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On finite gain $\mathcal{L}_p$ stability of nonlinear sampled-data systems

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Abstract

It is shown that uniform global exponential stability of the input-free discrete-time model of a globally Lipschitz sampled-data time-varying nonlinear system with inputs implies finite gain $\mathcal{L}_p$ stability of the sampled-data system for all $p \in [1, \infty]$. This result generalizes results on $\mathcal{L}_p$ stability of sampled-data linear systems and it is an important tool for analysis of robustness of sampled-data nonlinear systems with inputs.

Keywords: discrete-time, $\mathcal{L}_p$ stability, sampled-data, nonlinear systems, time-varying

1 Introduction

Prevalence of computer controlled systems strongly motivates investigation of sampled-data control systems. Moreover, due to the fact that the plant model or the control law are often nonlinear, we often need to consider nonlinear sampled-data systems. While the area of linear sampled-data systems has matured into a well understood and developed discipline (see [1]), a range of open problems still remains in the area of nonlinear sampled-data systems. In particular, a complete analysis of $\mathcal{L}_p$ stability properties of nonlinear sampled-data systems with inputs appears to be lacking in the literature.

One of the first results on $\mathcal{L}_2$ stability of nonlinear sampled-data systems that we are aware of can be found in [7]. A result on $\mathcal{L}_\infty$ of linear sampled-data systems can be found in [3] and a complete characterization of $\mathcal{L}_p$ stability for any $p \in [1, \infty]$ of linear time-invariant and time-varying sampled-data systems can be found respectively in [2] and [5]. Related results on integral stability properties with nonlinear gains, such as input-to-state stability (ISS) and integral input-to-state stability (iISS), for sampled-data systems with inputs were addressed respectively in [10, 9, 13] and [8]. In particular, preservation of the ISS property under discretization (emulation) of the dynamic controllers for nonlinear sampled-data systems were presented in [9, 13]. Results on achieving iISS and ISS for nonlinear sampled-data systems via their approximate discrete-time models were considered respectively in [8] and [10].

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It is the purpose of this paper to present a result on $\mathcal{L}_p$ stability of globally Lipschitz nonlinear sampled-data systems with inputs. In particular, it is shown that if the discrete-time model of the input-free sampled-data system is uniformly globally exponentially stable, then the sampled-data nonlinear system with inputs is $\mathcal{L}_p$ stable for any $p \in [1, \infty]$. This result generalizes similar results on $\mathcal{L}_p$ stability of linear time-invariant and time-varying sampled-data systems in [2] and [5], respectively, and it is an important tool in analysis of robustness properties of sampled-data nonlinear systems. Moreover, our proof technique is based on Lyapunov arguments and it is different from the proof technique exploited in [2, 5]. We present detailed proofs only for global results and then comment on how the same proof technique applies to local results. We also apply our results to the case where the sampled-data system arises in feedback control schemes using discrete-time, dynamic controllers.

The paper is organized as follows. Preliminaries are presented in Section 2. Section 3 contains the main result and a discussion on how the same technique can be used to address several related problems. The proof of the main result is presented in Section 4 and proofs of some auxiliary results can be found in the appendix.

1.1 Notation

We use $\mathbb{Z}_{\geq j}$ to denote all integers greater than or equal to the integer j. For a function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, we define the $\mathcal{L}_p$ norm of $v(\cdot)$ as follows:

$$\|v(\cdot)\|_{\mathcal{L}_p} := \left(\int_0^\infty |v(t)|^p dt \right)^{1/p} \qquad \text{for } p \in [1, \infty)$$

and $\|v(\cdot)\|_{\mathcal{L}_\infty} := \text{ess.sup.}_{t \geq 0} |v(t)|$, where the underlying vector norm is, without loss of generality, the Euclidean norm. Similarly, but in the discrete-time setting, given a sequence $\nu : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$, we define the ℓ_p norm of $\nu(\cdot)$ as:

$$\|\nu(\cdot)\|_{\ell_p} := \left(\sum_{k=0}^{\infty} |\nu(k)|^p \right)^{1/p} \qquad \text{for } p \in [1, \infty)$$

and $\|\nu(\cdot)\|_{\ell_\infty} := \sup_{k \geq 0} |\nu(k)|$.

2 Preliminaries

In this paper, we consider explicitly time-varying sampled-data systems with inputs

$$\begin{aligned} \dot{x} &= f(x(t), x(\lfloor t \rfloor_T), t, u(t)) \\ \lfloor t \rfloor_T &= T \max \left\{ j \in \mathbb{Z} : \; j \leq \frac{t}{T} \right\} \; . \end{aligned} \qquad (1)$$

In this system, T is the sampling period, u is an exogenous input and x is the "state" (more precisely, values of x at the initial time $t_\circ$ and at the possibly earlier time $\lfloor t_\circ \rfloor_T$ are needed to compute the solution forward in time) which, in a closed-loop control problem, may include some (possibly discrete-time) controller dynamics. The right-hand side's dependence on $x(\lfloor t \rfloor_T)$ may be due to the sample and hold nature of the control system whose sampling times are fixed along the t axis. See, for example, Section 3.2.

An equivalent representation of (1) which we will use is given by

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t_s(t)), p(t), u(t)) \\ \dot{p}(t) &= 1 \\ t_s(t) &= \lfloor p(t) \rfloor_T - p(0) \end{aligned} \tag{2}$$

where the initial time t_0 is taken to be zero without loss of generality. For this system, the sampling times are fixed along the p axis but their locations along the t axis depend on the initial value $p(0)$. We can enumerate the sampling times of interest as

$$t_k := t_s((k+1)T) = (k+1)T - \sigma \qquad k \in \mathbb{Z}_{\geq -1} \tag{3}$$

where $\sigma := -t_s(0)$ represents the distance between the nearest sampling time not in the future and $t = 0$. See Figure 1 for further clarification.

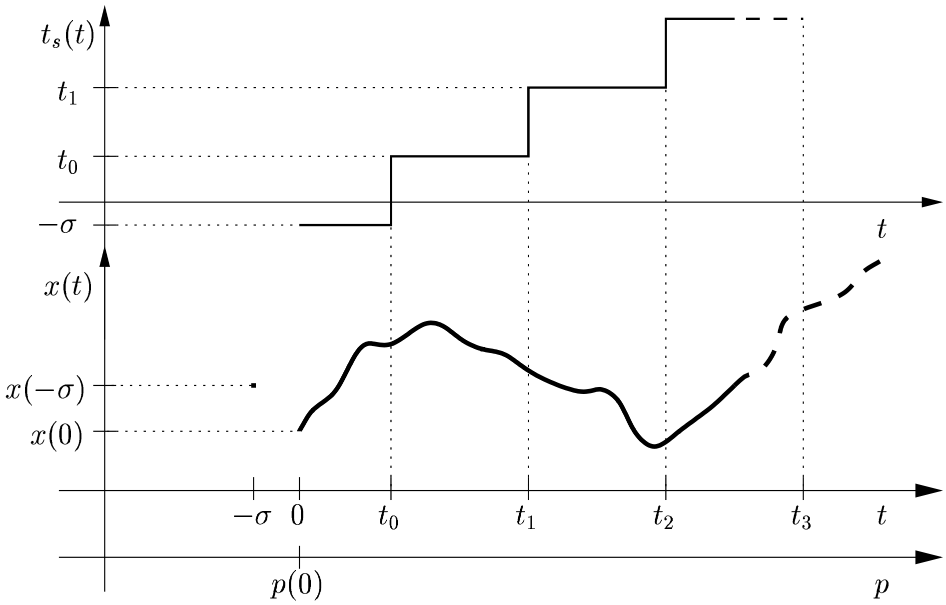

Figure 1: A trajectory of the sampled-data system with indications of its initial conditions.

Our main results for the system (1), equivalently (2), will establish different finite gain $\mathcal{L}_p$ stability ($p \in [1, \infty)]$) properties from $u(\cdot)$ to $x(\cdot)$. In particular, we will use the following:

Definition 1 *The sampled-data system (2) is said to be*

1. *finite gain $\mathcal{L}_p$ to $(\mathcal{L}_p, \mathcal{L}_\infty)$ stable if there exists $c > 0$ such that for all $p(0) \geq 0$, $x(0) \in \mathbb{R}^n$, $x(t_s(0)) \in \mathbb{R}^n$ and u,*

$$\max\{\|x(\cdot)\|_{\mathcal{L}_p}, \|x(\cdot)\|_{\mathcal{L}_\infty}\} \leq c\left(|x(0)| + |x(t_s(0))| + \|u\|_{\mathcal{L}_p}\right) . \tag{4}$$

2. *finite gain $\mathcal{L}_p$ to (ℓ_p, ℓ_∞) stable if there exists $c > 0$ such that for all $p(0) \geq 0$, $x(0) \in \mathbb{R}^n$, $x(t_s(0)) \in \mathbb{R}^n$ and u,*

$$\max\{\|\xi(\cdot)\|_{\ell_p}, \|\xi(\cdot)\|_{\ell_\infty}\} \leq c\left(|x(0)| + |x(t_s(0))| + \|u\|_{\mathcal{L}_p}\right), \tag{5}$$

where $\xi(k) := x(t_k)$, for all $k \geq 0$.

We will need the following assumption on the regularity and growth of f:

Assumption 1 *The function $f(\cdot,\cdot,\cdot,\cdot)$ is globally Lipschitz in its first two arguments uniformly in its third and fourth arguments, measurable in its third argument, continuous in its fourth argument, $f(0,0,p,0)=0$ for all $p\geq 0$ and, for all (x_1,x_2,u) and $p\geq 0$,*

$$|f(x_1,x_2,p,u)-f(x_1,x_2,p,0)|\leq L|u| \ . \tag{6}$$

In order to state our main results we use the stability properties of the zero-input discrete-time model of (1) or (2), which is generated by (1) with initial times satisfying $t_\circ=\lfloor t_\circ\rfloor_T$ or by (2) with initial times satisfying $\lfloor p(0)\rfloor_T - p(0) = 0$. The discrete-time model of (2) with $u\equiv 0$ uses

$$\begin{array}{rcl} \phi(\tau,\xi,\varrho) & := & \xi+\displaystyle\int_0^\tau f(\phi(s,\xi,\varrho),\xi,q(s,\xi,\varrho),0)ds \\ q(\tau,\xi,\varrho) & := & \varrho+\tau \end{array} \tag{7}$$

(these definitions are well-posed due to Assumption 1) and is defined by

$$\left.\begin{array}{rcl} \xi^+ & = & \phi(T,\xi,\varrho) \\ \varrho^+ & = & \varrho+T \end{array}\right\} =: G(\xi,\varrho) \tag{8}$$

The motivation for calling this the zero-input discrete-time model corresponding to (2) is that, with the definition of sampling times given in (3), the trajectories of (2) with $u\equiv 0$ satisfy

$$\left[\begin{array}{c} x(t_{k+1}) \\ p(t_{k+1}) \end{array}\right] = G(x(t_k),p(t_k)) \qquad \forall k\in\mathbb{Z}_{\geq 0} \ . \tag{9}$$

To state our main results, we will use the following stability property of (8):

Definition 2 *The system (8) is uniformly globally exponentially stable (UGES) if there exist $M>0$ and $\lambda\in(0,1)$ such that for all $\xi(0)\in\mathbb{R}^n$, $\varrho(0)\geq 0$, and all $k\in\mathbb{Z}_{\geq 0}$, the solutions of (8) satisfy*

$$|\xi(k)|\leq M|\xi(0)|\lambda^k \ . \tag{10}$$

3 Main results

In this section we present our main result, which states that uniform global exponential stability of the zero-input discrete-time model of (2) implies $\mathcal{L}_p$ stability of the sampled-data system for any $p\in[1,\infty]$. The proof of this result is postponed until the next section. Then, in the second part of this section we discuss several possible generalizations of our results and relation to some existing results in the literature. The main result of this section is stated next.

Theorem 1 *Suppose that Assumption 1 holds and the system (8) is UGES. Then, the system (2) is*

1. *finite gain $\mathcal{L}_p$ to (ℓ_p,ℓ_∞) stable and*
2. *finite gain $\mathcal{L}_p$ to $(\mathcal{L}_p,\mathcal{L}_\infty)$ stable.*

Proof. See Section 4.1. •

In the next two subsections, we state two local versions of the global result established in Theorem 1, and we explicitly discuss the application of the results to the case where the sampled-data system is characterized by a dynamic (discrete-time) feedback function. In all cases, we also discuss how our results compare to some existing results in the literature.

3.1 Local results

Our methods also apply to the investigation of local stability properties of nonlinear systems that are only locally Lipschitz and whose zero-input discrete-time model is only uniformly locally exponentially stable, according to the following definition, which generalizes the property in Definition 2.

Definition 3 *The system (8) is uniformly locally exponentially stable (ULES) if there exist $M > 0$, $c > 0$ and $\lambda \in (0,1)$ such that the solutions of (8) satisfy (10) for all $\xi(0) \in \mathbb{R}^n$ with $|\xi(0)| \leq c$, $\varrho(0) \geq 0$, and all $k \in \mathbb{Z}_{\geq 0}$.*

A first natural extension of Theorem 1 is to give sufficient conditions for the sampled-data system (2) to satisfy the following local version of the stability property in Definition 1.

Definition 4 *The sampled-data system (2) is said to be*

1. *small signal finite gain $\mathcal{L}_p$ to $(\mathcal{L}_p, \mathcal{L}_\infty)$ stable if there exist $c > 0$, $d_1 > 0$ and $d_2 > 0$ such that (4) holds for all $p(0) \geq 0$, $|x(0)| \leq d_1$, $|x(t_s(0))| \leq d_1$, $\|u(\cdot)\|_{\mathcal{L}_p} \leq d_2$.*
2. *small signal finite gain $\mathcal{L}_p$ to (ℓ_p, ℓ_∞) stable if there exist $c > 0$, $d_1 > 0$ and $d_2 > 0$ such that (5) holds for all $p(0) \geq 0$, $|x(0)| \leq d_1$, $|x(t_s(0))| \leq d_1$, $\|u(\cdot)\|_{\mathcal{L}_p} \leq d_2$.*

The local result can then be stated based on the following relaxed version of Assumption 1. Based on this assumption, we are able to prove the forthcoming Theorem 2, which is a first local version of Theorem 1.

Assumption 2 *The function f is locally Lipschitz in its first two arguments, uniformly in its third and fourth arguments, measurable in its third argument, continuous in its fourth argument, $f(0,0,p,0) = 0$ for all $p \geq 0$, and there exists $\delta > 0$ such that for all $|x_1| \leq \delta$ and $|x_2| \leq \delta$, $\varrho \geq 0$ and for all u, the bound (6) holds.*

Theorem 2 *Suppose that Assumption 2 holds and the system (8) is ULES. Then, the system (2) is*

1. *small signal finite gain $\mathcal{L}_p$ to (ℓ_p, ℓ_∞) stable and*
2. *small signal finite gain $\mathcal{L}_p$ to $(\mathcal{L}_p, \mathcal{L}_\infty)$ stable.*

Proof. See Section 4.2. •

To establish Theorem 2 we impose that the bound (6) holds for small x_1, x_2 and *all* u. This is assumed because signals that have small $\mathcal{L}_p$ norm ($p < \infty$) may have arbitrarily large $\mathcal{L}_\infty$ norm. We can relax Assumption 2, asking that the bound (6) holds for small x_1, x_2 and small u, if we change the input-output stability definition so that only inputs with sufficiently small $\mathcal{L}_\infty$ norm are considered:

Definition 5 *The sampled-data system (2) is said to be*

1. *small signal finite gain $\mathcal{L}_{p,\infty}$ to $(\mathcal{L}_p, \mathcal{L}_\infty)$ stable if there exist $c > 0$, $d_1 > 0$ and $d_2 > 0$ such that (4) holds for all $p(0) \geq 0$, $|x(0)| \leq d_1$, $|x(t_s(0))| \leq d_1$, $\|u(\cdot)\|_{\mathcal{L}_\infty} \leq d_2$.*
2. *small signal finite gain $\mathcal{L}_{p,\infty}$ to (ℓ_p, ℓ_∞) stable if there exist $c > 0$, $d_1 > 0$ and $d_2 > 0$ such that (5) holds for all $p(0) \geq 0$, $|x(0)| \leq d_1$, $|x(t_s(0))| \leq d_1$, $\|u(\cdot)\|_{\mathcal{L}_\infty} \leq d_2$.*

Definition 5 enforces an alternative small signal bound on the infinity norm of u. (Note that when $p = \infty$, Definition 5 coincides with Definition 4.) The stability property in Definition 5 allows to draw conclusions that have interesting connections with standard continuous time input-output stability results (see the following Remark 1). As a matter of fact, if we are interested in guaranteeing the input-output properties in Definition 5 for the sampled-data system (2), then the following relaxed version of Assumption 2 is sufficient, as stated in the forthcoming Theorem 3.

Assumption 3 *The function f is locally Lipschitz in its first two arguments, uniformly in its third and fourth arguments, measurable in its third argument, continuous in its fourth argument, $f(0,0,p,0) = 0$ for all $p \geq 0$ and there exists $\delta > 0$ such that for all $|x_1| \leq \delta$, $|x_2| \leq \delta$, $\varrho \geq 0$ and $|u| \leq \delta$, the bound (6) holds.*

Theorem 3 *Suppose that Assumption 3 holds and the system (8) is ULES. Then, the system (2) is*

1. *small signal finite gain $\mathcal{L}_{p,\infty}$ to (ℓ_p, ℓ_∞) stable and*
2. *small signal finite gain $\mathcal{L}_{p,\infty}$ to $(\mathcal{L}_p, \mathcal{L}_\infty)$ stable.*

Proof. See Section 4.2. •

Remark 1 Theorem 3 is a sampled-data version of the continuous-time result [6, Theorem 6.1]. Moreover, it was shown in [4] that ULES of (1) can be deduced from ULES of the zero-input discrete-time model of the linearization of the system (1). Hence, combining our results in Theorem 3 with results of [4] we conclude that ULES of the discrete-time model of the linearization of the system (1) implies small signal finite gain $\mathcal{L}_{p,\infty}$ to (ℓ_p, ℓ_∞) stability and $\mathcal{L}_{p,\infty}$ to $(\mathcal{L}_p, \mathcal{L}_\infty)$ stability of the nonlinear sampled-data system (1) for any $p \in [1, \infty]$. ◦

3.2 Application: dynamic feedback case

Our results cover the case of dynamic feedback if the system has the following form:

$$\begin{aligned} \dot{x}_P(t) &= f_P(x_P(t), \psi(t), t, u(t)) \\ z(\lfloor t \rfloor_T + T) &= f_C(x_P(\lfloor t \rfloor_T), z(\lfloor t \rfloor_T)) \\ \psi(t) &= \Psi(x_P(\lfloor t \rfloor_T), z(\lfloor t \rfloor_T)) \ , \end{aligned} \tag{11}$$

where the first equation models the plant dynamics, the second equation models the discrete-time controller dynamics and ψ is the control signal (output of the controller that is passed through a zero order hold). Under appropriate assumptions on f_P, f_C and Ψ we can apply our results. Indeed, to see this we introduce $\widetilde{f}_C(x_P, z) := \frac{1}{T}(f_C(x_P, z) - z)$ and $\zeta(t) := z(\lfloor t \rfloor_T) + (t - \lfloor t \rfloor_T)\widetilde{f}_C(x_P(\lfloor t \rfloor_T), z(\lfloor t \rfloor_T))$. The definition of the variable ζ is similar to that of the "numerical interpolant" that has found a widespread use in numerical analysis literature (see [12, Definition 7.2.1]). Note that $\zeta(\lfloor t \rfloor_T) = z(\lfloor t \rfloor_T)$ and the variable ζ is piecewise linear in t and hence it is absolutely continuous in t. Hence, we can write for almost all t:

$$\dot{\zeta} = \widetilde{f}_C(x_P(\lfloor t \rfloor_T), \zeta(\lfloor t \rfloor_T)) \ . \tag{12}$$

Consider now the system

$$\begin{aligned} \dot{x}_P &= f_P(x_P(t), \Psi(x_P(\lfloor t \rfloor_T), \zeta(\lfloor t \rfloor_T)), t, u(t)) \\ \dot{\zeta} &= \widetilde{f}_C(x_P(\lfloor t \rfloor_T), \zeta(\lfloor t \rfloor_T)) \ , \end{aligned} \tag{13}$$

which has the same form as (1) if we identify a new "state" $x := (x_P, \zeta)$ and a new right hand side of continuous-time part of the model

$$f(x(t), x(\lfloor t \rfloor_T), t, u(t)) := \begin{pmatrix} f_P(x_P(t), \Psi(x_P(\lfloor t \rfloor_T), \zeta(\lfloor t \rfloor_T)), t, u(t)) \\ \widetilde{f}_C(x_P(\lfloor t \rfloor_T), \zeta(\lfloor t \rfloor_T)) \end{pmatrix} . \tag{14}$$

Then we can state the following result:

Corollary 1 *Suppose that Assumption 1 holds for the function (14) and the zero-input discrete-time model of the system (11) is UGES. Then, the system (11) is*

1. *finite gain* $\mathcal{L}_p$ *to* (ℓ_p, ℓ_∞) *stable and*
2. *finite gain* $\mathcal{L}_p$ *to* $(\mathcal{L}_p, \mathcal{L}_\infty)$ *stable.*

Note that a sufficient condition for Assumption 1 to hold for the function (14) is that $f_P, \widetilde{f}_C, \Psi$ are all globally Lipschitz and zero at zero, uniformly in t.

Remark 2 The above corollary generalizes [2, Corollary 4] when $f_P, \widetilde{f}_C, \Psi$ are linear and time invariant and [5, Propositions 6 and 7] when $f_P, \widetilde{f}_C, \Psi$ are linear and time varying. To see this, we note that the state of the system (11) that was used in [2, 5] is:

$$\tilde{x}(t) := (x_P^T(t)\ \Psi^T(x_P(\lfloor t \rfloor_T), z(\lfloor t \rfloor_T))\ z^T(\lfloor t \rfloor_T))^T$$

and the cited results prove that stability of the input-free discrete-time model of (11) implies that $u \in \mathcal{L}_p$ yields $\tilde{x} \in \mathcal{L}_p$ for all $p \in [1, \infty]$. Note that in the linear case Ψ is globally Lipschitz and zero at zero since it is linear. Moreover, whenever $u \in \mathcal{L}_p$, then item 2 of Corollary 1 implies that $x_P \in \mathcal{L}_p$ and since Ψ is linear item 1 of Corollary 1 implies that $\Psi(x_P(\lfloor t \rfloor_T), z(\lfloor t \rfloor_T))$ and $z(\lfloor t \rfloor_T)$ are ℓ_p. Hence, we can conclude that $\tilde{x} \in \mathcal{L}_p, p \in [1, \infty]$. Note that our conclusions are somewhat stronger than the results of the cited references since, for instance, we can also conclude from Corollary 1 that $u \in \mathcal{L}_p$ for $p \in [1, \infty)$ implies $\tilde{x} \in \mathcal{L}_\infty$, which is not explicitly stated in [2, 5]. $\circ$

The following corollaries are the local generalizations of Corollary 1 corresponding, respectively, to Theorems 2 and 3.

Corollary 2 *Suppose that Assumption 2 holds for the function (14) and the zero-input discrete-time model of the system (11) is ULES. Then, the system (11) is*

1. *small signal finite gain* $\mathcal{L}_p$ *to* (ℓ_p, ℓ_∞) *stable and*
2. *small signal finite gain* $\mathcal{L}_p$ *to* $(\mathcal{L}_p, \mathcal{L}_\infty)$ *stable.*

Corollary 3 *Suppose that Assumption 3 holds for the function (14) and the zero-input discrete-time model of the system (11) is ULES. Then, the system (11) is*

1. *small signal finite gain* $\mathcal{L}_{p,\infty}$ *to* (ℓ_p, ℓ_∞) *stable and*
2. *small signal finite gain* $\mathcal{L}_{p,\infty}$ *to* $(\mathcal{L}_p, \mathcal{L}_\infty)$ *stable.*

Note that a sufficient condition for Assumption 3 to hold for the function (14) is that $f_P, \widetilde{f}_C, \Psi$ are all locally Lipschitz and zero at zero, uniformly in t. On the other hand, for Assumption 2 to hold, we can impose a uniform sector growth property on $f_P(x_P, \psi, t, \cdot) - f_P(x_P, \psi, t, 0)$.

Remark 3 We emphasize that the plant in (11) is strictly proper with respect to the disturbance input u (i.e., the disturbance u does not affect the controller dynamics directly). This structure is crucial since there exists a linear counterexample (see [2]) which shows that if the plant is not strictly proper, then $\mathcal{L}_p$ stability can not be achieved. If the plant is not strictly proper, then one can insert a continuous-time strictly proper stable filter at the output of the plant to make the plant+filter system strictly proper with respect to the disturbance and then our results apply. This approach was taken, for instance, in [2, 5]. We also note that input-output results for strictly proper plants with outputs easily follow from our input-to-state results. ◦

4 Proof of Main Results

We will make use of the following fact which is proved using Holder's inequality.

Fact 1 *Let the sequence t_k, $k \in \mathbb{Z}_{\geq -1}$ be such that $t_{-1} \leq 0$ and $t_{k+1} - t_k = T$ for all $k \in \mathbb{Z}_{\geq -1}$. Given a function $u(\cdot)$ defined on $[t_{-1}, \infty)$ with $u(t) = 0$ for all $t \in [t_{-1}, 0)$, define*

$$\widetilde{\nu}(k) := \int_0^T |u(t_k + \tau)| d\tau \qquad \forall k \in \mathbb{Z}_{\geq -1} \; . \tag{15}$$

Then, for each $p \in [1, \infty]$ (where for $p = \infty$ we let $\frac{p-1}{p} = 1$),

$$||\widetilde{\nu}(\, \cdot \, - 1)||_{\ell_p} \leq T^{(p-1)/p} ||u(\cdot)||_{\mathcal{L}_p} \; . \tag{16}$$

Proof. *See Appendix A.1.* •

The proof of our results will rely heavily on the input to state properties of the discrete-time system

$$\begin{bmatrix} \xi^+ \\ \varrho^+ \end{bmatrix} = G(\xi, \varrho) + \begin{bmatrix} \nu \\ 0 \end{bmatrix} \tag{17}$$

where G was defined in (8).

4.1 Proof of the global result

The following can be established:

Proposition 1 *Suppose Assumption 1 holds and the system (8) is UGES. Then the discrete-time system (17) is finite gain ℓ_p stable from $\nu(\cdot)$ to $\xi(\cdot)$ for all $p \in [1, \infty]$, i.e., for each $p \in [1, \infty]$, there exists c_p such that, for each $\xi(0) \in \mathbb{R}^n$ and $\varrho(0) \geq 0$,*

$$||\xi(\cdot)||_{\ell_p} \leq c_p \left(|\xi(0)| + ||\nu(\cdot)||_{\ell_p} \right) \; . \tag{18}$$

Proof. See Appendix A.2. •

We are now ready to prove our global result.
Proof of Theorem 1 Recall the definition of t_k for $k \in \mathbb{Z}_{\geq -1}$ given in (3). Define

$$\begin{aligned} \xi(k) &:= x(t_k) \qquad \forall k \in \mathbb{Z}_{\geq -1} \\ \varrho(k) &:= p(t_k) \qquad \forall k \in \mathbb{Z}_{\geq 0} \; . \end{aligned} \tag{19}$$

Then define

$$\widetilde{\xi}(-1) = \begin{bmatrix} \xi(-1) \\ x(0) \end{bmatrix} \ , \quad \widetilde{\xi}(k) := \begin{bmatrix} \xi(k) \\ 0_n \end{bmatrix} \quad \forall k \in \mathbb{Z}_{\geq 0} \ . \tag{20}$$

If $t_{-1} < 0$ then, for all $t \in [t_{-1}, 0)$, define $x(t) := x(t_{-1})$ and $u(t) := 0$. Also define

$$\widetilde{\nu}(k) := \int_0^T |u(t_k + \tau)| d\tau \qquad \forall k \in \mathbb{Z}_{\geq -1} \tag{21}$$

and

$$\nu(k) := \widetilde{\phi}(T, \xi(k), \varrho(k), u(t_k + .)) - \widetilde{\phi}(T, \xi(k), \varrho(k), 0) \qquad \forall k \in \mathbb{Z}_{\geq 0} \tag{22}$$

where

$$\widetilde{\phi}(t, \xi, \varrho, w(\cdot)) = \xi + \int_0^t f(\widetilde{\phi}(\tau, \xi, \varrho, w(\cdot)), \xi, \varrho + \tau, w(\tau)) d\tau \ . \tag{23}$$

It follows from Assumption 1 that this definition is well-posed. From the definition of $\nu(k)$ in (22), $G(\cdot, \cdot)$ in (8) and $\xi(k)$ and $\varrho(k)$ in (19), it follows that

$$\begin{bmatrix} \xi(k+1) \\ \varrho(k+1) \end{bmatrix} = G(\xi(k), \varrho(k)) + \begin{bmatrix} \nu(k) \\ 0 \end{bmatrix} \qquad k \in \mathbb{Z}_{\geq 0} \ . \tag{24}$$

The following two claims (whose proofs are reported in Appendix A.1 for completeness) are based on a simple application of the Gronwall lemma and will serve to complete our proof.

Claim 1 *Under Assumption 1, if $\nu(\cdot)$ and $\widetilde{\nu}(\cdot)$ are defined as in (21) and (22), then*

$$|\nu(k)| \leq c\,\widetilde{\nu}(k) \qquad \forall k \in \mathbb{Z}_{\geq 0} \ , \tag{25}$$

with $c := L \exp(LT)$. ■

Claim 2 *Under Assumption 1, if $\widetilde{\nu}(\cdot)$ and $\widetilde{\xi}(\cdot)$ are defined as in (20) and (21), then*

$$|x(t)| \leq c_2 |\widetilde{\xi}(k)| + c\,\widetilde{\nu}(k) \ , \quad \forall k \in \mathbb{Z}_{\geq -1}, \ t \in [t_k, t_{k+1}], \tag{26}$$

with $c := L \exp(LT)$ and $c_2 := (1 + LT) \exp(LT)$. ■

It follows from inequality (26), that

$$\begin{aligned} ||x(\cdot)||_{\mathcal{L}_\infty} &\leq c_2 ||\widetilde{\xi}(\cdot - 1)||_{\ell_\infty} + c||\widetilde{\nu}(\cdot - 1)||_{\ell_\infty} \\ &\leq c_2 |\widetilde{\xi}(-1)| + c_2 ||\xi(\cdot)||_{\ell_\infty} + c||\widetilde{\nu}(\cdot - 1)||_{\ell_\infty} \end{aligned} \tag{27}$$

and

$$\begin{aligned} ||x(\cdot)||^p_{\mathcal{L}_p} &\leq \sum_{k \in \mathbb{Z}_{\geq -1}} \int_{t_k}^{t_{k+1}} |x(t)|^p dt \\ &\leq T \sum_{k \in \mathbb{Z}_{\geq -1}} \Big((2c_2)^p |\widetilde{\xi}(k)|^p + (2c)^p \, \widetilde{\nu}(k)^p \Big) \\ &\leq T \left((2c_2)^p |\widetilde{\xi}(-1)|^p + (2c_2)^p ||\xi(\cdot)||^p_{\ell_p} + (2c)^p ||\widetilde{\nu}(\cdot - 1)||^p_{\ell_p} \right) \ . \end{aligned} \tag{28}$$

We also have, using the definition of $\xi(\cdot)$, the relation (24), inequality (25), Proposition 1, and (26) with $k = -1$ and $t = t_\circ$, that for all $p \in [1, \infty]$,

$$\begin{aligned} ||\xi(\cdot)||_{\ell_p} &\leq c_p \left(|x(t_0)| + ||\nu(\cdot)||_{\ell_p} \right) \\ &\leq c_p \left(c_2 |\widetilde{\xi}(-1)| + 2c||\widetilde{\nu}(\cdot - 1)||_{\ell_p} \right) \ . \end{aligned} \tag{29}$$

Combining (29) with (25), (27), (28), Fact 1 and using the fact that, for each $\zeta : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$,

$$||\zeta(\cdot)||_{\ell_\infty} \leq ||\zeta(\cdot)||_{\ell_p} \qquad \forall p \in [1, \infty] \tag{30}$$

establishes the results of the theorem. •

4.2 Proof of the local results

The proofs of Theorems 2 and 3 are similar in nature to the proof of the global result. A key tool for these proofs is the following local version of Proposition 1.

Proposition 2 *Suppose that the function $f(\cdot,\cdot,\cdot,0)$ is locally Lipschitz in its first two arguments uniformly in its third argument and that the system (8) is ULES. Then the discrete-time system (17) is small signal finite gain ℓ_p stable from $\nu(\cdot)$ to $\xi(\cdot)$ for all $p \in [1,\infty]$, i.e., for each $p \in [1,\infty]$, there exist positive constants c_p and d such that, for each $|\xi(0)| \leq d$, $\varrho(0) \geq 0$ and $\|\nu(\cdot)\|_{\ell_\infty} \leq d$,*

$$||\xi(\cdot)||_{\ell_\infty} \leq ||\xi(\cdot)||_{\ell_p} \leq c_p \left(|\xi(0)| + ||\nu(\cdot)||_{\ell_p}\right) \ . \tag{31}$$

Proof. See Appendix A.3. •

Based on Fact 1 and Proposition 2, the proof of Theorems 2 and 3 can be carried out in a similar way as the proof of Theorem 1 reported in Section 4.1, with a special attention to the fact that since the right hand side of (2) is only locally Lipschitz, solutions may escape in finite time and the function $G(\xi,\varrho)$ in (8), (7) may be not well defined.

Proof of Theorems 2 and 3 The following two claims (whose proofs are omitted due to space constraints) can be easily proven by contradiction, using Gronwall's inequality and Fact 1.

Claim 3 *Under Assumption 2 (respectively, Assumption 3), consider a value $\xi(k)$ and a function $u(\cdot)$ such that*

$$|\xi(k)| < \xi_M := \frac{\delta e^{-LT}}{2(1+LT)}, \qquad \|u(\cdot)\|_{\mathcal{L}_p} < u_M := \frac{\delta e^{-LT}}{2LT^{\frac{p-1}{p}}}, \tag{32}$$

$$\left(\textit{respectively,} \qquad \|u(\cdot)\|_{\mathcal{L}_\infty} < u_M := \min\left\{\frac{\delta e^{-LT}}{2LT}, \delta\right\} \right).$$

Then, the value $\nu(k)$ in (22), (23) is well defined and satisfies the bound

$$|\nu(k)| \leq c_{\nu,p}\|u(\cdot)\|_{\mathcal{L}_p}, \quad p \in [1,\infty].$$

■

Claim 4 *Under Assumption 2 (respectively, Assumption 3), given the discrete-time system (24) with the selection (22), if*

$$|\xi(0)| < \min\left\{d, \frac{\xi_M}{2c_p}\right\}, \qquad \|u(\cdot)\|_{\mathcal{L}_p} < \frac{e^{-LT}}{LT^{\frac{p-1}{p}}} \min\left\{d, \frac{\xi_M}{2c_p}, \frac{\delta}{2}\right\},$$

$$\left(\textit{respectively,} \qquad \|u(\cdot)\|_{\mathcal{L}_\infty} < \min\left\{ \frac{e^{-LT}}{LT^{\frac{p-1}{p}}} \min\left\{d, \frac{\xi_M}{2c_p}, \frac{\delta}{2}\right\}, \delta \right\} \right),$$

then $\xi(k)$ and $\nu(k)$ are well defined and $\xi(k)$ satisfies (32) for all $k \geq 0$. ■

Based on the two claims above, the proof can be completed following the guidelines of the proof of Theorem 1. In particular, if $t_{-1} < 0$ then, for all $t \in [t_{-1}, 0)$, define $x(t) := x(t_{-1})$ and $u(t) := 0$. By the definitions in (19), (20), (21), setting $\xi(0) = x(t_0)$, ensures that, if

$$|x(t_0)| < \min\left\{d, \frac{\xi_M}{2c_p}\right\}, \tag{33}$$

then the discrete-time solution $\xi(\cdot)$ to (24), (22) satisfies $\xi(k) = x(t_k)$ for all $k \geq 0$, and the combination of Claims 3 and 4 implies that the samples $x(t_k)$ are well defined and bounded by ξ_M for all $k \geq 0$.

To ensure that (33) holds, by Fact 1 we can enforce an arbitrarily small bound on $\int_0^{t_0} |u(\tau)|d\tau$ by fixing a sufficiently small bound on $\|u(\cdot)\|_{\mathcal{L}_p}$ (respectively, $\|u(\cdot)\|_{\mathcal{L}_\infty}$). Moreover, following the same approach as in the proof of Claim 3, it can be shown by contradiction that if $|x(t_{-1})|$ and $|x(0)|$ are sufficiently small, equation (33) holds.

Finally, the proof is completed as in the proof of Theorem 1 using Proposition 2, (25), (26), (27), and (28). •

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A Proofs of technical results

A.1 Proof of Fact 1 and Claims 1 and 2

Proof of Fact 1 Case 1: $p < \infty$. By the integral version of the Holder's inequality written by substituting $q = p/(p-1)$ (see, *e.g.*, [14, page 274]), we get

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} 1\,|u(\tau)|d\tau &\leq \left(\int_{t_k}^{t_{k+1}} d\tau\right)^{(p-1)/p} \left(\int_{t_k}^{t_{k+1}} |u(\tau)|^p d\tau\right)^{1/p} \\
&= |T|^{(p-1)/p} \left(\int_{t_k}^{t_{k+1}} |u(\tau)|^p d\tau\right)^{1/p} .
\end{aligned}$$

Taking the p-th power of both sides and summing between -1 and ∞, we get

$$\begin{aligned}
\|\widetilde{\nu}(\cdot - 1)\|_{\ell_p}^p &= \sum_{k=-1}^{\infty} \left(\int_{t_k}^{t_{k+1}} |u(\tau)|d\tau\right)^p \\
&\leq T^{p-1} \left(\sum_{k=-1}^{\infty} \int_{t_k}^{t_{k+1}} |u(\tau)|^p d\tau\right) \\
&= T^{p-1} \int_0^{\infty} |u(\tau)|^p d\tau.
\end{aligned}$$

Case 2: $p = \infty$. This case follows easily from the fact that (15) implies

$$|\widetilde{\nu}(k)| \leq T\,\|u(\cdot)\|_{\mathcal{L}_\infty}, \qquad \forall k \in \mathbb{Z}_{\geq -1} .$$

•

Proof of Claim 1 First note that, by Assumption 1, we can write

$$\begin{aligned}
|f(z_1, x, p, u) - f(z_2, x, p, 0)| &\leq |f(z_1, x, p, u) - f(z_1, x, p, 0)| + \\
&\qquad |f(z_1, x, p, 0) - f(z_2, x, p, 0)| \\
&\leq L(|u| + |z_1 - z_2|).
\end{aligned} \tag{34}$$

Given any $k \geq 0$, define $z_1(t) = \widetilde{\phi}(t, \xi(k), \varrho(k), u(t_k + \cdot))$ and $z_2(t) = \widetilde{\phi}(t, \xi(k), \varrho(k), 0)$ for all $t \in [0, T]$ and note that, by equation (22), $\nu(k) = z_1(T) - z_2(T)$. Hence, equation (23) with (34) implies that

$$\begin{aligned}
|z_1(t) - z_2(t)| &\leq L \int_0^t \left(|z_1(\tau) - z_2(\tau)| + |u(t_k + \tau)|\right) d\tau \\
&\leq L \int_0^t |u(t_k + \tau)|d\tau + L \int_0^t |z_1(\tau) - z_2(\tau)|d\tau.
\end{aligned}$$

Finally, by the Gronwall-Bellman inequality, we can write

$$|z_1(t) - z_2(t)| \leq \exp(Lt) \left(L \int_0^t |u(t_k + \tau)|d\tau\right),$$

which can be evaluated for $t = T$ to get inequality (25). $\bullet$

Proof of Claim 2 By the definition (19), given any $k \geq 1$, we can write

$$\begin{aligned} x(t) &= x(t_k) + \int_{t_k}^{t} f(x(\tau), x(t_k), p(t_k), u(\tau))d\tau \\ &= \xi(k) + \int_{t_k}^{t} f(x(\tau), \xi(k), \rho(k), u(\tau))d\tau, \quad \forall t \in [t_k, t_{k+1}]. \end{aligned}$$

Since by Assumption 1

$$\begin{aligned} |f(x_1, x_2, p, u)| &\leq |f(x_1, x_2, p, u) - f(x_1, x_2, p, 0)| + \\ &\qquad\qquad |f(x_1, x_2, p, 0) - f(0, 0, p, 0)| \\ &\leq L(|x_1| + |x_2| + |u|), \end{aligned}$$

then the proof can be completed using Gronwall-Bellman inequality and definition (21) in a similar way as the proof of Claim 1. $\bullet$

A.2 Proof of Proposition 1

The following two lemmas (whose proofs are omitted due to space constraints) will be useful for the proof of Proposition 1. The first is a straightforward application of Gronwall's lemma (similar in nature to the proof of Claim 2). The second one is a converse Lyapunov statement (a similar lemma can be found, *e.g.*, in [7]).

Lemma 1 *Under Assumption 1 the map $\xi \mapsto G(\xi, \varrho)$ defined in (8) is globally Lipschitz, uniformly in ϱ.*

Lemma 2 *(Converse Lyapunov theorem) If the system (8) is UGES and Assumption 1 holds then there exists a (Lyapunov) function $(\xi, \varrho) \mapsto V(\xi, \varrho)$ which is globally Lipschitz in ξ, uniformly in $\varrho \geq 0$, and there exist $M > 0$ and $\overline{\alpha} > 0$ such that the following holds for all $\varrho \in \mathbb{R}_{\geq 0}$ and for all $\xi \in \mathbb{R}^n$:*

$$\begin{aligned} |\xi| \leq \ V(\xi, \varrho) &\ \leq M|\xi| \\ V(G(\xi, \varrho)) - V(\xi, \varrho) &\ \leq -\overline{\alpha}|\xi|. \end{aligned} \tag{35}$$

Proof of Proposition 1 <u>Case 1</u>: $p < \infty$. By Assumption 1 and Lemma 1, the map $(\xi, \varrho) \mapsto G(\xi, \varrho)$ in (8) is globally Lipschitz in ξ, uniformly in ϱ. Then, by UGES of the system (8) and Lemma 2, there exists a Lyapunov function $V(\cdot, \cdot)$ globally Lipschitz in the first argument (uniformly in the second), satisfying (35). Then, by the global Lipschitz property of $V(\cdot, \cdot)$, the following holds:

$$\begin{aligned} V(G(\xi, \varrho) + (\nu, 0)) - V(\xi, \varrho) &\leq -\overline{\alpha}|\xi| + V(G(\xi, \varrho) + (\nu, 0)) - V(G(\xi, \varrho)) \\ &\leq -\overline{\alpha}|\xi| + L_V|\nu|. \end{aligned} \tag{36}$$

Consider now the Lyapunov function $\rho(V) := V^p$. Applying [11, Lemma 1] with $\alpha_1(s) = s$, $\alpha_2(s) = Ms$, $\gamma(s) = L_V s$, $\alpha(s) = \overline{\alpha}s$ and $T = 1$, equation (36) and the first equation in (35) are sufficient for the following to hold:

$$V(G(\xi, \varrho) + (\nu, 0))^p - V(\xi, \varrho)^p \leq -c_\xi|\xi|^p + c_\nu|\nu|^p, \tag{37}$$

where

$$c_\xi = \frac{p\gamma}{2^p}, \qquad c_\nu = pL_V \left[L_V \left(\frac{2M}{\gamma} + 1 \right) \right]^{p-1}.$$

Substituting in equation (37) the trajectories of system (17) and taking the sum from zero to infinity of both sides, we get

$$0 \leq V(\xi(0), \varrho(0))^p + \sum_{k=0}^{\infty} \left(-c_\xi |\xi(k)|^p + c_\nu |\nu(k)|^p \right),$$

which implies

$$\begin{aligned} \|\xi(\cdot)\|_{\ell_\infty}^p \leq \|\xi(\cdot)\|_{\ell_p}^p &= \sum_{k=0}^{\infty} |\xi(k)|^p \\ &\leq \frac{c_\nu}{c_\xi} \|\nu(\cdot)\|_{\ell_p}^p + \frac{M^p}{c_\xi} |\xi(0)|^p. \end{aligned} \tag{38}$$

<u>Case 2</u>: $p = \infty$. Consider inequality (36). By the first equation in (35) and since $\nu(\cdot) \in \ell_\infty$, this can be written as follows:

$$V(G(\xi, \varrho) + (\nu, 0)) - V(\xi, \varrho) \leq -\frac{\overline{\alpha}}{M} V(\xi, \varrho) + L_V \|\nu(\cdot)\|_{\ell_\infty}, \tag{39}$$

By equation (39), it follows that if $V(\xi, \varrho) > \frac{ML_V}{\overline{\alpha}} \|\nu(\cdot)\|_{\ell_\infty}$, then $V(G(\xi, \varrho) + (\nu, 0)) < V(\xi, \varrho)$. Conversely, if $V(\xi, \varrho) \leq \frac{ML_V}{\overline{\alpha}} \|\nu(\cdot)\|_{\ell_\infty}$, then

$$\begin{aligned} V(G(\xi, \varrho) + (\nu, 0)) &\leq \left(1 - \tfrac{\overline{\alpha}}{M}\right) V(\xi, \varrho) + L_V \|\nu(\cdot)\|_{\ell_\infty} \\ &\leq \left(1 - \tfrac{\overline{\alpha}}{M}\right) \tfrac{ML_V}{\overline{\alpha}} \|\nu(\cdot)\|_{\ell_\infty} + L_V \|\nu(\cdot)\|_{\ell_\infty} \\ &= \tfrac{ML_V}{\overline{\alpha}} \|\nu(\cdot)\|_{\ell_\infty}. \end{aligned} \tag{40}$$

Hence, combining the two bounds derived above,

$$V(G(\xi, \varrho) + (\nu, 0)) \leq \max \left\{ V(\xi, \varrho), \frac{ML_V}{\overline{\alpha}} \|\nu(\cdot)\|_{\ell_\infty} \right\}, \quad \forall \varrho \geq 0, \forall \xi. \tag{41}$$

Finally, writing equation (41) along the trajectories of the system (17) and applying iteratively the bound on $V(\xi(k+1), \varrho(k+1)) = V(G(\xi(k), \varrho(k)) + (\nu(k), 0))$, the following is proven:

$$V(\xi(k), \varrho(k)) \leq \max \left\{ V(0), \frac{ML_V}{\overline{\alpha}} \|\nu(\cdot)\|_{\ell_\infty} \right\}, \quad \forall k \geq 0,$$

which, from the first equation in (35), yields

$$\|\xi(\cdot)\|_{\ell_\infty} \leq M|\xi(0)| + \frac{ML_V}{\overline{\alpha}} \|\nu(\cdot)\|_{\ell_\infty}.$$

•

A.3 Proof of Proposition 2

In this appendix we give guidelines to show Proposition 2, based on the steps of the proof in Appendix A.2. The following two results are straightforward generalizations of Lemmas 1 and 2 for the local case.

Lemma 3 *If the function $f(\cdot,\cdot,\cdot,0)$ is locally Lipschitz in its first two arguments uniformly in its third argument, then there exists $r > 0$ such that the map $\xi \mapsto G(\xi,\varrho)$ defined in (8) is well defined and Lipschitz for all ξ such that $|\xi| \le r$, uniformly in ϱ.*

Lemma 4 *(Converse Lyapunov theorem) If the system (8) is ULES and the map $(\xi,\varrho) \mapsto G(\xi,\varrho)$ is defined and Lipschitz for all ξ such that $|\xi| \le r$, uniformly in ϱ, then there exists a (Lyapunov) function $(\xi,\varrho) \mapsto V(\xi,\varrho)$ and $\delta > 0$, $M > 0$, $\overline{\alpha} > 0$, with $\delta < r$, such that for all $|\xi| \le \delta$ and $\varrho \in \mathbb{R}_{\ge 0}$, V is locally Lipschitz in ξ, uniformly in $\varrho \ge 0$, and*

$$\begin{array}{rcl} |\xi| \;\le\; V(\xi,\varrho) & \le & M|\xi| \\ V(G(\xi,\varrho)) - V(\xi,\varrho) & \le & -\overline{\alpha}|\xi|. \end{array}$$

Based on Lemmas 3 and 4, Proposition 2 can be proven following the steps of the proof of Proposition 1 as follows:

Proof of Proposition 2 Consider the constant δ introduced in Lemma 4 and take $d = \frac{\delta}{2M}\min\left\{\frac{\overline{\alpha}}{L_V}, 1\right\}$. Based on the combination of Lemmas 3 and 4, if $\|\nu(\cdot)\|_{\ell_\infty} < d$ and $|\xi(0)| < d$, then the proof of Proposition 1 for the case $p = \infty$ can be followed verbatim [1] to show that $\|\xi(\cdot)\|_{\ell_\infty} < \delta/2$.

Hence, since $|\xi(k)| < \delta/2$, and $|\nu(k)| < \delta/2$ for all $k \ge 0$, equation (36) holds and, by a local version of [11, Lemma 1] (whose proof is the straightforward generalization of the proof in [11]), also equation (37) can be proven to hold.

Finally, following the same steps as in the proof of Proposition 1, (38) holds too, thus completing the proof. •

[1] Note that, by equation (39), necessarily $|\xi(k)| \le \delta$, $\forall k \ge 0$, as a matter of fact, whenever $V(\xi(k),\varrho(k)) \ge \delta/2$, the inequalities

$$V(\xi(k),\varrho(k)) \ge \frac{\delta}{2} \ge \frac{ML_V}{\overline{\alpha}} d > \frac{ML_V}{\overline{\alpha}}\|\nu(\cdot)\|_{\ell_\infty},$$

imply that $|\xi(k+1)| \le |\xi(k)|$ (this actually also holds whenever $V(\xi(k),\varrho(k)) \ge \frac{ML_V}{\overline{\alpha}}\|\nu(\cdot)\|_{\ell_\infty}$, which is an even larger set). Moreover, if $V(\xi(k),\varrho(k)) < \frac{ML_V}{\overline{\alpha}}\|\nu(\cdot)\|_{\ell_\infty}$, then by the inequalities (40), $|\xi(k+1)| \le V(\xi(k+1),\varrho(k+1)) \le \frac{ML_V}{\overline{\alpha}}\|\nu(\cdot)\|_{\ell_\infty} < \frac{\delta}{2}$.