. \tag{22}$$

The second step is to test, using the $\epsilon - \delta$ definition if there exists a controller in the family (22) which is also stabilizing. The answer to this question was not possible to solve using the version of QEPCAD that the authors had. However, we show below that we do not have to resort to QEPCAD in order to obtain affirmative answer to this question. However, stability of all controllers in the family (22) can be deduced.

The control u is obtained as a real solution to a set of polynomial equations if $x \in \hat{S}_1$ or $\hat{S}_2$. On the other hand, a polynomial equation and an inequality should be solved for $x \in \hat{S}_3$. We can first solve the equation and then check which solutions satisfy the inequality. Notice that any controller $u = u(x)$ in the family (22), must satisfy an equation of the form:

$$u^{m_k} + a^k_{m_k-1}(x)u^{m_k-1} + \ldots + a^k_1(x)u + a^k_0(x) = 0, x \in \hat{St}_k, k = 1, 2, 3$$

For any fixed $x \in \mathbb{R}^n$ we let $r(u) = u^{m_k}$ and $p(u) = \sum_{i=0}^{m_k-1} a^k_i u^i$. All roots of $r(u) = 0$ are equal to 0. If we regard the (analytic) function $p(u)$ as a perturbation to the original (analytic) function $r(u)$, the Ruche's Theorem [15] states that all roots to $r(u) + p(u) = 0$, denoted as u_i, are inside a ball $B_\epsilon = \{u : |u| < \epsilon\}$ if on the boundary of the ball $u = \epsilon$, we have that $p(\epsilon) < r(\epsilon)$. This can be used to show that $\forall \epsilon_k > 0, \exists \delta_k > 0$ such that if $x \in \hat{St}_k$ and $||x|| < \delta_k$ then $||u(x)|| < \epsilon_k$. Finally, using an induction argument, it can be shown that any controller in the family (22) renders the closed loop system globally stable.

Although we could not solve the stabilizability problem using the "brute force" approach of Section 4, QEPCAD was still able to provide us with enough information to design a family of stabilizers (22) for the system, which could not be obtained by any other method. QEPCAD is an interactive package [6] and its efficiency can be greatly improved if a problem is reformulated. Moreover, whenever a theoretical result which simplifies computations can be used, one should use it, as it was the case in the above example. In this sense, the procedures we presented can be regarded as a brute force approach since they rely entirely on the computational capabilities of QEPCAD. Combining QEPCAD with available theoretical results that exist in the literature can enhance our ability to design new classes of stabilizing controllers for the class of polynomial systems. To cover all possibilities, however, appears to be impossible in this note and the example is supposed to illustrate one such situation.

7 Conclusion

In this paper, we have presented a novel approach to testing stability and stabilizability for a class of polynomial discrete-time systems. The symbolic computation approach based on QEPCAD offers several possibilities to tackle the stability/stabilizability problems. We presented solutions to stability, stabilizability and constrained stabilizability problems for polynomial systems. The constrained stabilizability problem arises in the context of stability of zero output constrained dynamics. Constraints on states and controls and MIMO systems can be tackled in a straightforward way. Computational complexity of the problem is very large but non-trivial problems can be tackled in this way. We emphasize that utilizing the structure of a polynomial system and perhaps some other tools, we may be able to reduce the computational complexity.

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