A note on input-to-state stability and averaging of systems with inputs

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Abstract

Two different definitions of an average for time-varying systems with inputs and a small parameter that were recently introduced in the literature are considered: "strong" and "weak" averages. It is shown that if the strong average is input-to-state stable (ISS), then the solutions of the actual system satisfy an integral bound in a semi-global practical sense. The integral bound that we prove can be viewed as a generalization of the notion of finite-gain L_2 stability, that was recently introduced in the literature. A similar result is proved for weak averages but the class of inputs for which the integral bound holds is smaller (Lipschitz inputs) than in the case of strong averages (measurable inputs).

1 Introduction

Averaging is an important tool used in the analysis of time-varying systems. An auxiliary time-invariant dynamical system $\dot{x} = f_{av}(x)$, called the average, is used to investigate properties of a time-varying dynamical system $\dot{x} = f\left(\frac{t}{\epsilon}, x\right)$ that depends on a small parameter ϵ. Averaging has been instrumental in solving a wide range of important control problems, such as vibrational control or adaptive control (for classical results on averaging see [1, 4, 9] and for some more recent results see [15] and references therein). We emphasize that classical averaging results apply only to input-free systems although systems with inputs are prevalent in control theory. In this paper we concentrate on averaging of systems with inputs.

Among many different stability notions for analysis of properties of systems with inputs, L_∞ and L_2 stability play a central role for their practical importance and intuitive appeal: the former captures the notion of "bounded inputs imply bounded outputs" whereas the latter guarantees that "bounded energy inputs imply bounded energy outputs". A recently introduced notion of input-to-state stability (ISS) [10] provides a particularly useful framework for analysis of L_∞ stability of nonlinear systems that is fully compatible with Lyapunov theory [5, 10, 11]. ISS was originally defined for systems of the form

$$\dot{x} = f(x, w) \ , \tag{1}$$

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where f is locally Lipschitz, using the L_∞ framework (see [10]). A necessary condition for a system to be ISS is that the origin of the system is globally asymptotically stable in the absence of inputs. Hence, systems that exhibit limit cycles, multiple equilibria or chaotic attractors in the absence of inputs cannot be ISS. One way to overcome this is to consider ISS with respect to compact sets [13]. Another possibility is to consider the so called input-to-state practical stability (ISpS) first considered in [2] and which is shown to be equivalent to set-ISS in [13]. ISpS is defined in the following way [2]:

Property I_1: There exists [1] $\gamma \in \mathcal{K}, \beta \in \mathcal{KL}$ and $\lambda > 0$ such that the solutions $x(t)$ of the system (1) satisfy $|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|w\|_\infty) + \lambda$, $\forall t \geq 0$. □

It was proved in [13][Section VI] that ISpS is equivalent to the following property:

Property I_2: There exist $\alpha_1, \alpha_2, \alpha_3, \gamma \in \mathcal{K}_\infty$, $\lambda > 0$ and a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2}$$

$$|x| \geq \gamma(|w|) \;\Rightarrow\; \frac{\partial V}{\partial x} f(x, w) \leq -\alpha_3(|x|) + \lambda, \qquad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{3}$$

(equivalently there exist $\alpha_1, \alpha_2, \tilde{\alpha}_3, \tilde{\gamma} \in \mathcal{K}_\infty$, $\lambda > 0$ and a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that (2) holds and $\frac{\partial V}{\partial x} f(x, w) \leq -\tilde{\alpha}_3(|x|) + \tilde{\gamma}(|w|) + \lambda$, $\forall x \in \mathbb{R}^n, w \in \mathbb{R}^m$). □

Recently, yet another important integral characterization of ISS was proved in [14] (see Theorem 1). Although ISpS was not considered in [14], it can be shown that an appropriate integral version of ISpS for forward complete systems (1) is given by the following property:

Property I_3: There exist $\alpha, \kappa, \gamma \in \mathcal{K}_\infty$ and $\lambda > 0$ such that the solutions of the system (1) exists and satisfy: $\int_0^t \alpha(|x(s)|)ds \leq \kappa(|x(0)|) + \int_0^t \gamma(|w(s)|)ds + \lambda t$, $\forall t \geq 0$. □

Note that if the state is regarded as the output of the system, then Property I_1 represents a generalization of finite-gain L_∞ stability, Property I_3 represents a generalization of finite-gain L_2 stability and Property I_2 provides a tool for simultaneously verifying both of these important properties.

In this paper we investigate properties of time-varying systems with inputs via averaging. The systems we consider are of the form $\dot{x} = f\left(\frac{t}{\epsilon}, x, w\right)$, where ϵ is a small parameter. We make use of "weak" and "strong" averages recently introduced in [8] for dealing with systems with inputs. In [8] it was shown under appropriate conditions on inputs that if Property I_2 holds with $\lambda = 0$ for the weak or strong average, then Property I_1 holds for the actual time-varying system in a semiglobal practical sense (ϵ is the parameter that we need to adjust). This showed that ISS Lyapunov techniques for the time invariant strong and weak averages of [8] provide a set of tools for the analysis of "L_∞-stability" (more precisely ISpS) of time-varying systems with inputs. A related result proved in [16] shows that under appropriate conditions on inputs, solutions of the weak or strong average can be made arbitrarily close to the

[1] A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ ($\gamma \in \mathcal{G}$) if it is continuous, zero at zero and nondecreasing. It is of class-$\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$.

solutions of the actual system on bounded time intervals if the parameter ϵ is sufficiently small (the result is derived without assuming that Property I_2 holds for the strong or weak average).

We emphasize that Properties I_1, I_2 and I_3 can be shown to be equivalent for forward complete time-invariant systems (1). Some results in this direction for general time-varying systems $\dot{x} = f(t, x, w)$ were shown in [6]. The implication "I_2 for weak or strong average $\Longrightarrow I_1$ for the actual time-varying system" was proved in [8] for a class of time-varying systems for which averages (strong or weak) exist. It is the purpose of this paper to show that if Property I_2 holds for the strong or weak average, then Property I_3 also holds for the actual time-varying system in a semiglobal practical sense and under appropriate conditions on inputs. Hence, the results of this paper show that the combination of strong and weak averages of [8] with ISS Lyapunov functions provides also a tool for "L_2-stability" analysis of time-varying systems. (For related results see also [7, 17].)

The paper is organized as follows. In Section 2 we present definitions and the main assumption. The main results are presented in Section 3 along with some related comments. All proofs are presented in Section 4.

2 Preliminaries

Given a measurable function w, we define its infinity norm $\|w\|_\infty := \operatorname{ess\,sup}_{t\geq 0} |w(t)|$. If we have $\|w\|_\infty < \infty$, then we write $w \in L_\infty$. If $w(\cdot)$ is Lipschitz, its derivative is defined almost everywhere and we can write $w(t) - w(t_\circ) = \int_{t_\circ}^{t} \dot{w}(\tau) d\tau$. Consider the time-varying system:

$$\dot{x} = f(t, x, w) \tag{4}$$

where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ is the input. We will use the following:

Assumption 1 *f is locally Lipschitz in x, w, uniformly in t, and there exists $c \geq 0$ such that $|f(t, 0, 0)| \leq c, \forall t \geq 0$.* □

Hence, we are guaranteed the existence of solutions and can use certain results on continuity of solutions with respect to initial conditions. The solution of the system (4) at time t, starting from an initial condition $x_\circ$ at initial time $t_\circ$ and under the action of input $w_{[t_\circ, t]}$ is denoted as $x(t)$ (since $t_\circ, x_\circ, w_{[t_\circ, t]}$ are usually clear from the context). We also investigate the time-varying system that depends on a small parameter $\epsilon > 0$:

$$\dot{x} = f\left(\frac{t}{\epsilon}, x, w\right). \tag{5}$$

We recall the definition of strong and weak averages for time-varying systems with inputs [8]:

Definition 1 (strong average) *A locally Lipschitz function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the strong average of f if there exist$\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that $\forall t \geq 0$, $\forall T \geq T^*$ the following holds:*

$$\left| \frac{1}{T} \int_t^{t+T} [f_{sa}(x, w(s)) - f(s, x, w(s))] \, ds \right| \leq \beta_{av}(\max\{|x|, \|w\|_\infty, 1\}, T), \ \forall x \in \mathbb{R}^n, w \in L_\infty. \tag{6}$$

The strong average of system (4) is then defined as $\dot{x} = f_{sa}(x, w)$. □

Definition 2 (weak average) *A locally Lipschitz function* $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is said to be the weak average of* f *if there exist* $\beta_{av} \in \mathcal{KL}$ *and* $T^* > 0$ *such that* $\forall T \geq T^*, \forall t \geq 0$ *we have*[2]

$$\left| f_{wa}(x, w) - \frac{1}{T} \int_t^{t+T} f(s, x, w) ds \right| \leq \beta_{av}(\max\{|x|, |w|, 1\}, T), \ \forall x \in \mathbb{R}^n, w \in \mathbb{R}^m.$$

The weak average of system (4) is then defined as $\dot{x} = f_{wa}(x, w)$. □

It was shown in [8, 16] that weak averages exist for a strictly larger class of systems than strong averages, but using strong averages we can prove stronger results for the actual time-varying system. However, we emphasize that both definitions have been found to be useful in different situations (for more details see [8, 16]).

3 Main results

In this section we state in Theorems 1 and 2 the main results of the paper.

Theorem 1 *Let Assumption 1 hold and suppose* $f(t, x, w)$ *has the strong average* $f_{sa}(x, w)$. *If there exists a differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *whose gradient* $\frac{\partial V}{\partial x}$ *is locally Lipschitz with* $\frac{\partial V}{\partial x}(0) = 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, *with* α_3 *locally Lipschitz*[3], *and* $\gamma \in \mathcal{G}$ *such that, for all* (x, w):

$$\begin{array}{rcl} \alpha_1(|x|) \leq & V(x) & \leq \alpha_2(|x|) \\ \dfrac{\partial V}{\partial x} f_{sa}(x, w) & \leq & -\alpha_3(|x|) + \gamma(|w|), \end{array} \tag{7}$$

then given any strictly positive real numbers Ω_x, Ω_w, ν, *there exists* $\epsilon^* > 0$ *such that for all* $\epsilon \in (0, \epsilon^*)$ *and for all* $t \geq t_\circ \geq 0$ *the solutions of (5) exist and satisfy:*

$$\int_{t_0}^t \alpha_3(|x(s)|) ds \leq \alpha_2(|x(t_\circ)|) + \int_{t_0}^t \gamma(|w(s)|) ds + \nu(t - t_\circ), \tag{8}$$

whenever $|x(t_\circ)| \leq \Omega_x, \|w\|_\infty \leq \Omega_w$. □

Theorem 2 *Let Assumption 1 hold and suppose* $f(t, x, w)$ *has the weak average* $f_{wa}(x, w)$. *If there exists a differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *whose gradient* $\frac{\partial V}{\partial x}$ *is locally Lipschitz with* $\frac{\partial V}{\partial x}(0) = 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, *with* α_3 *locally Lipschitz, and* $\gamma \in \mathcal{G}$ *such that for all* (x, w):

$$\begin{array}{rcl} \alpha_1(|x|) \leq & V(x) & \leq \alpha_2(|x|) \\ \dfrac{\partial V}{\partial x} f_{wa}(x, w) & \leq & -\alpha_3(|x|) + \gamma(|w|), \end{array} \tag{9}$$

and if only Lipschitz inputs w *are acting on the system (5), then given any strictly positive real numbers* $\Omega_x, \Omega_w, \Omega_{\dot{w}}, \nu$, *there exists* $\epsilon^* > 0$ *such that for all* $\epsilon \in (0, \epsilon^*)$ *and* $t \geq t_\circ \geq 0$ *the solutions of (5) exist and satisfy (8) whenever* $|x(t_\circ)| \leq \Omega_x, \|w\|_\infty \leq \Omega_w, \|\dot{w}\|_\infty \leq \Omega_{\dot{w}}$. □

[2] Note that w in the integral is a constant vector.

[3] Without loss of generality we may assume that α_3 is locally Lipschitz since if it is not, we can always find a locally Lipschitz function $\tilde{\alpha}_3$ so that $-\alpha_3(s) \leq -\tilde{\alpha}_3(s), \forall s \geq 0$ (see footnote on pg. 139 in [4]) and replace α_3 in (7) with $\tilde{\alpha}_3$. Moreover, the proofs can be carried out with some additional work if we use only the continuity of α_3 instead of the local Lipschitz assumption.

Remark 1 *The condition on derivatives of inputs in Theorem 2 cannot be relaxed, as the following example shows (the example is taken from [8]). The system* $\dot{x} = -cx^3 + \cos\left(\frac{t}{\epsilon}\right) x^3 w$*, where* $c \in (0, 0.5)$*, has a weak average* $\dot{x} = -cx^3$ *which is ISS. However, with the input* $w_\epsilon := \cos\left(\frac{t}{\epsilon}\right)$ *the actual system exhibits finite escape times from* $x_\circ = 1$ *and for any* $t_\circ \geq 0, \epsilon > 0$ *and hence does not satisfy (8) in a semiglobal practical sense.* □

Remark 2 *The bound (8) that we prove in Theorems 1 and 2 contains the offset term* $\nu(t - t_\circ)$ *which indicates that system (5) may contain a "finite power source" such as a limit cycle. The "finite power gain property" (with linear gains and in the input-output setting) was introduced and analyzed in [3] and the bound (8) can be viewed as its nonlinear generalizations for the case when the whole state is the output of the system. Moreover, similar results can be proved when the weak or strong average systems are ISpS instead of ISS.* □

4 Proofs

We start with a set of "continuity of solutions" results whose proofs are standard and are omitted (see [4, Section 2.5]).

Lemma 1 *Under Assumption 1, given any pair of strictly positive real numbers* (r, r_1)*, there exists* $d > 0$ *and* $M > 0$ *such that, for each* $\epsilon > 0$ *and for each* $t_\circ \geq 0$*, if* $|x(t_\circ)| \leq r$, $\|w\|_\infty \leq r_1$ *then the following property holds:*

Property A: *for all* $t \in [t_\circ, t_\circ + d]$ *the solution* $x(t)$ *of (5) exists and satisfies* $|x(t) - x(t_\circ)| \leq M(t - t_\circ)$*.* □

For notational convenience, we state an obvious corollary:

Corollary 1 *Under Assumption 1, given any continuous* V *and any quadruple of strictly positive real numbers* (r, r_1, μ_0, μ_1)*, there exists* $d > 0$ *such that for each* $\epsilon > 0$ *and for each* $t_\circ \geq 0$*, if* $|x(t_\circ)| \leq r$, $\|w\|_\infty \leq r_1$ *then the following properties hold:*

Property B: *for all* $t \in [t_\circ, t_\circ + d]$ *the solution* $x(t)$ *of (5) exists and satisfies* $|x(t)| \leq |x(t_\circ)| + \mu_0$*.*

Property C: *for all* $t \in [t_\circ, t_\circ + d]$ *the solution* $x(t)$ *of (5) exists and satisfies* $V(x(t)) \leq V(x(t_\circ)) + \mu_1$*.* □

Lemma 2 *Under the assumptions of Theorem 1, given any quadruple of strictly positive real numbers* $(\Delta, \Delta_1, \mu_1, \mu_2)$ *and* $\mu_0 = 1$*, there exists* $d^* > 0$ *such that for any fixed* $d \in (0, d^*)$*, there exists* $\epsilon^* > 0$ *such that for all* $\epsilon \in (0, \epsilon^*)$*,* $t_\circ \geq 0$*,* $|x(t_\circ)| \leq \Delta$*,* $\|w\|_\infty \leq \Delta_1$*,*

(i) Property A holds with some $M > 0$*;*

(ii) Property B holds for $\mu_0 = 1$*;*

(iii) Property C holds for μ_1*;*

(iv) the following property holds:

Property L: *for all* $t \in [t_\circ, t_\circ + d]$ *the solution* $x(t)$ *of (5) exists and satisfies*

$$\frac{V(x(t_\circ + d)) - V(x(t_\circ))}{d} \leq -\alpha_3(|x(t_\circ)|) + \frac{1}{d}\int_{t_\circ}^{t_\circ + d} \gamma(|w(s)|)ds + \mu_2. \tag{10}$$

$\square$

Proof of Lemma 2: Let $(\Delta, \Delta_1, \mu_1, \mu_2)$ be a given quadruple of strictly positive real numbers and let $\mu_0 = 1$. Let $L > 0$ be a (uniform) Lipschitz constant for $\frac{\partial V}{\partial x}, f(\frac{t}{\epsilon}, x, w), f_{sa}(x, w)$ over the set where $|x| \leq \Delta + 1 =: \tilde{\Delta}, |w| \leq \Delta_1$. Define $r := \Delta$ and $r_1 := \Delta_1$ and $K := 2L\tilde{\Delta} + L\Delta_1 + c$. With the pair (r, r_1) and the quadruple (r, r_1, μ_0, μ_1), apply Lemma 1 and Corollary 1 to generate M and $d_1^* > 0$ such that Properties A, B and C hold for any $d \in (0, d_1^*)$. Let $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ be such that (6) holds for all $T \geq T^*$. Let $T \geq T^*$ be such that

$$L\Delta\beta_{av}(\max\{\Delta, \Delta_1, 1\}, T) \leq \frac{\mu_2}{2} \ . \tag{11}$$

Throughout the rest of the proof we assume that $|x(t_\circ)| \leq \Delta$ and $\|w\|_\infty \leq \Delta_1$. Let $d_2^* := \frac{\mu_2}{KLM}$ and define $d^* := \min\{d_1^*, d_2^*\}$. Fix $d \in (0, d^*)$ and define $\epsilon^* := \frac{d}{T}$. Let $\epsilon \in (0, \epsilon^*)$. Given any $t_\circ \geq 0$ we can write that for all $t \in [t_\circ, t_\circ + d]$:

$$\begin{aligned} \frac{\partial V}{\partial x}(x(t)) f\left(\frac{t}{\epsilon}, x(t), w(t)\right) &= \frac{\partial V}{\partial x}(x(t_\circ)) f_{sa}(x(t_\circ), w(t)) \\ &\quad - \frac{\partial V}{\partial x}(x(t_\circ)) f_{sa}(x(t_\circ), w(t)) + \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{t}{\epsilon}, x(t_\circ), w(t)\right) \\ &\quad + \frac{\partial V}{\partial x}(x(t)) f\left(\frac{t}{\epsilon}, x(t), w(t)\right) - \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{t}{\epsilon}, x(t_\circ), w(t)\right) \ . \end{aligned} \tag{12}$$

Integrate both sides of the inequality (12) along the solution $x(t)$ over the interval $[t_\circ, t_\circ + d]$ and divide by d to obtain

$$\begin{aligned} \frac{V(x(t_\circ + d)) - V(x(t_\circ))}{d} &\leq \underbrace{\frac{1}{d}\int_{t_\circ}^{t_0+d} \frac{\partial V}{\partial x}(x(t_\circ)) f_{sa}(x(t_\circ), w(t)) ds}_{1} \\ &\quad + \underbrace{\frac{1}{d}\int_{t_\circ}^{t_0+d} \left| \frac{\partial V}{\partial x}(x(t_\circ)) f_{sa}(x(t_\circ), w(s)) - \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{s}{\epsilon}, x(t_\circ), w(s)\right) \right| ds}_{2} \\ &\quad + \underbrace{\frac{1}{d}\int_{t_\circ}^{t_0+d} \left| \frac{\partial V}{\partial x}(x(s)) f\left(\frac{s}{\epsilon}, x(s), w(s)\right) - \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{s}{\epsilon}, x(t_\circ), w(s)\right) \right| ds}_{3} \ . \end{aligned} \tag{13}$$

Now we turn to bounding each of the terms on the right-hand side of (13).

1 : From ISS of the strong average we can write:

$$\begin{aligned} \frac{1}{d}\int_{t_\circ}^{t_0+d} \frac{\partial V}{\partial x}(x(t_\circ)) f_{sa}(x(t_\circ), w(t)) ds &\leq \frac{1}{d}\int_{t_\circ}^{t_\circ+d} \left[-\alpha_3(|x(t_\circ)|) + \gamma(|w(s)|)\right] ds \\ &= -\alpha_3(|x(t_\circ)|) + \frac{1}{d}\int_{t_\circ}^{t_\circ+d} \gamma(|w(s)|) ds \ . \end{aligned} \tag{14}$$

2 : Since $\left|\frac{\partial V}{\partial x}(x(t_\circ))\right| \leq L|x(t_\circ)| \leq L\Delta$ and $d = \epsilon^* T$, Term **2** can be bounded above by:

$$\frac{\epsilon L\Delta}{\epsilon^* T} \left| \int_{t_\circ}^{t_0+T\epsilon^*} \left(f_{sa}(x(t_\circ), w(s)) - f\left(\frac{s}{\epsilon}, x(t_\circ), w(s)\right) \right) d\left(\frac{s}{\epsilon}\right) \right|$$

Introduce the change of variables $\tau = s/\epsilon$ in the above integral and introduce $w_1(\tau) := w(\epsilon\tau)$ (note that $\|w_1\|_\infty = \|w\|_\infty \le \Delta_1$) and $T_1 := \dfrac{\epsilon^* T}{\epsilon} > T$. Then by the definition of strong average we have:

$$\begin{aligned} L\Delta \left| \frac{1}{T_1} \int_{\frac{t_\circ}{\epsilon}}^{\frac{t_0}{\epsilon}+T_1} \left(f_{sa}(x(t_\circ), w_1(\tau)) - f(\tau, x(t_\circ), w_1(\tau)) \right) d\tau \right| &\le L\Delta\beta_{av}(\max\{|x(t_\circ)|, \|w_1\|_\infty, 1\}, T_1) \\ &\le L\Delta\beta_{av}(\max\{\Delta, \Delta_1, 1\}, T) \le \frac{\mu_2}{2}, \end{aligned} \tag{15}$$

which follows from the fact that $\epsilon T_1 = d$ and (11).

3: Using Assumption 1 and the definition for $L > 0$, for all x, w with $\max\{|x_1|, |x_2|\} \le \tilde{\Delta}$, $|w| \le \Delta_1$ we have[4]:

$$\left| \frac{\partial V}{\partial x}(x_1) f\left(\frac{s}{\epsilon}, x_1, w\right) - \frac{\partial V}{\partial x}(x_2) f\left(\frac{s}{\epsilon}, x_2, w\right) \right| \le (2L\tilde{\Delta} + L\Delta_1 + c)L\,|x_1 - x_2| = KL\,|x_1 - x_2| \ . \tag{16}$$

Using Properties A and B it follows that we can over bound the term **3** by $\frac{KLMd}{2}$. Finally, from our choice of d^* (in particular the choice of d_2^*), we can bound Term **3** by $\frac{\mu_2}{2}$. From the bounds on terms **1-3** on the right-hand side of (13), it follows that (10) holds, which completes the proof. $\square$

Lemma 3 *Under the assumptions of Theorem 2, given any* 5*-tuple of strictly positive real numbers* $(\Delta, \Delta_1, \Delta_2, \mu_1, \mu_2)$ *and* $\mu_0 = 1$*, there exists* $d^* > 0$ *such that for any fixed* $d \in (0, d^*)$*, there exists* $\epsilon^* > 0$ *such that for all* $\epsilon \in (0, \epsilon^*)$*,* $t_\circ \ge 0$*,* $|x(t_\circ)| \le \Delta$*,* $\|w\|_\infty \le \Delta_1$*,* $\|\dot{w}\|_\infty \le \Delta_2$*,*

(i) Property A holds with some $M > 0$*;*

(ii) Property B holds for $\mu_0 = 1$*;*

(iii) Property C holds for μ_1*;*

(iv) Property L holds for μ_2*.* $\square$

Sketch of proof of Lemma 3: The proof of Lemma 3 is very similar to the proof of Lemma 2 and we only point out the differences. The main difference comes form the fact that instead of (13) we use the following inequality:

$$\begin{aligned} \frac{V(x(t_\circ + d)) - V(x(t_\circ))}{d} \le\ & \underbrace{\frac{1}{d}\int_{t_\circ}^{t_0+d} \frac{\partial V}{\partial x}(x(t_\circ)) f_{wa}(x(t_\circ), w(t)) ds}_{1} \\ & + \underbrace{\frac{1}{d}\int_{t_\circ}^{t_0+d} \left| \frac{\partial V}{\partial x}(x(t_\circ)) f_{wa}(x(t_\circ), w(s)) - \frac{\partial V}{\partial x}(x(t_\circ)) f_{wa}(x(t_\circ), w(t_\circ)) \right| ds}_{2} \\ & + \underbrace{\frac{1}{d}\int_{t_\circ}^{t_0+d} \left| \frac{\partial V}{\partial x}(x(t_\circ)) f_{wa}(x(t_\circ), w(t_\circ)) - \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{s}{\epsilon}, x(t_\circ), w(t_\circ)\right) \right| ds}_{3} \\ & + \underbrace{\frac{1}{d}\int_{t_\circ}^{t_0+d} \left| \frac{\partial V}{\partial x}(x(s)) f\left(\frac{s}{\epsilon}, x(s), w(s)\right) - \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{s}{\epsilon}, x(t_\circ), w(t_\circ)\right) \right| ds}_{4} \ . \end{aligned} \tag{17}$$

[4]The Lipschitz assumption on $\frac{\partial V}{\partial x}$ may be relaxed to continuity with some additional work.

Term 1 is bounded in the same way as in proof of Lemma 2 and since $|w(s) - w(t_\circ)| \le \Omega_{\dot w} |s - t_\circ|$, we can bound Term 2 by $\frac{1}{d}\int_{t_\circ}^{t_\circ+d} L^2 \Delta |w(s) - w(t_\circ)| \, ds \le L^2 \Delta \Omega_{\dot w} d/2$. Term 3 is bounded in a similar way as Term 2 in proof of Lemma 2 but we use the definition of the weak average. Term 4 is bounded using the fact that:

$$\left| \frac{\partial V}{\partial x}(x_1) f\left(\frac{s}{\epsilon}, x_1, w_1\right) - \frac{\partial V}{\partial x}(x_2) f\left(\frac{s}{\epsilon}, x_2, w_2\right) \right| \le KL |x_1 - x_2| + L^2 \tilde\Delta |w_1 - w_2| ,$$

where $K, L, \tilde\Delta$ are defined in proof of Lemma 2 and then using calculations similar to bounding the above given Term 2 and Term 3 from Lemma 2. The proof is then completed in a similar way as proof of Lemma 2 (with appropriate modifications). □

Remark 3 *Let d^* be as in Lemma 2 (respectively Lemma 3). Then we can prove that given any strictly positive numbers δ_1 and δ_2 such that $\delta_1 < \delta_2 \le d^*$ there exists an $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ Property L holds uniformly for any $d \in [\delta_1, \delta_2]$. This is immediate from the proof of Lemma 2 (Lemma 3) if T is defined in the same way as in Lemma 2 (respectively Lemma 3) and we take $\epsilon^* := \frac{\delta_1}{T}$.*

First, we use a continuity of solutions argument in Lemma 4 to show that the required integral bound on solutions of the actual system holds on small time intervals. Then the proof is extended to arbitrary large time intervals.

Lemma 4 *If the strong average (respectively weak average) exists and (7) (respectively (9)) holds, then given any triple of strictly positive numbers (r, r_1, ν), there exists $d^* > 0$ such that for any $d \in (0, d^*]$, $\epsilon > 0$ and any $t_\circ \ge 0$, if $|x(t_\circ)| \le r$, $\|w\|_\infty \le r_1$ then the following property holds:*

Property D: *for all $t \in [t_\circ, t_\circ + d]$ the solution of (5) exists and satisfies (8).* □

Proof of Lemma 4: In this Lemma, f_a denotes either strong or weak average (the proof relies on continuity of f_a). Fix any $t_\circ \ge 0$. Let $x(\cdot)$ and $y(\cdot)$ denote respectively the solutions of the time varying system (5) and the corresponding average given respectively by $x(t) = x(t_\circ) + \int_{t_\circ}^t f\left(\frac{s}{\epsilon}, x(s), w(s)\right) ds$ and $y(t) = y(t_\circ) + \int_{t_\circ}^t f_a(y(s), w(s)) ds$. Let d_1^* denote the maximum sampling interval allowable for Properties A, B, and C to hold with $\mu_0 = 1$. Property B and (7) imply that for any $d \in (0, d_1^*]$, the following inequalities hold for any $t \in [t_\circ, t_\circ + d]$, $\|w\|_\infty \le r_1$, $|x(t_\circ)| \le r$ where $x(t_\circ) = y(t_\circ)$: $|x(t)| \le r + 1$, $|y(t)| \le \beta(r, 0) + \tilde\gamma(r_1)$, where $\beta \in \mathcal{KL}$ and $\tilde\gamma := \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(p\gamma)$ with $p > 1$. Define L to be the maximum of the Lipschitz constants of α_3 and f on the set $|x| \le \Delta_1 := \max\{r + 1, \beta(r, 0) + \tilde\gamma(r_1)\}$. Define $d^* := \min\left(d_1^*, \frac{\nu}{LB}\right)$, where $B := \max_{|x| \le \Delta_1, |w| \le r_1, t \ge 0} \max\{|f(t, x, w)|, |f_a(x, w)|\}$ (note that $B < \infty$ by Assumption 1 and Definition 1). Let $d \in (0, d^*]$. Then, integrating inequality (7) or (9) along the solution of the average $\dot y = f_a(y, w)$ initialized at $y(t_\circ) = x(t_\circ)$ and adding $\int_{t_\circ}^t \alpha_3(|x(s)|) ds$ to both sides yields

$$\begin{aligned} \int_{t_\circ}^t \alpha_3(|x(s)|) ds &\le \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^t \gamma(|w(s)|) ds + \int_{t_\circ}^t [\alpha_3(|x(s)|) - \alpha_3(|y(s)|)] ds \\ &\le \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^t \gamma(|w(s)|) ds + L \int_{t_\circ}^t |x(s) - y(s)| ds \, . \end{aligned} \quad (18)$$

Using the definition of B and $x(t_\circ) = y(t_\circ)$, we can write

$$\begin{aligned} |x(s) - y(s)| &= \left| \int_{t_\circ}^{s} [f\left(\frac{\sigma}{\epsilon}, x(\sigma), w(\sigma)\right) - f_a(y(\sigma), w(\sigma))] d\sigma \right| \leq \int_{t_\circ}^{s} \left[\left| f\left(\frac{\sigma}{\epsilon}, x(\sigma), w(\sigma)\right) \right| + |f_a(y(\sigma), w(\sigma))| \right] d\sigma \\ &\leq \int_{t_\circ}^{s} 2B d\sigma = 2B(s - t_\circ). \end{aligned} \tag{19}$$

Combining inequalities (18) and (19), noting that $t - t_\circ \leq d$ and using the definition of d^* (in particular $\nu \geq LBd$) we have

$$\begin{aligned} \int_{t_\circ}^{t} \alpha_3(|x(s)|) ds &\leq \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^{t} \gamma(|w(s)|) ds + 2LB \int_{t_\circ}^{t} (s - t_\circ) ds = \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^{t} \gamma(|w(s)|) ds + LB(t - t_\circ)^2 \\ &\leq \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^{t} \gamma(|w(s)|) ds + LBd(t - t_\circ) \leq \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^{t} \gamma(|w(s)|) ds + \nu(t - t_\circ), \end{aligned}$$

for all $t \in [t_\circ, t_\circ + d]$ thereby completing the proof. □

Lemma 5 *Let $(\Omega_x, \Omega_w, \nu)$ be given and $\mathcal{W} := \{w : \|w\|_\infty \leq \Omega_w\}$. Let $\mathcal{W}_1$ be an arbitrary subset of $\mathcal{W}$. Let V be a continuous function such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ for all $x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Let α_3 be a locally Lipschitz $\mathcal{K}_\infty$ function and define*

$$\Delta := \max \left\{ \alpha_1^{-1} \left(\alpha_2 \circ \alpha_3^{-1} \left(2\gamma(\Omega_w) + \nu \right) + \frac{\nu}{2} \right), \alpha_1^{-1} \circ \alpha_2(\Omega_x) \right\} . \tag{20}$$

Suppose the following property holds:

Property E: *There exists $d^* > 0$ and such that for any $\delta_1 \in (0, d^*)$ and $\delta_2 \in (\delta_1, d^*]$, there exists an $\epsilon^* > 0$, such that for all $d \in [\delta_1, \delta_2]$, $\epsilon \in (0, \epsilon^*)$, $t_\circ \geq 0$, $|x(t_\circ)| \leq \Delta$, $w \in \mathcal{W}_1$:*

(i) Property A holds with some $M > 0$;

(ii) Property B holds with $\mu_0 = 1$;

(iii) Property C holds with $\mu_1 = \frac{\nu}{2}$;

(iv) Property D holds;

(v) Property L holds with α_3 and $\mu_2 = \frac{\nu}{2}$.

Then the following property holds:

Property F: *For all $\epsilon \in (0, \epsilon^*)$, $|x(t_\circ)| \leq \Omega_x$, $w \in \mathcal{W}_1$ and $t \geq t_\circ \geq 0$ the solution $x(t)$ exists and satisfies (8).* □

Proof of Lemma 5: Let the triple $(\Omega_x, \Omega_w, \nu)$ and $\mathcal{W}_1 \subseteq \mathcal{W}$ be given. Let Δ be defined by (20) and let Property E hold for some d^*. Denote $L_\alpha > 0$ a number that satisfies $|\alpha_3(s_1) - \alpha_3(s_2)| \leq L_\alpha |s_1 - s_2|$, $\forall s_1, s_2 \in [0, \Delta + 1]$. Define $d_1^* := \min \left\{ d^*, \frac{\nu}{L_\alpha M} \right\}$, where $M > 0$ comes from (i). Let $\delta_1 = \frac{d_1^*}{2}$ and $\delta_2 = d_1^*$ determine ϵ^* using Property E. Consider arbitrary $\epsilon \in (0, \epsilon^*)$, $|x(t_\circ)| \leq \Omega_x$ (note that this implies $|x(t_\circ)| \leq \Delta$), $w \in \mathcal{W}_1$ and $t, t_\circ$, such that $t \geq t_\circ \geq 0$. Define[5] $\ell = \ell(t_\circ, t) := \min\{n \geq 1 : n d_1^* \geq t - t_\circ\}$.

[5] Integer ℓ is the minimum number of intervals of length d_1^* required to cover the interval $[t_\circ, t]$.

Case 1, $\ell = 1$: $\ell = 1$ implies that $t \in [t_\circ, t_\circ + d_1^*]$, which by (iv) implies that Property F holds. This completes the proof for Case 1.

Case 2, $\ell > 1$: Define the sampling interval $d = d(t_\circ, t) := \frac{t - t_\circ}{\ell}$. Using the definition of ℓ we have $(\ell - 1)d_1^* < t - t_\circ$ and since $\ell > 1$, we can write

$$d = \frac{t - t_\circ}{\ell} > d_1^* \left(\frac{\ell - 1}{\ell} \right) \geq \frac{d_1^*}{2}, \quad \forall \ell > 1.$$

That is, $d \in (\frac{d_1^*}{2}, d_1^*]$ for arbitrary $t > t_\circ + d_1^*$. From (v) it follows that for all $\epsilon \in (0, \epsilon^*)$, inequality (10) holds uniformly for any sampling interval $d \in [\frac{d_1^*}{2}, d_1^*]$ and hence uniformly for all $t > t_\circ + d_1^*$.

Introduce the following sequence of numbers $t_k := t_\circ + kd, k = 0, 1, 2, \ldots$ and the following notation $x(k) := x(t_k)$, $V(k) := V(x(t_k))$ and $w[k] := \{w(t) : t \in [t_k, t_{k+1}]\}$. First, we show that if we have $V(0) \leq \alpha_1(\Delta)$, then $V(k) \leq \alpha_1(\Delta), \forall k \geq 0$. Indeed, consider arbitrary $k \geq 0$, arbitrary $x(k)$ such that $V(k) \leq \alpha_1(\Delta)$ (which implies $|x(k)| \leq \Delta$) and arbitrary $w(\cdot) \in \mathcal{W}_1$. Then, we have either that $|x(k)| \geq \alpha_3^{-1}(2\gamma(\Omega_w) + \nu)$, in which case (from (v) and our supposition that $V(k) \leq \alpha_1(\Delta)$) we have

$$V(k+1) - V(k) \leq -\frac{d}{2}\alpha_3(|x(k)|) \ \Rightarrow \ V(k+1) \leq V(k) \leq \alpha_1(\Delta) \ , \tag{21}$$

or we have that $|x(k)| < \alpha_3^{-1}(2\gamma(\Omega_w) + \nu)$, which implies (from (iii) and the definition of Δ)

$$V(k+1) \ \leq \ V(k) + \frac{\nu}{2} \ \leq \ \alpha_2(|x(k)|) + \frac{\nu}{2} \ < \ \alpha_2 \circ \alpha_3^{-1}(2\gamma(\Omega_w) + \nu) + \frac{\nu}{2} \ \leq \ \alpha_1(\Delta) \ . \tag{22}$$

By induction, we have that if $V(0) \leq \alpha_1(\Delta)$ then $V(k) \leq \alpha_1(\Delta), \forall k \geq 0$. Using the above discussion and (ii), it follows that for all $|x(k)| \leq \Delta, k \geq 0$ and $w \in \mathcal{W}_1$ the following inequalities hold for all $k \geq 0$:

$$|x(k)| \ \leq \ \Delta, \ \forall k = 0, 1, 2, \ldots \tag{23}$$

$$|x(t)| \ \leq \ |x(k)| + 1, \ \forall t \in [t_k, t_{k+1}] \tag{24}$$

$$V(x(t)) \ \leq \ V(k) + \frac{\nu}{2}, \ \forall t \in [t_k, t_{k+1}] \tag{25}$$

$$\frac{V(k+1) - V(k)}{d} \ \leq \ -\alpha_3(|x(k)|) + \frac{1}{d}\int_{t_k}^{t_{k+1}} \gamma(|w(s)|)ds + \frac{\nu}{2} \ . \tag{26}$$

Hence, (23) and (24) guarantee existence of solutions of the actual system for all $t \geq t_\circ \geq 0$. Iteratively add (26) for $k \in [0, \ell - 1]$. Noting that $t_\ell - t_\circ = t - t_\circ = \ell d$ and using (7) we obtain:

$$d\sum_{k=0}^{\ell-1} \alpha_3(|x(k)|) \ \leq \ V(0) - V(\ell) + \int_{t_\circ}^{t_\ell} \gamma(|w(s)|)ds + \frac{\nu}{2}d\ell \ \leq \ \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^{t} \gamma(|w(s)|)ds + \frac{\nu}{2}(t - t_\circ) \ .$$

Hence, adding $\int_{t_\circ}^{t} \alpha_3(|x(s)|)ds$ to both sides of (27),

$$\int_{t_\circ}^{t} \alpha_3(|x(s)|)ds \ \leq \ \alpha_2(|x(t_\circ)|) + \int_{t_\circ}^{t} \gamma(|w(s)|)ds + \frac{\nu}{2}(t - t_\circ) + \underbrace{\left(\int_{t_\circ}^{t} \alpha_3(|x(s)|)ds - d\sum_{k=0}^{\ell-1} \alpha_3(|x(k)|) \right)}_{Term\ 1} \ . \tag{27}$$

We now bound Term 1 in (27). Since $|x(t)| \leq \Delta + 1, \forall t \geq t_\circ$ and using the definitions of M in (i) and L_α we can write $|x(s) - x(t_k)| \leq M(s - t_k), \forall s \in [t_k, t_{k+1}], k \in [0, \ell - 1]$, and hence we have:

$$\begin{aligned}
\left| \int_{t_\circ}^{t} \alpha_3(|x(s)|)ds - d \sum_{k=0}^{\ell-1} \alpha_3(|x(k)|) \right| &= \sum_{k=0}^{\ell-1} \left\{ \int_{t_k}^{t_{k+1}} [\alpha_3(|x(s)|) - \alpha_3(|x(k)|)]ds \right\} \leq L_\alpha \sum_{k=0}^{\ell-1} \int_{t_k}^{t_{k+1}} |x(s) - x(k)|\, ds \\
&\leq L_\alpha M \sum_{k=0}^{\ell-1} \int_{t_k}^{t_{k+1}} |s - t_k|\, ds = L_\alpha M \ell \int_0^d \tau d\tau = \frac{L_\alpha M \ell d^2}{2} \\
&\leq \frac{\nu}{2}(t - t_\circ) \, . \qquad (28)
\end{aligned}$$

Combining (28) with (27), we have that (8) holds. Since this holds for arbitrary $t > t_\circ + d_1^*$, this proves inequality (8) for all $t \geq t_\circ \geq 0$ and hence completes the proof. □

Lemma 5 can now be used to prove the main results.

Proof of Theorem 1: Let the triple $(\Omega_x, \Omega_w, \nu)$ be given. Fix $\mathcal{W}_1 = \mathcal{W} = \{w : \|w\|_\infty \leq \Omega_w\}$. Let V be the differentiable Lyapunov function with locally Lipschitz gradient for the strong average, as per (7). Let Δ be defined by (20), $\mu_0 = 1$, $\mu_1 = \frac{\nu}{2}$, $\mu_2 = \frac{\nu}{2}$. Let the quadruple $(\Delta, \Omega_w, \mu_1, \mu_2)$ define $d_1^* > 0$ using Lemma 2. Let (Δ, Ω_w, ν) determine $d_2^* > 0$ using Lemma 4. Take $d^* := \min\{d_1^*, d_2^*\}$. Then, from Lemmas 2 and 4 and Remark 3 we have that Property E holds. Using Lemma 5 we have that Property F holds, which completes the proof. □

Proof of Theorem 2: Let the quadruple $(\Omega_x, \Omega_w, \Omega_{\dot{w}}, \nu)$ be given. Let $\mathcal{W}_1 = \{w : \|w\|_\infty \leq \Omega_w, \|\dot{w}\|_\infty \leq \Omega_{\dot{w}}\} \subset \mathcal{W} = \{w : \|w\|_\infty \leq \Omega_w\}$. Let V be the differentiable Lyapunov function with locally Lipschitz gradient for the weak average, as per (9). Let Δ be defined by (20), $\mu_0 = 1$, $\mu_1 = \frac{\nu}{2}$, $\mu_2 = \frac{\nu}{2}$. Let the 5-tuple $(\Delta, \Omega_w, \Omega_{\dot{w}}, \mu_1, \mu_2)$ define $d_1^* > 0$ using Lemma 3. Let (Δ, Ω_w, ν) determine $d_2^* > 0$ using Lemma 4. Take $d^* := \min\{d_1^*, d_2^*\}$. Then, from Lemmas 3 and 4 and Remark 3 it follows that Property E holds. Using Lemma 5 we have that Property F holds, which completes the proof. □

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