Robust Stability of Packetized Predictive Control of Nonlinear Systems with Disturbances and Markovian Packet Losses ⋆

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Abstract

We study a predictive control formulation for uncertain discrete-time non-linear uniformly continuous plant models where controller output data is transmitted over an unreliable communication channel. The channel introduces Markovian data-loss and does not provide acknowledgments of receipt. To achieve robustness with respect to dropouts, at every sampling instant the controller transmits packets of data. These contain possible control inputs for a finite number of future time instants, and minimize a finite horizon cost function. At the plant actuator side, received packets are buffered, providing the plant inputs. Within this context, we adopt a stochastic Lyapunov function approach to establish stability results of the networked control system. A distinguishing aspect of this work is that it considers situations where the maximum number of consecutive packet dropouts has unbounded support.

Key words: Control over networks, predictive control, packet dropouts, non-linear systems, stochastic stability.

1 Introduction

Motivated by both practical and also theoretical aspects, significant research has concentrated on Networked Control Systems (NCSs), as documented, e.g., in [1, 2]. In a NCS, plant and controller communicate via a network which may be shared with other applications. The sharing of a network simplifies the cabling (especially if the network is wireless) and, thus, increases overall system reliability. However, since general purpose network platforms were not originally designed for applications with critical timing requirements, their use for closed-loop control presents some serious challenges. The network itself is a dynamical system that exhibits characteristics which traditionally have not been taken into account in control system design. In addition to being quantized, transmitted data may be affected by time delays and data-dropouts. Thus, in a NCS links are not transparent, often constituting a significant performance bottleneck.

Various models have been utilized to describe time-delays and packet dropouts in NCSs. For example, suitable deterministic boundedness assumptions can be used to derive sufficient conditions for deterministic stability; see, e.g., [3–9]. On the other hand, in the communication literature, stochastic models are generally preferred, since they allow one to describe phenomena more accurately [10, 11]. A simple approach considers network effects as independent and identically distributed (i.i.d.) random variables. In particular, stochastic stability results for various NCS architectures have been established, e.g., in [12–25]. Whilst i.i.d. models will often give better insight into NCSs than deterministic ones, fading communication channel gains and network congestion levels are, in general, correlated [26–29]. This motivates the adoption of, more general, Markov chain models. In particular, [30] studies NCSs where time delays are described by a finite Markov chain; [31, 32] consider stochastic packet dropouts by modeling the times between successful transmission as a finite Markov chain. By way of contrast, [33–37] adapt the Gilbert-Elliott fading channel model [26, 27] to describe the dropout process directly as a Markov chain and allow for the time between successful transmission instants to have unbounded support.

An important feature of many communication protocols is that data is sent in large time-stamped packets. This opens the possibility to conceive NCS architectures in which *packets of data* containing finite sequences, rather than individ-

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ual values, are sent through the network. Through buffering and appropriate selection logic at the receiver node, time delays and packet dropouts can to some extent be compensated for; see, e.g., [38–40]. Here, model predictive control (MPC) [41] becomes a natural choice for tackling controller to actuator links, since potential plant input values over a finite horizon are readily available. Not surprisingly, this *packetized predictive control* (PPC) idea often gives good performance; see, e.g., [42, 43].

Deterministic stability results of PPC have been obtained in various works, including [44–48]. In particular, stability of nonlinear systems with disturbances is the focus of [45, 46, 48] for cases where the maximum number of consecutive packets dropouts is bounded, and of [47] for networks with bounded time-delays. Given the widespread use of stochastic models in the communications community, it is surprising that these have only recently been taken into account when studying stability of PPC formulations. To the best of our knowledge, only our own recent publications [49,50] deal with this issue. The work presented in [49] studies quantized control of perturbed LTI systems with i.i.d. packet dropouts and establishes sufficient conditions for mean-square stability, whereas [50] studies stochastic stability of nonlinear plant models without disturbances in the presence of i.i.d. dropouts.

In the present work we study a PPC formulation for discrete-time non-linear plant models with disturbances, where optimizing sequences are transmitted over an unreliable communication channel, see Fig. 1. The controller is designed without knowledge of the packet dropout distribution and does not require acknowledgments of receipt. We combine elements of the PPC model of [46] with stochastic stability analysis [51,52] to establish sufficient conditions for the optimal MPC value function to constitute a stochastic Lyapunov function of the NCS *at the successful transmission instants*. We then show how this property ensures stochastic stability of the NCS. Our stability results apply to NCSs with Markovian packet dropouts and nonlinear plant models with disturbances. Disturbances and times between successful transmissions are allowed to have unbounded support.

The remainder of this work is organized as follows: In Section 2, we present the NCS architecture to be studied. Section 3 describes how the control packets are designed. Section 4 presents a model for the NCS at the successful transmission instants. Stochastic stability results are then established in Section 5. Section 6 gives guidelines on how to choose design parameters. The proofs of the main results are given in Sections 7 and 8. Section 9 draws conclusions.

Notation We write $\mathbb{R}$ for the real numbers, $\mathbb{R}_{>0}$ for $(0,\infty)$, $\mathbb{N}$ for $\{1,2,\dots\}$, and $\mathbb{N}_0$ for $\mathbb{N}\cup\{0\}$. The $p\times p$ identity matrix

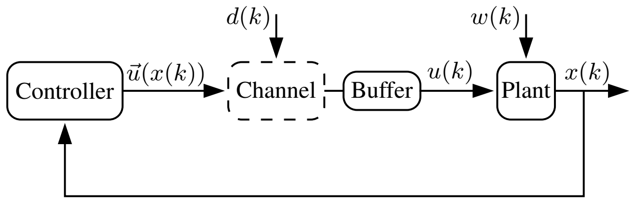

Fig. 1. NCS Architecture with Packet Dropouts and Buffering

is denoted via I_p; $0_p \triangleq 0\cdot I_p$; $\{y\}_{\mathbb{K}} = \{y(\ell) : \ell \in \mathbb{K}\}$, and

$$\{y\}_{\ell_1}^{\ell_2} = \begin{cases} \{y(\ell_1),\dots,y(\ell_2)\} & \text{if } \ell_1 \le \ell_2, \\ \{\,\} & \text{if } \ell_1 > \ell_2. \end{cases}$$

We adopt the convention $\sum_{k=\ell_1}^{\ell_2} a_k = 0$, if $\ell_1 > \ell_2$ and irrespective of a_k. The norm of a vector x is denoted $|x|$. To denote the unconditional probability of an event Ω, we write $\mathbf{Pr}\{\Omega\}$. The conditional probability of Ω given Γ is denoted via $\mathbf{Pr}\{\Omega\,|\,\Gamma\}$. The expected value of a random variable ν given Γ, is denoted by $\mathbf{E}\{\nu\,|\,\Gamma\}$, whereas for the unconditional expectation we will write $\mathbf{E}\{\nu\}$. We use the same notation for random variables and their realizations. What is meant will depend upon the context.

2 NCS Architecture

We consider discrete-time nonlinear (and possibly unstable) plant models with state $x \in \mathbb{R}^n$ and constrained input $u \in \mathbb{U} \subseteq \mathbb{R}^p$, $0 \in \mathbb{U}$, described via:

$$x(k+1) = f(x(k), u(k), w(k)), \quad k \in \mathbb{N}_0, \tag{1}$$

where $f(0,0,0) = 0$. The initial state $x(0)$ is arbitrarily distributed (with possibly unbounded support) and the disturbance $\{w\}_{\mathbb{N}_0}$ is i.i.d., but otherwise arbitrarily distributed. [1]

Network effects Our interest lies in clock-driven networks situated between controller output and plant input. All data to be transmitted is sent in large time-stamped packets. The network is affected by transmission errors (for example, due to channel fading and congestion), which are in general correlated in time and introduce packet-dropouts [26–29]. This motivates us to model the network as an erasure channel, which operates at the same sampling rate as the plant model (1), and to characterize transmission effects via the following time-homogeneous binary Markov process $\{d\}_{\mathbb{N}_0}$:

$$d(k) \triangleq \begin{cases} 1 & \text{if packet-dropout occurs at instant } k, \\ 0 & \text{if packet-dropout does not occur at instant } k, \end{cases} \tag{2}$$

[1] Note that our disturbance model serves to describe a class of NCSs with quantized inputs [49]. Our results in Section 5 require that $\mathbf{E}\{|x(0)|^s\}$ and $\mathbf{E}\{|w(k)|^s\}$ be bounded for some $s > 0$.

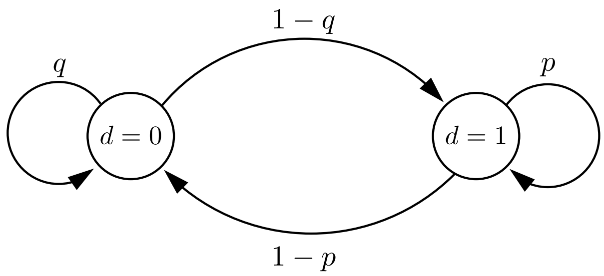

Fig. 2. Markov Packet Dropout Model

where the transition probabilities are given by, see Fig. 2:

$$\begin{aligned}&\mathbf{Pr}\{d(k+1)=0\,|\,d(k)=0\}=q\\&\mathbf{Pr}\{d(k+1)=1\,|\,d(k)=0\}=1-q\\&\mathbf{Pr}\{d(k+1)=1\,|\,d(k)=1\}=p\\&\mathbf{Pr}\{d(k+1)=0\,|\,d(k)=1\}=1-p.\end{aligned}\tag{3}$$

The associated *failure rate* is $1-q\in(0,1)$, whereas the *recovery rate* is given by $1-p\in(0,1)$. The values $q\approx 1$ and $p\approx 0$, thus, describe a more reliable network; $q\approx 0$ and $p\approx 1$ refer to a network more prone to dropouts. The model (3) incorporates temporal correlations of network conditions. It is therefore more general and realistic than i.i.d. Bernoulli models. In fact, the i.i.d. dropout model corresponds to the special case where $q=1-p$, so that:

$$\mathbf{Pr}\{d(k)=1\}=p,\quad \mathbf{Pr}\{d(k)=0\}=1-p,\tag{4}$$

where p is the *dropout-rate*. In practice, p and q are not known exactly. Accordingly, in the present work our focus is on situations where the controller does not have knowledge about p and q. (Of course, closed loop stability will depend upon these parameters, see Sections 5 and 6.)

As foreshadowed in the introduction, at each time instant k and for plant state $x(k)$, the packetized predictive controller sends a control packet, say $\vec{u}(x(k))$, to the plant input node. To achieve good performance despite unreliable communication, $\vec{u}(x(k))$ contains constrained tentative control inputs for a finite number of N future time instants, i.e., we have

$$\vec{u}(x(k))=\begin{bmatrix}u_0(x(k))\\u_1(x(k))\\\vdots\\u_{N-1}(x(k))\end{bmatrix}\in\mathbb{U}^N\subseteq\mathbb{R}^{pN}.\tag{5}$$

At the actuator side, the received packets are buffered, providing the plant inputs, see Fig. 1.

Buffering The buffering mechanism amounts to a parallel-in serial-out shift register, which acts as a safeguard against dropouts. The buffer state, $b(k)\in\mathbb{U}^N$, is overwritten whenever a valid (i.e., error-free and undelayed) control packet arrives. Actuator values are sequentially passed on to the plant until the next valid control packet is received. Thus,

$$\begin{aligned}b(k)&=d(k)Sb(k-1)+(1-d(k))\vec{u}(k),\\u(k)&=e_1^Tb(k)\end{aligned}\tag{6}$$

where $b(0)=0$ and where S and e_1 are defined via:

$$S\triangleq\begin{bmatrix}0_p & I_p & 0_p & \cdot\,\cdot & 0_p\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0_p & \dots & 0_p & I_p & 0_p\\ 0_p & \dots\dots & & 0_p & I_p\\ 0_p & \dots\dots\dots & & & 0_p\end{bmatrix},\quad e_1\triangleq\begin{bmatrix}I_p\\0_p\\\vdots\\0_p\end{bmatrix}.\tag{7}$$

Remark 1 (Holding the control input) *The choice of the matrix S in (7) corresponds to setting the buffer state to zero if no data is received over N consecutive instants. Alternatively, if one wished to hold the latest value, one could set the "last" element of S equal to I_p; see also [31, 53].* ☐

Remark 2 (Dropouts of State Measurements) *Our ideas can be readily extended to also encompass dropouts of the state measurements. In fact, if the controller only calculates control packets whenever state measurements are successfully received, then our subsequent analysis still holds. The only modification needed is that the dropout process $\{d\}_{\mathbb{N}_0}$ include both links, i.e., we set $d(k)=0$ if and only if both $x(k)$ and $\vec{u}(k)$ are successfully transmitted; c.f.,* (2). ☐

3 Packetized Predictive Control

The control packets $\vec{u}(x(k))$ in (5) are formed by adapting the ideas underpinning MPC. More precisely, at each time instant k, the following cost function is minimized:

$$J(\vec{u}\,',x(k))\triangleq F(x'(N))+\sum_{\ell=0}^{N-1}L(x'(\ell),u'(\ell)).\tag{8}$$

The cost function in (8) examines predictions of the plant model (1) over a finite horizon of length N, which is taken equal to the buffer size. To provide a method with manageable computational complexity, the predicted state trajectories do not take into account packet-losses or disturbances and are, thus, generated by the nominal model

$$x'(\ell+1)=f(x'(\ell),u'(\ell),0),\quad \ell\in\{0,1,\dots,N-1\}$$

starting from $x'(0)=x(k)$ and where the constrained entries in $\vec{u}\,'=\big[(u'(0))^T\ \dots\ (u'(N-1))^T\big]^T\in\mathbb{U}^N$ are the associated tentative plant inputs. Predicted quantities are penalized via the per-stage weighting function $L(\cdot,\cdot)$ and the terminal weighting $F(\cdot)$. These design variables allow one to trade-off control performance versus control effort. As in control loops without dropouts [41], the choices made for $L(\cdot,\cdot)$, $F(\cdot)$ and N influence closed loop stability. This issue will be further examined in Sections 5 and 6.

The control packet $\vec{u}(x(k))$ is set equal to the optimizer

$$\vec{u}(x(k)) \triangleq \arg\min_{\vec{u}' \in \mathbb{U}^N} J(\vec{u}', x(k)) \tag{9}$$

and is sent through the network to the buffer, see Fig. 1. Following the receding horizon optimization idea, at the next sampling step and given $x(k+1)$, the horizon is shifted by one and another optimization is carried out, providing

$$\vec{u}(x(k+1)) = \arg\min_{\vec{u}' \in \mathbb{U}^N} J(\vec{u}', x(k+1)),$$

sequence, which is transmitted to the buffer. This procedure is repeated *ad infinitum*. Note that $\vec{u}(x(k))$ in (9) contains constrained tentative plant input values for instants $\{k, \dots, k+N-1\}$. If $\vec{u}(x(k))$ is received at time k, then these values are written into the buffer and implemented sequentially until some future (valid) control packet arrives. Since the plant model and cost function are time-invariant, the optimization in (9) gives rise to a time-invariant mapping, say $\kappa_N : \mathbb{R}^n \to \mathbb{U}^N$, which characterizes $\vec{u}(x(k))$ via:

$$\vec{u}(x(k)) = \kappa_N(x(k)), \quad \forall k \in \mathbb{N}_0. \tag{10}$$

In the NCS architecture studied, the plant input design is done dynamically to optimize performance, under the constraint that the controller has no knowledge on the dropout distribution parameters p and q and on whether previous packets were successfully transmitted. It is important to note that whilst $\vec{u}(x(k))$ is found by evaluating open-loop predictions (and not closed-loop policies), the resultant control policy is a closed-loop one. Indeed, the loop is closed at all successful transmission instants, i.e., where $d(k) = 0$.

The above yields that the NCS which results from using packetized predictive control over a Markovian erasure channel can be described by the jump non-linear system

$$\begin{aligned}
\theta(k+1) &= \mathcal{G}_{d(k)}(\theta(k), w(k)), \quad \theta(k) \triangleq \begin{bmatrix} x(k) \\ b(k-1) \end{bmatrix} \\
\mathcal{G}_0(\theta(k), w(k)) &= \begin{bmatrix} f(x(k), e_1^T \kappa_N(x(k)), w(k)) \\ \kappa_N(x(k)) \end{bmatrix}, \\
\mathcal{G}_1(\theta(k), w(k)) &= \begin{bmatrix} f(x(k), e_1^T S b(k-1), w(k)) \\ S b(k-1) \end{bmatrix},
\end{aligned} \tag{11}$$

where the jump variable $\{d\}_{\mathbb{N}_0}$ is Markovian. Related *linear* characterizations of NCSs have been extensively studied within the context of Markov jump linear systems; see, e.g., [34, 38]. To treat the non-linear case (11), we will use an alternative model, presented in the following section.

4 NCS Model at successful transmission instants

Our subsequent analysis extends the approach of [46] to a stochastic setting. [2] We denote the time instants where there are no packet-dropouts (i.e., where $d(k) = 0$) via

$$\mathbb{K} = \{k_i\}_{i \in \mathbb{N}_0} \subseteq \mathbb{N}_0, \quad k_{i+1} > k_i, \ \forall i \in \mathbb{N}_0. \tag{12}$$

and define $\Delta(i) \triangleq k_{i+1} - k_i$, where $i \in \mathbb{N}_0$. For ease of exposition, we will assume that the first successful transmission instant occurs at $k = 0$, thus, $k_0 = 0$ and $\Delta(0) = k_1$.

Interestingly, despite the fact that the dropout process $\{d\}_{\mathbb{N}_0}$ is correlated, the process $\{\Delta\}_{\mathbb{N}_0}$ is i.i.d.. More precisely, since we assume that $k_0 = 0$, it follows from (3) and (12), that $\{\Delta\}_{\mathbb{N}_0}$ is i.i.d. with geometric-like distribution

$$\mathbf{Pr}\{\Delta(i) = j\} = \begin{cases} q & \text{if } j = 1, \\ (1-q)(1-p)p^{j-2} & \text{if } j \geq 2, \end{cases} \tag{13}$$

see also [36]. As a particular case of (13), if the dropout process is i.i.d. Bernoulli distributed as in (4), then we have:

$$\mathbf{Pr}\{\Delta(i) = j\} = (1-p)p^{j-1}, \quad \forall j \in \mathbb{N}. \tag{14}$$

For our subsequent analysis, it is convenient to introduce the following iterated mappings with inputs [3] $\{\varpi\}_0^{j-1}$:

$$f^j(x, \{\varpi\}_0^{j-1}) \triangleq f(f^{j-1}(x, \{\varpi\}_0^{j-2}), u_{j-1}(x), \varpi(j-1)), \tag{15}$$

for $j \in \{1, \dots, N\}$, and where we set $f^0(x, \{\varpi\}_0^{-1}) \triangleq x$.

We also introduce the iterated *open-loop* mapping:

$$f_{ol}^j(x, \{\varpi\}_0^{j-1}) \triangleq f(f_{ol}^{j-1}(x, \{\varpi\}_0^{j-2}), 0, \varpi(j-1))$$

for $j \in \mathbb{N}$, and where $f_{ol}^0(x, \{\varpi\}_0^{-1}) \triangleq x$.

Given the buffering mechanism, see (6), (9) and (5), it is easy to see that the plant state at $k_i \in \mathbb{K}$ is characterized via:

$$x(k_{i+1}) = \begin{cases} f^{\Delta(i)}\big(x(k_i), \{w\}_{k_i}^{k_{i+1}-1}\big) & \\ \qquad\qquad\qquad \text{if } \Delta(i) \leq N, \\ f_{ol}^{\Delta(i)-N}\big(x(k_i+N), \{w\}_{k_i+N}^{k_{i+1}-1}\big) & \\ \qquad\qquad\qquad \text{if } \Delta(i) \geq N. \end{cases} \tag{16}$$

where $x(k_i + N) = f^N\big(x(k_i), \{w\}_{k_i}^{k_i+N-1}\big)$.

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[2] Our ideas are also related to methods used to study randomly sampled systems [36,54], and to the averaging technique of [8,9].

[3] For example, we have $f^1(x, \varpi(0)) = f(x, u_0(x), \varpi(0))$ and $f^2(x, \{\varpi\}_0^1) = f(f(x, u_0(x), \varpi(0)), u_1(x), \varpi(1))$.

Remark 3 (Relationship to previous works) *A key difference between the current situation and that studied in articles such as [23, 24, 32, 40, 45, 46] is that the results in the latter works require that $\Delta(i)$ be bounded, for all $i \in \mathbb{N}_0$. In the present work, we remove this assumption by allowing the maximum number of consecutive packet dropouts to have unbounded support, see (14). For that purpose, we extend our recent work documented in [49] to encompass non-linear plant models and Markovian dropouts.* $\square$

Having presented a NCS model, in the sequel we will elucidate the effect of the system parameters on stability.

5 Stochastic Stability of the NCS

Due to disturbances and packet dropouts, the system state $\{x\}_{\mathbb{N}_0}$ becomes a random process. To establish closed loop stability results, we will adopt the stochastic Lyapunov approach, as described in [51], and study the optimal costs $V(x) \triangleq J(\vec{u}(x), x)$. These are based upon predictions which use the nominal plant model (1) without disturbances and with inputs taken from $\vec{u}(x)$, see (9). In fact, we have:

$$V(x) = F\big(\bar{f}^N(x)\big) + \sum_{\ell=0}^{N-1} L\big(\bar{f}^\ell(x), u_\ell(x)\big), \tag{17}$$

where, see (15),

$$\bar{f}^\ell(x) \triangleq f^\ell\big(x, \{0, \dots, 0\}\big), \quad \forall \ell \in \{0, 1, \dots, N\}. \tag{18}$$

5.1 Assumptions

To derive our results, we will make some assumptions on the cost function and on the class of plant models considered.

Assumption 1 *There exist $\alpha_F, \alpha_L, b, c \in \mathbb{R}_{>0}$, such that, for all $(x, u) \in \mathbb{R}^n \times \mathbb{U}$,*

$$\alpha_F |x|^b \leq F(x), \quad F(0) = 0, \tag{19a}$$
$$\alpha_L |x|^c \leq L(x, u), \quad L(0, 0) = 0. \tag{19b}$$

Assumption 2 *The plant model and weighting functions are uniformly continuous, i.e., $\exists \lambda_x, \lambda_w, \lambda_L, \lambda_F, s \in \mathbb{R}_{>0}$:*

$$|f(x, u, w) - f(z, u, 0)|^s \leq \lambda_x |x - z|^s + \lambda_w |w|^s, \tag{20a}$$
$$|L(x, u) - L(z, u)| \leq \lambda_L |x - z|^s, \tag{20b}$$
$$|F(x) - F(z)| \leq \lambda_F |x - z|^s. \tag{20c}$$

for all $(x, z, u, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^m$. $\square$

Assumption 3 *There exists a constrained control law $\kappa_f : \mathbb{R}^n \to \mathbb{U}$ such that for the nominal plant model:*

$$F\big(f(x, \kappa_f(x), 0)\big) - F(x) + L(x, \kappa_f(x)) \leq 0, \tag{21}$$

for all $x \in \mathbb{R}^n$. $\square$

Assumptions 1 to 3 (and variations thereof) have been extensively used for establishing stability of predictive control loops (without dropouts); see, e.g., [41]. The policy $\kappa_f(\cdot)$ in (21) can be regarded as a stabilizing law for the nominal plant model, which is not necessarily implemented.

Assumption 4, stated below, is specific to the NCS architecture studied. It amounts to an upper bound of p, see (3), for a given plant model, or, conversely, to an upper bound on the rate of growth of x when left in open loop, for a given p.

Assumption 4 *There exist $\gamma \in [1, 1/p)$ and $\eta, v \in \mathbb{R}_{>0}$ such that*

$$F(f(x, 0, w)) \leq \gamma F(x) + \eta |w|^v, \tag{22}$$

for all $(x, w) \in \mathbb{R}^n \times \mathbb{R}^m$. $\square$

Before proceeding, we make the following observations:

Lemma 4 *Suppose that Assumption 3 holds. Then:*

$$L(x, u_0(x)) \leq V(x) \leq F(x), \quad \forall x \in \mathbb{R}^n. \tag{23}$$

PROOF. Lemma 2.15 in [41] establishes $V(x) \leq F(x)$. The other inequality in (23) follows directly from (17). $\square$

Lemma 5 *Suppose that Assumptions 1 and 2 hold with $b = s$ and that*

$$p < \frac{\alpha_F}{\lambda_F \lambda_x}. \tag{24}$$

Then Assumption 4 is satisfied with $\gamma = \lambda_F \lambda_x / \alpha_F$, $\eta = \lambda_F \lambda_w$ and $v = s$.

PROOF. Since $f(0, 0, 0) = 0$, we can use (20) to obtain $F\big(f(x, 0, w)\big) = \big|F\big(f(x, 0, w)\big)\big| \leq \lambda_F |f(x, 0, w)|^s = \lambda_F |f(x, 0, w) - f(0, 0, 0)|^s \leq \lambda_F \lambda_x |x|^s + \lambda_F \lambda_w |w|^s$. The result follows from using (19a) and (23). $\square$

A particular case of (24) results when the plant model is scalar LTI with system pole a and $F(x) = |x|$, the absolute value. Here, one can choose $\lambda_F = \alpha_F = s = b = 1$, and $\lambda_x = |a|$. Thus, (24) becomes $|a| p < 1$, which is necessary and sufficient for mean-square stabilizability in the case where dropouts are i.i.d. as in (4); see, e.g., [18, 37, 55].

5.2 Main Results

To study stability of the NCS, we first note that, by (16), and since $\{w\}_{\mathbb{N}_0}$ and $\{\Delta\}_{\mathbb{N}_0}$ are i.i.d., the system state sequence, at the instants of successful transmission $\{x\}_{\mathbb{K}}$, is Markovian. ($\{x\}_{\mathbb{N}_0}$ is in general not Markovian.) Our first results, stated as Proposition 6 and Corollary 7, establish sufficient conditions for exponential convergence of $\big\{\mathbf{E}\{|x(k_i)|^c\}\big\}$ for $k_i \in \mathbb{K}$. This property is then used, in Theorem 8, for establishing asymptotic convergence of $\big\{\mathbf{E}\{|x(k)|^c\}\big\}$ *at all time instants* $k \in \mathbb{N}_0$. In both cases, convergence is to a bounded set containing the origin, whose size is proportional to the moment $\mathbf{E}\{|w(k)|^s\}$. To formulate our results, we define the following quantities:

$$\begin{aligned}\psi_\ell &\triangleq \frac{1-p\gamma}{p^{\ell-1}(1-q)(\gamma-1)}, \quad \ell \in \mathbb{N} \\ \phi(x) &\triangleq L(x,u_0(x)) - \psi_N^{-1}F(\bar{f}^N(x)), \quad x \in \mathbb{R}^n.\end{aligned} \tag{25}$$

Proposition 6 *Suppose that Assumptions 2 to 4 hold with* $v = s$. *Then there exists* $\sigma \in \mathbb{R}_{>0}$ *such that, at* $k_1 \in \mathbb{K}$,

$$\begin{aligned}\mathbf{E}\big\{V(x(k_1))\,\big|\,x(0)\big\} - V(x(0)) \\ \leq \sigma\mathbf{E}\{|w(k)|^s\} - \phi(x(0)).\end{aligned} \tag{26}$$

PROOF. See Section 7. □

Corollary 7 *Suppose that* (19b) *and Assumptions 2 to 4 hold with* $v = s$, *that*

$$\mathbf{E}\{|x(0)|^s\} < \infty, \quad \mathbf{E}\{|w(k)|^s\} < \infty$$

and that there exists $\rho \in (0,1]$ *such that*

$$L(x,u_0(x)) - \psi_N^{-1}F(\bar{f}^N(x)) \geq \rho F(x), \quad \forall x \in \mathbb{R}^n. \tag{27}$$

Then the expectation of $|x(k_i)|^c$, $k_i \in \mathbb{K}$ *converges exponentially to a bounded set, i.e., for all* $k_i \in \mathbb{K}$,

$$\mathbf{E}\{|x(k_i)|^c\} \leq \frac{\lambda_F(1-\rho)^i}{\alpha_L}\mathbf{E}\{|x(0)|^s\} + \frac{\sigma}{\rho\alpha_L}\mathbf{E}\{|w(k)|^s\},$$

with $\sigma \in \mathbb{R}$. *Here,* α_L *is given in (19b) and* λ_F, *in (20c).*

PROOF. Proposition 6 establishes that $V(x(k_i))$ is a stochastic Lyapunov-like function for the closed loop at the time instants $k_i \in \mathbb{K}$. In fact, if (27) holds, then Lemma 4 gives that $\phi(x(k_i)) \geq \rho V(x(k_i)), \forall k_i \in \mathbb{K}$, so that, by (26),

$$\mathbf{E}\{V(x(k_1))\,|\,x(0)\} - V(x(0)) \leq \sigma\mathbf{E}\{|w(k)|^s\} - \rho V(x(0)).$$

Given (19) and since $\{x\}_\mathbb{K}$ is Markovian, we can use [52, Prop. 3.2] to conclude that:

$$\begin{aligned}\mathbf{E}\big\{V(x(k_i))\,\big|\,x(0)\big\} \leq\ & (1-\rho)^i V(x(0)) \\ &+ \sigma\mathbf{E}\{|w(k)|^s\}\sum_{\ell=0}^{i-1}(1-\rho)^\ell, \quad \forall i \geq 1.\end{aligned} \tag{28}$$

The result now follows from using (19b), (20c), Lemma 4 and taking expectation; see, e.g., [56, p.341]. □

Whilst Corollary 7 examines only $\{x\}_\mathbb{K}$, i.e., the plant state *at the instants of successful transmission*, the following theorem establishes boundedness of $\mathbf{E}\{|x(k)|^s\}$ for all $k \in \mathbb{N}_0$.

Theorem 8 *Suppose that* (27) *and Assumptions 1 to 4 hold with* $b = c = v = s$. *Then there exist* $C_1, C_2 \in \mathbb{R}_{>0}$ *such that, for all* $i \in \mathbb{N}_0$,

$$\begin{aligned}\max_{k\in\{k_i,k_1+1\dots,k_i+\Delta(i)-1\}} \mathbf{E}\{|x(k)|^s\} \\ \leq C_1(1-\rho)^i\mathbf{E}\{|x(0)|^s\} + C_2\mathbf{E}\{|w(k)|^s\}.\end{aligned} \tag{29}$$

PROOF. See Section 8. □

Theorem 8 constitutes the main result of the present work. It gives a sufficient condition for stochastic stability of the NCS in the presence of disturbances and Markovian dropouts. Our result establishes that, if the conditions of the theorem are met, then $\mathbf{E}\{|x(k)|^s\}$ is bounded for all $k \in \mathbb{N}_0$. Furthermore, by taking $i \to \infty$, the bound in (29) provides:

$$\lim_{k\to\infty}\mathbf{E}\{|x(k)|^s\} \leq C_2\mathbf{E}\{|w(k)|^s\}. \tag{30}$$

Thus, if $\mathbf{E}\{|w(k)|^s\} = 0$, then one obtains that [4]

$$\lim_{k\to\infty}\mathbf{E}\{|x(k)|^s\} = 0. \tag{31}$$

The condition (27) involves the upper bound of the plant growth rate, γ, the dropout distribution parameters p and q, and the cost function parameters N, $F(\cdot)$ and $L(\cdot,\cdot)$. The result confirms that, it is desirable that $q \approx 1$, $p \approx 0$, $\gamma \approx 1$ and N be large. More details on how to choose the design parameters are given in the following section.

Remark 9 *Since (21) is a* global *condition (see also [9]), Assumption 3 will, in general, not be satisfied if the plant model (1) is open-loop unstable outside a bounded region and the constraint set* $\mathbb{U}$ *is bounded; c.f., [25]. To examine such situations and also to incorporate state constraints, it would be desirable to replace (21) by a* local *condition, which needs to hold only in some bounded set, say* $\mathbb{X}_f \subset \mathbb{R}^n$; *see, e.g., [57] and also [45, 46], which use similar ideas for NCSs where* $\Delta(i)$ *and the disturbances are uniformly bounded. In the present case, where disturbances and consecutive dropouts are unbounded, the situation is significantly more difficult to handle. In fact, to prove boundedness of* $\mathbf{E}\{|x(k)|^s\}$ *following along the lines adopted in the present work, one would require that* $\mathbb{X}_f$ *be a robust invariant set of* $\{x\}_\mathbb{K}$, *see (16). Since the maximum number of consecutive dropouts is unbounded, for plants which are open-loop unstable outside a bounded region, this will in general not hold. We conclude that the issue of formulating* local conditions *deserves further study.* □

[4] As shown in [50], if there are no disturbances, then Assumption 2 is not needed to establish (31).

6 Choice of Design Parameters

The stability result in Theorem 8 suggests that one may incorporate (27) as an additional constraint on $\bar{f}^N(x)$ in the minimization of the cost function. The following corollary shows how to design the cost function parameters in (8) such that stochastic stability can be guaranteed without requiring additional constraints in the optimization. Our result also sheds some light into closed loop performance by quantifying the convergence factor $1-\rho$ in (29).

Corollary 10 *If the assumptions of Theorem 8 hold and*

$$F(x) < (1+\psi_N)L(x,u_0(x)), \quad \forall x \in \mathbb{R}^n, \tag{32}$$

then $\{x\}_{\mathbb{N}_0}$ *satisfies (29) for all* $\rho \in (0, \rho_N^\star]$, *where*

$$\rho_N^\star \triangleq \frac{L(x,u_0(x))}{F(x)} + \psi_N^{-1}\left(\frac{L(x,u_0(x))}{F(x)} - 1\right) > 0. \tag{33}$$

PROOF. If (32) is satisfied, and since $\psi_N > 0$, for all $\rho \in (0, \rho_N^\star]$ and all $x \in \mathbb{R}^n$, we have $F(x) \leq (1+\psi_N)L(x,u_0(x))/(1+\rho\psi_N)$, which is equivalent to

$$\rho F(x) \leq \psi_N^{-1}(L(x,u_0(x)) - F(x)) + L(x,u_0(x)). \tag{34}$$

On the other hand, (23) and (17) provide the bound $F\big(\bar{f}^N(x)\big) \leq F(x) - L(x,u_0(x)) - \sum_{\ell=1}^{N-1} L\big(\bar{f}^\ell(x), u_\ell(x)\big) \leq F(x) - L(x,u_0(x))$, which upon substitution into (34) implies (27). The result follows from Theorem 8. □

Corollary 10 allows us to conclude that the NCS will be stable if the weighting functions $F(\cdot)$ and $L(\cdot,\cdot)$ are chosen to be compatible with Assumptions 1 to 4 and the bound in (32). To further elucidate this result, we observe that $\psi_{\ell+1} \geq \psi_\ell$, $\forall \ell \in \mathbb{N}$, and $\lim_{\ell\to\infty} \psi_\ell = \infty$. Thus, choosing larger horizon lengths N is beneficial for fulfilling (32) and hence guaranteeing stability, in the sense of (29). Moreover, for any design which satisfies Assumptions 1 to 4, stability can be ensured if a sufficiently large horizon length is used. In fact, $\{x\}_{\mathbb{N}_0}$ satisfies (29) if $N \in \mathbb{N}$ is chosen such that

$$N > 1 + \log\left(\frac{\tau(1-p\gamma)L(x,u_0(x))}{F(x) - L(x,u_0(x))}\right) \Big/ \log(p), \ \forall x \in \mathbb{R}^n,$$

where $\tau^{-1} \triangleq (1-q)(\gamma-1)$.

The influence of the horizon length N on the convergence factor of the bound (29) can be studied by using (25) as follows: Direct calculations yield that

$$1 - \rho_N^\star = \left(1 - \frac{L(x,u_0(x))}{F(x)}\right)\left(1 + \frac{p^N}{\tau p(1-p\gamma)}\right). \tag{35}$$

Thus, the convergence factor $1-\rho_N^\star$ is exponentially decreasing in N with limiting value

$$\lim_{N\to\infty} 1 - \rho_N^\star = 1 - L(x,u_0(x))/F(x).$$

Example 1 (Adapted from [58, Example 2.3]) *Consider an open-loop unstable plant model of the form (1), where*

$$f(x,u,w) = \begin{bmatrix} x_2 + u_1 \\ -\mathrm{sat}(x_1+x_2) + u_2 \end{bmatrix} + \begin{bmatrix} \sqrt{w^2+5} - \sqrt{5} \\ 0 \end{bmatrix}, \tag{36}$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathrm{sat}(\nu) = \begin{cases} -1, & \textit{if } \nu < -1, \\ \nu & \textit{if } \nu \in [-1,1], \\ 1, & \textit{if } \nu > 1. \end{cases}$$

We fix $|\cdot|$ *as the Euclidean norm. The second component of the plant input is constrained via*

$$|u_2(k)| \leq 0.8, \quad \forall k \in \mathbb{N}_0.$$

The network introduces Markovian packet dropouts with failure rate $1-q=0.4$ *and recovery rate* $1-p=0.45$. *To control this NCS, we wish to use the PPC formulation of Section 3. The weighting functions are chosen as* $F(x) = 2|x|$, $L(x,u) = |x|$, *which satisfy (19) and (20) with* $\alpha_F = \lambda_F = 2$, *and* $\alpha_L = \lambda_L = b = c = s = 1$. *Since*

$$\left|\frac{\mathrm{d}\sqrt{w^2+5}}{\mathrm{d}w}\right| \leq 1, \quad \forall w \in \mathbb{R}$$

and by proceeding as in [58, p.73], we obtain that (36) satisfies (20a) *with* $s=1$, $\lambda_x = 1.618$ *and* $\lambda_w = 1$ *uniformly. Since* $p = 0.55 < 1/\lambda_x \approx 0.618$, *Lemma 5 gives that Assumption 4 holds with* $\eta = 2$, $v = 1$ *and open-loop rate of growth bounded by* $\gamma = 1.618$. *Assumption 3 is also satisfied, since for* $\kappa_f \colon \mathbb{R}^2 \to \mathbb{R} \times [-0.8, 0.8]$ *chosen as* $\kappa_f(x) = \begin{bmatrix} -x_2 & 0.8\mathrm{sat}(x_1+x_2) \end{bmatrix}^T$, *we have*

$$\begin{aligned} &F\big(f(x,\kappa_f(x),0)\big) - F(x) + L(x,\kappa_f(x)) \\ &\quad = 0.4|\mathrm{sat}(x_1+x_2)| - |x| \leq 0.4|x_1+x_2| - |x| \\ &\quad \leq 0.8\max\{|x_1|,|x_2|\} - \max\{|x_1|,|x_2|\} \leq 0. \end{aligned}$$

Given that Assumptions 1 to 4 hold with $b=c=v=s$, *the results established in the present work can be readily applied. In fact, direct calculations give that* $\psi_\ell = 0.245/p^\ell$, *see (25), and condition (32) becomes*

$$2|x| < \big(1 + 0.245/p^N\big)|x|, \ \forall x \in \mathbb{R}^2 \iff p^N < 0.245.$$

Thus, if the horizon and buffer length satisfy $N \geq 3$, *then Corollary 10 establishes that the NCS is stable in the sense of (29). For example, if one chooses* $N=3$, *then the convergence factor provided by (35) is given by* $1 - \rho_3^\star \approx 0.84$. □

7 Proof of Proposition 6

Throughout this section, we will write x instead of $x(0)$, Δ for $\Delta(0)$, and u_ℓ for $u_\ell(x(0))$, see (5). To prove Proposition 6, we will first state, and prove, some technical lemmas.

Lemma 11 *If* (20a) *holds, then:*

$$\mathbf{E}\big\{|x(\ell)-\bar{f}^\ell(x)|^s \,\big|\, x\big\} \leq \lambda_w \sum_{j=0}^{\ell-1} \lambda_x^j \,\mathbf{E}\{|w(k)|^s\} \qquad (37)$$

for all $\ell \in \{1,2,\dots,\Delta\}$*, with* $\Delta \leq N$.

PROOF. Since $x(\ell) = f^\ell(x,\{w\}_0^{\ell-1})$, (20a) gives that

$$\begin{aligned}
&\big|f^\ell(x,\{w\}_0^{\ell-1}) - \bar{f}^\ell(x)\big|^s \\
&\quad = \big|f(f^{\ell-1}(x,\{w\}_0^{\ell-2})), u_{\ell-1}(x), w(\ell-1)) \\
&\qquad\qquad - f(\bar{f}^{\ell-1}(x), u_{\ell-1}(x), 0)\big|^s \\
&\quad \leq \lambda_x \big|f^{\ell-1}(x,\{w\}_0^{\ell-2}) - \bar{f}^{\ell-1}(x)\big|^s + \lambda_w |w(\ell-1)|^s.
\end{aligned}$$

The result follows from noting that $\{w\}_{\mathbb{N}_0}$ is i.i.d. $\square$

Lemma 12 *Define*

$$\beta_{\Delta,\ell} \triangleq \lambda_w \lambda_x^{\ell-\Delta} \sum_{j=0}^{\Delta-1} \lambda_x^j, \quad \tilde{\lambda}_{\Delta,N} \triangleq \lambda_F \beta_{\Delta,N} + \lambda_L \sum_{\ell=\Delta}^{N-1} \beta_{\Delta,\ell}. \qquad (38)$$

If $\Delta \leq N$*, and Assumptions 2 and 3 hold, then:*

$$\begin{aligned}
&\mathbf{E}\big\{V(x(k_1)) \,\big|\, x\big\} - V(x) \\
&\qquad \leq \tilde{\lambda}_{\Delta,N} \mathbf{E}\{|w(k)|^s\} - \sum_{\ell=0}^{\Delta-1} L(\bar{f}^\ell(x), u_\ell(x)).
\end{aligned} \qquad (39)$$

PROOF. We adapt the approach of [46, Lemma 1]. We focus on time instant $k_1 = \Delta$ and introduce the sequence

$$\{x^\sharp\}_\Delta^{N+\Delta} \triangleq \{x_\Delta^\sharp, x_{\Delta+1}^\sharp, \dots, x_N^\sharp, x_{N+1}^\sharp, \dots, x_{N+\Delta}^\sharp\}. \qquad (40)$$

Its first $N-\Delta+1$ elements are defined recursively via:

$$x_{j+1}^\sharp = f\big(x_j^\sharp, u_j(x), 0\big), \quad j \in \{\Delta,\dots,N-1\}, \qquad (41)$$

with $x_\Delta^\sharp = x(k_1)$; the remaining Δ elements are given by

$$\begin{aligned}
x_{j+1}^\sharp &= f\big(x_j^\sharp, u_j^\sharp, 0\big), \\
u_j^\sharp &= \kappa_f(x_j^\sharp), \qquad j \in \{N,\dots,N+\Delta-1\},
\end{aligned} \qquad (42)$$

where $\kappa_f(\cdot)$ is chosen such that (21) holds. We next consider

$$\begin{aligned}
\vec{u}^\sharp = \big\{&u_\Delta(x), u_{\Delta+1}(x), \dots, u_{N-1}(x), \\
&u_N^\sharp, u_{N+1}^\sharp \dots, u_{N+\Delta-1}^\sharp\big\},
\end{aligned}$$

whose first $N-\Delta$ elements are taken from $\vec{u}(x)$, whereas the remaining Δ elements are provided by (42). It follows from (8) that the associated cost satisfies:

$$\begin{aligned}
J(\vec{u}^\sharp, x(k_1)) &= F(x_{N+\Delta}^\sharp) + \sum_{\ell=\Delta}^{N-1} L(x_\ell^\sharp, u_\ell(x)) \\
&\quad + \sum_{\ell=N}^{N+\Delta-1} L(x_\ell^\sharp, u_\ell^\sharp).
\end{aligned} \qquad (43)$$

Since $V(x(k_1)) \leq J(\vec{u}^\sharp, x(k_1))$, for $\Delta < N$ we can bound

$$\begin{aligned}
&\mathbf{E}\big\{V(x(k_1)) \,\big|\, x\big\} - V(x) \leq \mathbf{E}\big\{J(\vec{u}^\sharp, x(k_1)) \,\big|\, x\big\} - V(x) \\
&\quad = \sum_{\ell=\Delta}^{N-1} \Big(\mathbf{E}\big\{L(x_\ell^\sharp, u_\ell(x)) \,\big|\, x\big\} - L(\bar{f}^\ell(x), u_\ell(x))\Big) \\
&\qquad - F(\bar{f}^N(x)) - \sum_{\ell=0}^{\Delta-1} L(\bar{f}^\ell(x), u_\ell(x)) \\
&\qquad + \mathbf{E}\big\{F(x_{N+\Delta}^\sharp) \,\big|\, x\big\} + \sum_{\ell=N}^{N+\Delta-1} \mathbf{E}\big\{L(x_\ell^\sharp, u_\ell^\sharp) \,\big|\, x\big\},
\end{aligned} \qquad (44)$$

where we have used (17). On the other hand, (20a), (41) and Lemma 11 provide that, for $\ell \in \{\Delta, \Delta+1, \dots, N\}$,

$$\mathbf{E}\big\{|x_\ell^\sharp - \bar{f}^\ell(x)|^s \,\big|\, x\big\} \leq \beta_{\Delta,\ell}\, \mathbf{E}\{|w(k)|^s\}, \qquad (45)$$

where $\beta_{\Delta,\ell}$ are defined in (38). By rearranging terms and using (20b), (44) can be rewritten as:

$$\begin{aligned}
&\mathbf{E}\big\{V(x(k_1)) \,\big|\, x\big\} - V(x) \leq \lambda_L \sum_{\ell=\Delta}^{N-1} \beta_{\Delta,\ell}\, \mathbf{E}\{|w(k)|^s\} \\
&\quad - \sum_{\ell=0}^{\Delta-1} L(\bar{f}^\ell(x), u_\ell(x)) + \mathbf{E}\big\{F(x_N^\sharp) \,\big|\, x\big\} - F(\bar{f}^N(x)) \\
&\quad + \sum_{\ell=N}^{N+\Delta-1} \mathbf{E}\big\{L(x_\ell^\sharp, u_\ell^\sharp) + F(x_{\ell+1}^\sharp) - F(x_\ell^\sharp) \,\big|\, x\big\}.
\end{aligned}$$

Use of (42), (21) and (20c) provides (39).

For $\Delta = N$, we consider

$$\vec{u}^\sharp = \{u_N^\sharp, u_{N+1}^\sharp \dots, u_{2N-1}^\sharp\},$$

where all N elements of $\vec{u}^\sharp$ are calculated via (42), but with initial value $x_N^\sharp = x(k_1)$. The bound in (39) then follows as in the case $\Delta < N$ studied above. $\square$

Lemma 13 *Suppose that* $\Delta > N$*, that* (20a), (20c) *and Assumptions 3 and 4 hold with* $v = s$*. Furthermore, define*

$$\Gamma_{\ell,N} \triangleq \eta \sum_{j=0}^{\ell-N-1} \gamma^j, \quad \tilde{\eta}_{\Delta,N} \triangleq \Gamma_{\Delta,N} + \gamma^{\Delta-N}\lambda_F \beta_{N,N}, \tag{46}$$

where $\beta_{N,N}$ *is as in (38) with* $\Delta = \ell = N$*. Then:*

$$\begin{aligned}\mathbf{E}\big\{F(x(\ell))\,\big|\,x\big\} \leq{}& \gamma^{\ell-N}\mathbf{E}\big\{F(x(N))\,\big|\,x\big\} \\ &+ \Gamma_{\ell,N}\mathbf{E}\{|w(k)|^s\},\end{aligned} \tag{47}$$

for all $\ell \in \{N+1,\dots,\Delta\}$*, and*

$$\begin{aligned}&\mathbf{E}\big\{V(x(k_1))\,\big|\,x\big\} - V(x) \leq \tilde{\eta}_{\Delta,N}\mathbf{E}\{|w(k)|^s\} \\ &+ (\gamma^{\Delta-N}-1)F\big(\bar{f}^N(x)\big) - \sum_{\ell=0}^{N-1} L(\bar{f}^\ell(x), u_\ell(x)).\end{aligned} \tag{48}$$

PROOF. With $\ell \in \{N+1,\dots,\Delta\}$, (22) gives:

$$\begin{aligned}&\mathbf{E}\big\{F(f(x(\ell-1),0,w(\ell-1)))\,\big|\,x(\ell-1)\big\} \\ &\qquad \leq \gamma F(x(\ell-1)) + \eta\mathbf{E}\{|w(\ell)|^s\}.\end{aligned} \tag{49}$$

Since $\Delta > N$, by (16), $\{x\}_N^{k_1}$ is Markovian. We thus have:

$$\begin{aligned}\mathbf{E}\big\{F(x(\ell))\,\big|\,x(\ell-1)\big\} &= \mathbf{E}\big\{F(x(\ell))\,\big|\,\{x\}_N^{\ell-1}\big\} \\ &\leq \gamma F(x(\ell-1)) + \eta\mathbf{E}\{|w(\ell)|^s\},\end{aligned}$$

where we have used (49). By taking conditional expectation in the above and using the Markov property, we obtain:

$$\begin{aligned}\mathbf{E}\big\{\mathbf{E}\big\{F(x(\ell))\,\big|\,\{x\}_N^{\ell-1}\big\}\,\big|\,x(N)\big\} &= \mathbf{E}\big\{F(x(\ell))\,\big|\,x(N)\big\} \\ &\leq \gamma\mathbf{E}\big\{F(x(\ell-1))\,\big|\,x(N)\big\} + \eta\mathbf{E}\{|w(\ell)|^s\}.\end{aligned}$$

The above recursion can be applied $\ell - N$ times to give $\Gamma_{\ell,N}\mathbf{E}\{|w(\ell)|^s\} + \gamma^{\ell-N}F(x(N)) \geq \mathbf{E}\big\{F(x(\ell))\,\big|\,x(N)\big\} = \mathbf{E}\big\{F(x(\ell))\,\big|\,x(N),x\big\}$; thus, [51, Eq. 1.28], establishes (47).

To prove (48), we set $\ell = \Delta$ in (47) and use Lemma 4:

$$\begin{aligned}&\mathbf{E}\big\{V(x(k_1))\,\big|\,x\big\} - V(x) \\ &\quad\leq \gamma^{\Delta-N}\mathbf{E}\big\{F(x(N))\,\big|\,x\big\} + \Gamma_{\Delta,N}\mathbf{E}\{|w(k)|^s\} - V(x) \\ &\quad= \Gamma_{\Delta,N}\mathbf{E}\{|w(k)|^s\} + \gamma^{\Delta-N}\mathbf{E}\big\{F(x(N))\,\big|\,x\big\} \\ &\qquad\qquad - F(\bar{f}^N(x)) - \sum_{\ell=0}^{N-1} L(\bar{f}^\ell(x), u_\ell(x)) \\ &\quad\leq \Gamma_{\Delta,N}\mathbf{E}\{|w(k)|^s\} + (\gamma^{\Delta-N}-1)F(\bar{f}^N(x)) \\ &\qquad + \gamma^{\Delta-N}\big(\mathbf{E}\big\{F(x(N))\,\big|\,x\big\} - F(\bar{f}^N(x))\big) \\ &\qquad - \sum_{\ell=0}^{N-1} L(\bar{f}^\ell(x), u_\ell(x)).\end{aligned}$$

Lemma 11 and Equation (20c) now provide (48). □

Having established the above results, Proposition 6 can now be proven by using (13) and conditioning upon Δ as follows:

$$\begin{aligned}&\mathbf{E}\big\{V(x(k_1))\,\big|\,x\big\} = \mathbf{E}\big\{\mathbf{E}\big\{V(x(k_1))\,\big|\,x,\Delta\big\}\big\} \\ &\quad= q\mathbf{E}\big\{V(x(k_1))\,\big|\,x,\Delta=1\big\} \\ &\qquad + (1-q)(1-p)\sum_{i=2}^{\infty} p^{i-2}\mathbf{E}\big\{V(x(k_1))\,\big|\,x,\Delta=i\big\} \\ &\quad= q\mathbf{E}\big\{V(x(k_1))\,\big|\,x,\Delta=1\big\} \\ &\qquad + (1-q)(1-p)\sum_{i=2}^{N} p^{i-2}\mathbf{E}\big\{V(x(k_1))\,\big|\,x,\Delta=i\big\} \\ &\qquad + (1-q)(1-p)\sum_{i=N+1}^{\infty} p^{i-2}\mathbf{E}\big\{V(x(k_1))\,\big|\,x,\Delta=i\big\}.\end{aligned}$$

If we now utilize (39) and (48), then:

$$\begin{aligned}&\mathbf{E}\big\{V(x(k_1))\,\big|\,x\big\} - V(x) \\ &\quad\leq q\big(\tilde{\lambda}_{1,N}\mathbf{E}\{|w(k)|^s\} - L(x,u_0(x))\big) \\ &\qquad + (1-q)(1-p)\sum_{i=2}^{N} p^{i-2}\Big(\tilde{\lambda}_{i,N}\mathbf{E}\{|w(k)|^s\} \\ &\qquad\qquad - \sum_{\ell=0}^{i-1} L(\bar{f}^\ell(x), u_\ell(x))\Big) \\ &\qquad + (1-q)(1-p)\sum_{i=N+1}^{\infty} p^{i-2}\Big(\tilde{\eta}_{i,N}\mathbf{E}\{|w(k)|^s\} \\ &\qquad + (\gamma^{i-N}-1)F(\bar{f}^N(x)) - \sum_{\ell=0}^{N-1} L(\bar{f}^\ell(x), u_\ell(x))\Big).\end{aligned}$$

We next define

$$\begin{aligned}\sigma \triangleq{}& q\tilde{\lambda}_{1,N} + (1-q)(1-p)\sum_{i=2}^{N} p^{i-2}\tilde{\lambda}_{i,N} \\ &+ (1-q)(1-p)\sum_{j=N+1}^{\infty} p^{i-2}\tilde{\eta}_{j,N},\end{aligned} \tag{50}$$

which is finite, since, by Assumption 4, $p\gamma < 1$. Therefore,

$$\begin{aligned}&\mathbf{E}\big\{V(x(k_1))\,\big|\,x\big\} - V(x) \\ &\quad\leq \sigma\mathbf{E}\{|w(k)|^s\} - qL(x,u_0(x)) \\ &\qquad - (1-q)(1-p)\sum_{i=2}^{N} p^{i-2}\sum_{\ell=0}^{i-1} L(\bar{f}^\ell(x), u_\ell(x)) \\ &\qquad + (1-q)(1-p)\sum_{i=N+1}^{\infty} p^{i-2}(\gamma^{i-N}-1)F(\bar{f}^N(x)) \\ &\qquad - (1-q)(1-p)\sum_{i=N+1}^{\infty} p^{i-2}\sum_{\ell=0}^{N-1} L(\bar{f}^\ell(x), u_\ell(x)),\end{aligned}$$

which gives:

$$\begin{aligned}\mathbf{E}\big\{V(x(k_1))\,\big|\,x\big\}-V(x)&\le\sigma\mathbf{E}\{|w(k)|^s\}\\&\quad-\bigg(q+(1-q)(1-p)\sum_{i=2}^{\infty}p^{i-2}\bigg)L(x,u_0(x))\\&\quad+(1-q)(1-p)\sum_{i=N+1}^{\infty}p^{i-2}(\gamma^{i-N}-1)F(\bar f^N(x))\\&\quad-(1-q)(1-p)\sum_{i=N+1}^{\infty}p^{i-2}\sum_{\ell=1}^{N-1}L(\bar f^\ell(x),u_\ell(x))\\&=\sigma\mathbf{E}\{|w(k)|^s\}-(1-q)p^{N-1}\sum_{\ell=1}^{N-1}L(\bar f^\ell(x),u_\ell(x))\\&\quad-L(x,u_0(x))+(1-q)p^{N-1}\bigg(\frac{\gamma-1}{1-p\gamma}\bigg)F(\bar f^N(x)),\end{aligned}$$

since $p\gamma<1$. This completes the proof of Proposition 6. □

8 Proof of Theorem 8

We will denote $x(0)$ by x, and $\Delta(0)$ by Δ. By Corollary 7, (28) holds for all $k_i\in\mathbb{K}$. We next examine instants $k\notin\mathbb{K}$. For that purpose, we condition upon Δ to obtain:

$$\begin{aligned}\sum_{j=0}^{k_1-1}\mathbf{E}\big\{|x(j)|^s\,\big|\,x\big\}&=\sum_{\ell=0}^{\Delta-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x\big\}\\&=q\mathbf{E}\big\{|x|^s\,\big|\,x,\Delta=1\big\}\\&\quad+(1-q)(1-p)\sum_{i=2}^{\infty}p^{i-2}\sum_{\ell=0}^{i-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=i\big\}\\&=q\mathbf{E}\big\{|x|^s\,\big|\,x,\Delta=N\big\}\\&\quad+(1-q)(1-p)\sum_{i=2}^{N}p^{i-2}\sum_{\ell=0}^{i-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=N\big\}\\&\quad+(1-q)(1-p)\sum_{i=N+1}^{\infty}p^{i-2}\sum_{\ell=0}^{i-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=i\big\}\\&\le q\sum_{\ell=0}^{N-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=N\big\}\\&\quad+(1-q)(1-p)\sum_{i=2}^{\infty}p^{i-2}\sum_{\ell=0}^{N-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=N\big\}\\&\quad+(1-q)(1-p)\sum_{i=N+1}^{\infty}p^{i-2}\sum_{\ell=N}^{i-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=i\big\}\\&=\sum_{\ell=0}^{N-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=N\big\}\\&\quad+(1-q)(1-p)\sum_{i=N+1}^{\infty}p^{i-2}\sum_{\ell=N}^{i-1}\mathbf{E}\big\{|x(\ell)|^s\,\big|\,x,\Delta=i\big\}.\end{aligned}$$

Expression (19a), then provides the bound:

$$\begin{aligned}&\sum_{j=0}^{k_1-1}\mathbf{E}\big\{|x(j)|^s\,\big|\,x\big\}\\&\quad\le\sum_{\ell=0}^{N-1}\mathbf{E}\big\{|x(\ell)-\bar f^\ell(x)+\bar f^\ell(x)|^s\,\big|\,x,\Delta=N\big\}\\&\qquad+\frac{(1-p)}{\alpha_F/(1-q)}\sum_{i=N+1}^{\infty}p^{i-2}\sum_{\ell=N}^{i-1}\mathbf{E}\big\{F(x(\ell))\,\big|\,x,\Delta=i\big\}.\end{aligned}$$

On the other hand, for all $y,z\in\mathbb{R}^n$ and all $s\in\mathbb{R}_{>0}$,

$$\begin{aligned}|y+z|^s&\le|2\max(|y|,|z|)|^s+|2\min(|y|,|z|)|^s\\&=2^s\big(|y|^s+|z|^s\big)\end{aligned}$$

Lemmas 11 and 13 then give that

$$\begin{aligned}\sum_{j=0}^{k_1-1}\mathbf{E}\big\{|x(j)|^s\,\big|\,x\big\}&\le2^s\lambda_w\sum_{\ell=0}^{N-1}\sum_{j=0}^{\ell-1}\lambda_x^j\,\mathbf{E}\{|w(k)|^s\}\\&\quad+2^s\sum_{\ell=0}^{N-1}\mathbf{E}\big\{|\bar f^\ell(x)|^s\,\big|\,x,\Delta=N\big\}\\&\quad+\frac{1-p}{\alpha_F/(1-q)}\sum_{i=N+1}^{\infty}p^{i-2}\sum_{\ell=N}^{i-1}\gamma^{\ell-N}\mathbf{E}\big\{F(x(N))\,\big|\,x\big\}\\&\quad+\frac{1-p}{\alpha_F/(1-q)}\sum_{i=N+1}^{\infty}p^{i-2}\sum_{\ell=N}^{i-1}\Gamma_{\ell,N}\mathbf{E}\{|w(k)|^s\}.\end{aligned}\tag{51}$$

By Assumption 4, $p\gamma<1$. Therefore, it is easy to see that all sums in (51) are convergent. Furthermore, (20c) provides

$$F(x(N))\le\lambda_F|x(N)-\bar f^N(x)|^s+F(\bar f^N(x)).$$

Having established the above, we can use Lemma 11 to conclude that there exist finite constants C_1' and C_3, which are independent of x and provide the bound $\sum_{j=0}^{k_1-1}\mathbf{E}\big\{|x(j)|^s\,\big|\,x\big\}\le C_1'V(x)+C_3\mathbf{E}\{|w(k)|^s\}$. In a similar manner, it can be shown that $\exists C_1',C_3\in\mathbb{R}$ such that

$$\sum_{j=k_i}^{k_{i+1}-1}\mathbf{E}\big\{|x(j)|^s\,\big|\,x(k_i)\big\}\le C_1'V(x(k_i))+C_3\mathbf{E}\{|w(k)|^s\},$$

for all $k_i\in\mathbb{K}$. Since $\{x\}_{\mathbb{K}}$ is Markovian, by taking conditional expectation and using (28), we have

$$\sum_{j=k_i}^{k_{i+1}-1}\mathbf{E}\big\{|x(j)|^s\,\big|\,x\big\}\le C_1'(1-\rho)^iV(x)+C_2\mathbf{E}\{|w(k)|^s\},$$

for all $k_i \in \mathbb{K}$, and where

$$C_2 \triangleq \sigma C_1' \sum_{\ell=0}^{i-1} (1-\rho)^\ell + C_3 \leq \frac{\sigma C_1'}{\rho} + C_3 < \infty.$$

As a consequence, it holds that

$$\begin{aligned} &\max_{j \in \{k_i, k_i+1, \dots, k_i+\Delta(i)-1\}} \mathbf{E}\{|x(j)|^s \mid x\} \\ &\qquad\qquad \leq C_1'(1-\rho)^i V(x) + C_3 \mathbf{E}\{|w(k)|^s\}, \end{aligned}$$

which gives

$$\begin{aligned} &\max_{j \in \{k_i, k_i+1, \dots, k_i+\Delta(i)-1\}} \mathbf{E}\{|x(j)|^s\} \\ &\qquad \leq C_1'(1-\rho)^i \mathbf{E}\{V(x)\} + C_2 \mathbf{E}\{|w(k)|^s\} \\ &\qquad \leq C_1'(1-\rho)^i \lambda_F \mathbf{E}\{|x|^s\} + C_2 \mathbf{E}\{|w(k)|^s\}, \end{aligned}$$

where we have used (20c) and (23). Setting $C_1 = C_1' \lambda_F$, establishes (29).

9 Conclusions

This work has studied a NCS architecture where a packetized predictive controller uses an unreliable network affected by Markovian packet-dropouts to control a nonlinear plant with unbounded disturbances. It has been shown that, provided that the plant and network satisfy suitable conditions, stochastic stability of the closed loop can be ensured by appropriate choice of tuning parameters. Future research could include the study of more general NCSs, including where the controller does not have access to the plant state.

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