Input-Output Stability Properties of Networked Control Systems

D. Nešić[1]
Department of Electrical and Electronic Engineering,
The University of Melbourne, Parkville, 3052, Victoria, Australia.
A. R. Teel[2]
CCEC, Electrical and Computer Engineering Department,
University of California, Santa Barbara, CA, 93106-9560, USA.

Abstract

Results on input-output $\mathcal{L}_p$ stability of networked control systems (NCS) are presented for a large class of network scheduling protocols. It is shown that static protocols and a recently considered dynamical protocol called Try-Once-Discard (TOD) belong to this class. Our results provide a unifying framework for generating new scheduling protocols that preserve $\mathcal{L}_p$ stability properties of the system if a design parameter is chosen sufficiently small. The most general version of our results can be used to treat NCS with data packet dropouts. The model of NCS and, in particular, of the scheduling protocol that we use appears to be novel and we believe that it will be useful in further study of these systems. The proof technique we use is based on the small gain theorem and it lends itself to an easy interpretation. We prove that our results are guaranteed to be better than existing results in the literature and we illustrate this via an example of a batch reactor.

I. Introduction

Nowadays, many control applications have some control loops that are closed via a serial communication channel that transmits signals from many sensors and actuators in the system, as well as signals from other unrelated users that are connected to the network. Motivation for using this set-up comes from lower cost, ease of maintenance, great flexibility, as well as low weight and volume. For instance, this architecture is widely used in automobiles and aircraft. This motivates research into the emerging class of *Networked Control Systems (NCS).* The main issue in NCS is that the serial communication channel has many "nodes" (sensors and actuators) where only one node can report its value at a time and, hence, access to the channel needs to be scheduled in an appropriate manner for a proper operation of the control system.

NCS are currently receiving considerable attention in the literature as illustrated by recent articles [2], [3], [8], [11], [15], [24], [25], [26], [27], [28], [30] and references listed therein. The area of NCS is still in its infancy and there are at least two areas where existing results can be improved. First, most existing literature considers only stabilization of linear NCS whereas nonlinear NCS have received little attention (with few exceptions, such as [25]). Second, most results treat NCS without disturbances and we are aware only of limited results on stability of NCS with disturbances, such as the $\mathcal{L}_\infty$ to root-mean-square stability of a class of NCS considered in [10] and results on input-output stability of linear jump parameter systems in [6] that can be exploited for certain NCS with static protocols. Also, in some cases it is possible to use tools for linear sampled-data systems [7] for analysis and design of certain classes of linear NCS. In this paper we consider input-output $\mathcal{L}_p$ stability of *nonlinear NCS with disturbances.*

We follow the method proposed in [25], [26], in which one first designs the controller without taking into account the network and then in the second step one determines a design parameter called the maximum allowable transfer interval

1 Research supported by the Australian Research Council under the discovery grants scheme.
2 Research supported in part by the AFOSR under grant F49620-00-1-0106 and the NSF under grant ECS-9988813.

(MATI) so that the closed loop remains stable when some control and sensor signals are transmitted via the network. This approach was shown to produce stabilizing controllers for linear NCS in [25] and nonlinear NCS in [26]. Similar ideas were also used to analyse mixed traffic wireless networks in [28]. These references considered different types of network scheduling protocols, such as static protocols, TOD protocol (dynamic) and dynamic protocols that are using different types of prediction algorithms [1].

Our main result states for a large class of protocols that if the controller designed in the first step of the above described procedure achieves $\mathcal{L}_p$ stability in an appropriate sense, then for sufficiently small values of MATI the NCS preserves $\mathcal{L}_p$ stability properties. There are several important features of our result that are worth mentioning. First, the model of NCS that we exploit appears to be novel, providing us with a very powerful tool to describe a very general class of NCS with static and dynamic protocols in a unified and mathematically precise manner. We believe that insights gained from this model will be very useful for future work in this area. Second, our result is presented for a very general class of static and dynamic protocols in a unified manner. In particular, we show that both static and TOD protocols are special cases of the general class of protocols we consider. Moreover, our results can serve as a *framework* for generating new classes of protocols that have good properties. Third, our proof is based on the classical small gain theorem and it is very different from stability proofs provided in [25], [26]. Moreover, our proof distinguishes between static and dynamic protocols and it relies on their intrinsic underlying features. Fourth, the generality of our results permits analysis of networks with data packet dropouts motivated in [30]. Last but not least, the bound on MATI that we obtain has a very simple form and we prove that it is better than the bounds obtained in [26] when our result is specialized to the case of NCS without disturbances. In particular, we show that $\mathcal{L}_p$ stability of the system is preserved if MATI is smaller than (see Theorem 4)

$$\frac{1}{L}\ln\left(\frac{L+\gamma}{\rho L+\gamma}\right) ,$$

where ρ characterizes the stability properties of the protocol and it typically depends on the number of nodes ℓ in the network; L characterizes the possible growth of the error between the real values of inputs and outputs and their last transmitted values via the network; γ is the $\mathcal{L}_p$ disturbance gain that captures robustness properties of the system without the network. We show that, even when using an estimate of the $\mathcal{L}_p$ gain that is the most conservative one possible, the above bound is at least $2(\ell+1)$ times better than the bound in [26], [30] for linear NCS with TOD protocol and without disturbances, where ℓ is the number of nodes. Similarly, for the so called Round Robin (RR) static protocol our bound is at least $2\sqrt{\ell+1}$ times better than the bound in [26], [30] for linear NCS. Our bounds compare in a similar manner to the bounds from [25] for nonlinear NCS. Hence, the larger the number of nodes in the network, the less conservative our bound is when compared to the bound from [26]. This is also illustrated via an example of an unstable batch reactor with the TOD protocol considered in [26] where our results provide 1000 times larger bound on MATI than the bound obtained in [26] even though $\ell = 2$.

The paper is organized as follows. In Section 2 we present preliminary definitions and results. The model of the NCS that we use is formulated in Section 3. A large class of protocols is defined in Section 4 and it is shown that the recently considered TOD protocol and static protocols belong to this class. Section 5 contains results on $\mathcal{L}_p$ stability of the error dynamics for the protocols defined in Section 4. The main results on $\mathcal{L}_p$ stability of general NCS are presented in Section 6. Comparisons of our bounds with existing bounds in the literature is presented in Section 7 and an example is given in Section 8. Summary is given in the last section. Several technical lemmas are given in the Appendix.

II. Preliminaries

A. Notation

$\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the sets of real and natural numbers. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative integers. Given $t \in \mathbb{R}$ and a piecewise continuous function $f : \mathbb{R} \to \mathbb{R}^n$, we use the notation $f(t^+) := \lim_{s \to t, s > t} f(s)$. All vector norms, denoted as $|\cdot|$, are Euclidean norms unless otherwise stated. The same notation $|\cdot|$ is used for the induced 2-norm of a matrix. Given a measurable, locally integrable signal $\varphi : [t_0, t] \to \mathbb{R}^n$ and $p \in [1, \infty)$, we denote its $\mathcal{L}_p$ norm as follows:

$$\|\varphi[t_0, t]\|_{\mathcal{L}_p} := \left(\int_{t_0}^{t} |\varphi(s)|^p \right)^{\frac{1}{p}} .$$

If $p = \infty$, we denote the $\mathcal{L}_\infty$ norm as follows:

$$\|\varphi[t_0, t]\|_{\mathcal{L}_\infty} := \text{ess} \sup_{s \in [t_0, t]} |\varphi(s)| .$$

If $\varphi(\cdot)$ is defined on $[t_0, \infty)$ and for some $p \in [1, \infty]$ there exists $K \geq 0$ such that $\|\varphi[t_0, t]\|_{\mathcal{L}_p} \leq K, \forall t \geq t_0 \geq 0$, then we write $\varphi \in \mathcal{L}_p$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if for each $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing to zero in the second argument and for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$. β is said to be of class exp-$\mathcal{KL}$ if there exist $K, c > 0$ such that $\beta(s, t) = K \exp(-ct)$. To shorten notation we often use $(x, y) := (x^T \ y^T)^T$.

B. Underlying stability theory

For the systems we consider in this paper, a monotonically increasing sequence of times $t_{s_i} \in \mathbb{R}_{\geq 0}$ is given where $i \in \mathbb{N}$ and $t_{s_0} = 0$. Moreover, we assume[1] that there exists $\epsilon > 0$ such that $\epsilon \leq t_{s_{i+1}} - t_{s_i}, \forall i \in \mathbb{N}$. The systems are governed by the evolution equations

$$\begin{aligned} \dot{x} &= f(t, x, w) \qquad t \in [t_{s_i}, t_{s_{i+1}}] \\ x(t_{s_i}^+) &= h(i, x(t_{s_i})) , \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ and $w \in \mathbb{R}^{n_w}$ are respectively the state and disturbance input of the system. The trajectories of the system are generated as follows. Let the initial time $t_0 \geq 0$ be given and let $t_0 \in (t_{s_i}, t_{s_{i+1}})$. Then, from (t_0, x_0) and with a given $w(\cdot)$, let $x(\cdot)$ be any absolutely continuous function satisfying $x(t_0) = x_0$ and $\dot{x} = f(t, x, w)$ for almost all t in some maximal interval of definition $[t_0, t_0 + T_0)$. We assume enough regularity on f and $w(\cdot)$ to guarantee that such a function exists (see for instance [9]). If $t_0 + T_0 \leq t_{s_{i+1}}$, then $x(\cdot)$ is a (not necessarily unique) solution of (1) and $[t_0, t_0 + T_0)$ is the maximal interval of definition of this solution. If, on the other hand, t_0 is such that $t_0 + T_0 > t_{s_{i+1}}$ then $x(\cdot)$ is the solution of (1) on the interval $[t_0, t_{s_{i+1}}]$. Moreover, in this case we can extend the solution of (1) beyond $t_{s_{i+1}}$ by using the new initial condition $(t_{s_{i+1}}, h(i+1, x(t_{s_{i+1}})))$ and repeating the above procedure. For initial times t_0 such that $t_0 = t_{s_i}$ for some $i \in \mathbb{N}$, we consider as solutions what results following the above procedure from both (t_0, x_0) and $(t_0, h(i, x_0))$. In this way, to each initial condition (t_0, x_0) and each disturbance $w(\cdot)$ we associate a (not necessarily unique) solution maximally defined on an interval $[t_0, t_0 + T)$, where $T \in (0, \infty]$. We use $x(\cdot, t_0, x_0, w)$ to denote such a solution. When (t_0, x_0) and $w(\cdot)$ are clear from the context we use the shorthand notation $x(\cdot)$.

If for all x_0, t_0 and $w(\cdot)$, all corresponding solutions are right maximally defined on $[t_0, \infty)$, then we say that the system (1) is *forward complete.*

[1] We use this assumption to rule out a possibility of Zeno solutions.

In addition to evolution equations (1) we also usually associate an output function

$$y = H(t, x, w) \ , \tag{2}$$

where $y \in \mathbb{R}^{n_y}$. We use notation $y(\cdot, t_0, x_0, w) := H(\cdot, x(\cdot, t_0, x_0, w), w(\cdot))$ or shortly $y(\cdot)$ when (t_0, x_0) and $w(\cdot)$ are clear from the context.

We use the following definitions for the system (1)-(2).

Definition 1: Let $p \in [1, \infty]$ and $\gamma \geq 0$ be given. The system (1)-(2) is said to be $\mathcal{L}_p$ stable from w to y (with gain γ) if there exists $K > 0$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_x}$, $w \in \mathcal{L}_p$ and each corresponding solution $x(\cdot)$, we have that

$$\|y[t_0, t]\|_{\mathcal{L}_p} \leq K\,|x_0| + \gamma\,\|w[t_0, t]\|_{\mathcal{L}_p} \qquad \forall t \in [t_0, t_0 + T) \ ,$$

where $[t_0, t_0 + T)$ is the maximal interval of definition of $x(\cdot)$. ■

Definition 2: Let $p, q \in [1, \infty]$ and $\gamma \geq 0$ be given. The state x of the system (1)-(2) is said to be $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from output y (with gain γ) if there exists $K > 0$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_x}$, $w \in \mathcal{L}_p$ and each corresponding solution $x(\cdot)$, we have that

$$\|x[t_0, t]\|_{\mathcal{L}_q} \leq K\,|x_0| + \gamma\,\|y[t_0, t]\|_{\mathcal{L}_p} + \gamma\,\|w[t_0, t]\|_{\mathcal{L}_p} \qquad \forall t \in [t_0, t_0 + T) \ ,$$

where $[t_0, t_0 + T)$ is the maximal interval of definition of $x(\cdot)$. ■

Definition 3: Let $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ be given. The system (1)-(2) is Input-to-Output Stable (IOS) from w to y if for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_x}$, $w \in \mathcal{L}_\infty$ and each corresponding solution $x(\cdot)$, we have that

$$|y(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|w\|_{\mathcal{L}_\infty}) \qquad \forall t \in [t_0, t_0 + T) \ , \tag{3}$$

where $[t_0, t_0 + T)$ is the maximal interval of definition of $x(\cdot)$. If $y = x$, then we say that the system (1) is Input-to-State Stable (ISS). Moreover, if $\gamma(\cdot)$ is a linear function, $\beta(\cdot, \cdot)$ is an exp-$\mathcal{KL}$ function and the system (1)-(2) is IOS (ISS), then we say that the system (1)-(2) is IOS (ISS) with a linear gain and an exp-$\mathcal{KL}$ function. ■

Definition 4: Let $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ be given. The system (1)-(2) is Input-Output-to-State Stable (IOSS) from (w, y) to x if for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_x}$, $w \in \mathcal{L}_\infty$ and each corresponding solution $x(\cdot)$, we have that

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|y[t_0, t]\|_{\mathcal{L}_\infty}) + \gamma(\|w\|_{\mathcal{L}_\infty}) \qquad \forall t \in [t_0, t_0 + T) \ , \tag{4}$$

where $[t_0, t_0 + T)$ is the maximal interval of definition of $x(\cdot)$. Moreover, if $\gamma(\cdot)$ is a linear function, $\beta(\cdot, \cdot)$ is an exp-$\mathcal{KL}$ function and the system (1)-(2) is IOSS, then we say that the system (1)-(2) is IOSS with a linear gain and an exp-$\mathcal{KL}$ function. ■

Now we consider the feedback interconnection of two systems of the form (1)-(2):

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2, w) \qquad t \in [t_{s_i}, t_{s_{i+1}}] \\ y_1 &= H_1(t, x_1, y_2, w) \\ x_1(t_{s_i}^+) &= h_1(i, x_1(t_{s_i})) \ , \end{aligned} \tag{5}$$

and

$$\begin{aligned} \dot{x}_2 &= f_2(t, x_1, x_2, w) \qquad t \in [t_{s_i}, t_{s_{i+1}}] \\ y_2 &= H_2(t, y_1, x_2, w) \\ x_2(t_{s_i}^+) &= h_2(i, x_2(t_{s_i})) \ . \end{aligned} \tag{6}$$

We present next two small gain theorems for systems with jumps. The proofs of these results are very similar to the classical proof for systems without jumps and are therefore omitted. Only a sketch of a part of the proof of Theorem 2 on exp-$\mathcal{KL}$ functions and linear gains is provided in the appendix since we are not aware of a reference where these calculations are stated.

Theorem 1: Suppose that for some $p \in [1, \infty]$ we have that:

1. System (5) is $\mathcal{L}_p$ stable from (y_2, w) to y_1 with gain γ_1.
2. The state x_1 of system (5) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from (y_1, w).
3. System (6) is $\mathcal{L}_p$ stable from (y_1, w) to y_2 with gain γ_2.
4. The state x_2 of system (6) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from (y_2, w).
5. The small gain condition holds, that is $\gamma_1\gamma_2 < 1$.

Then, the system (5), (6) is $\mathcal{L}_p$ stable from w to (x_1, x_2). ■

Theorem 2: Suppose that:

1. System (5) is IOS stable from (y_2, w) to y_1.
2. System (5) is IOSS stable from (y_1, w) to x_1.
3. System (6) is IOS stable from (y_1, w) to y_2.
4. System (6) is IOSS stable from (y_2, w) to x_2.
5. The small gain condition holds, that is there exist a $\mathcal{K}_\infty$ function ρ such that

$$(Id + \rho) \circ \gamma_2 \circ (Id + \rho) \circ \gamma_1(s) \le s \qquad \forall s > 0 \; .$$

Then, all solutions of the system (5), (6) are right maximally defined on $[t_0, \infty)$ (i.e., the system is forward complete) and the system is ISS from w to (x_1, x_2). Moreover, if all properties in items 1-4 hold with exp-$\mathcal{KL}$ functions and linear gains $\gamma_1(s) = \bar{\gamma}_1 s$ and $\gamma_2(s) = \bar{\gamma}_2 s$ for some $\bar{\gamma}_1, \bar{\gamma}_2 \ge 0$ and the small gain condition $\bar{\gamma}_1 \cdot \bar{\gamma}_2 < 1$, holds, then the system is IOS from w to (x_1, x_2) with a linear gain and an exp-$\mathcal{KL}$ function. ■

We are often interested in stability properties of the system (1) when $w \equiv 0$. In particular, we use the following:

Definition 5: Consider system (1) and suppose that $w \equiv 0$. Let $\beta \in \mathcal{KL}$ be given. We say that the system is *uniformly globally asymptotically stable* (UGAS) if for all $x_0 \in \mathbb{R}^{n_x}$ and all corresponding solutions $x(\cdot)$, we have that

$$|x(t, t_0, x_0)| \le \beta(|x_0|, t - t_0) \qquad \forall t \ge t_0 \ge 0 \; .$$

The system (1) is *uniformly globally exponentially stable* (UGES) if the above holds with a class exp-$\mathcal{KL}$ function β. ■

In order to relate $\mathcal{L}_p$ stability for some $p \in [1, \infty)$ and UGES we need the following definition (see [21]):

Definition 6: The origin of the system (1) with $w \equiv 0$ is said to be *uniformly globally fixed time interval stable* (UGFTIS) with linear gain if there exist $T > 0$ and $\bar{\rho} > 0$ such that, for all $t_0 \ge 0$, $x_0 \in \mathbb{R}^{n_x}$ and the corresponding solutions $x(\cdot)$, we have that

$$t - t_0 \in [0, T] \implies |x(t, t_0, x_0)| \le \bar{\rho}\, |x_0| \; . \tag{7}$$

■

The following result relates $\mathcal{L}_p$ stability for some $p \in [1, \infty)$ and UGES[2].

Theorem 3: Suppose that the system (1) is $\mathcal{L}_p$ stable from w to x for some $p \in [1, \infty)$ and the origin of (1) with $w \equiv 0$ is UGFTIS with linear gain. Then, the system (1) with $w \equiv 0$ is UGES. ■

[2]Theorem 3 was proved in [21] for systems without jumps. However, the proof extends with minor changes to systems with jumps and it is therefore omitted.

Remark 1: We note that it is possible to assume a weaker condition than UGFTIS with linear gain (if we assume that a similar bound holds with a nonlinear gain) in Theorem 3 in order to obtain the weaker conclusion that the system is uniformly globally asymptotically stable (see [21, Section 3]). ■

We emphasize that conditions of Theorem 3 exclude the case $p = \infty$. In order to conclude exponential stability from $\mathcal{L}_\infty$ stability we can use ISS like properties that we defined earlier. In particular, a consequence of Theorem 2 is:

Corollary 1: Suppose that all conditions of Theorem 2 hold and the system (1) is ISS from w to (x_1, x_2). Then, the system (1) with $w \equiv 0$ is UGAS. Moreover, if all conditions of Theorem 2 hold with exp-$\mathcal{KL}$ functions and linear gains, then the system (1) with $w \equiv 0$ is UGES. ■

Finally, we present two different sufficient conditions for UGFTIS with linear gain that can be used in conjunction with Theorem 3. The proofs are given in the appendix. We note that under these conditions the system (1) is forward complete.

Proposition 1: Suppose that there exist $L_1, L_2, \epsilon > 0$ such that for all $x \in \mathbb{R}^{n_x}$, $t \in \mathbb{R}$ and $i \in \mathbb{N}$

$$
\begin{aligned}
|f(t,x,0)| &\leq L_1 |x|, &(8)\\
|h(i,x)| &\leq L_2 |x| &(9)\\
t_{s_{i+1}} - t_{s_i} &\geq \epsilon, \ . &(10)
\end{aligned}
$$

Then, the origin of the system (1) is UGFTIS with linear gain. ■

Proposition 2: Suppose that

1. The system (1) is $\mathcal{L}_p$ stable from w to y for some $p \in [1, \infty]$;
2. The state x is $\mathcal{L}_p$ to $\mathcal{L}_\infty$ detectable from the output y.

Then, the origin of the system (1) with $w \equiv 0$ is UGFTIS with linear gain. ■

III. Definition of Networked Control Systems

In this section we introduce a class of models with jumps that we will use to describe NCS. We augment the model of NCS proposed in [26] with an equation that describes the operation of the scheduling protocol and we show that two protocols (RR and TOD protocols) that were studied in [26] can be modelled in this manner.

Let the sequence $t_{s_i}, i \in \mathbb{N}$ of monotonically increasing transmission times satisfy $\epsilon \leq t_{s_{j+1}} - t_{s_j} \leq \tau$ for all $j \in \mathbb{N}$ and some fixed $\epsilon, \tau > 0$. At each t_{s_i}, the protocol gives access to the network to one of the nodes $i \in \{1, 2, \ldots, \ell\}$. We adopt terminology from [26] and refer to τ as the *maximum allowable transmission interval* (MATI). We consider general nonlinear NCS with disturbances of the following form

$$
\begin{array}{rcll}
\dot{x}_P &=& f_P(t, x_P, \hat{u}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\
y &=& g_P(t, x_P) & \\
\dot{x}_C &=& f_C(t, x_C, \hat{y}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\
u &=& g_C(t, x_C) & \\
\dot{\hat{y}} &=& \hat{f}_P(t, x_P, x_C, \hat{y}, \hat{u}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\
\dot{\hat{u}} &=& \hat{f}_C(t, x_P, x_C, \hat{y}, \hat{u}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\
\hat{y}(t_{s_i}^+) &=& y(t_{s_i}) + h_y(i, e(t_{s_i})) & \\
\hat{u}(t_{s_i}^+) &=& u(t_{s_i}) + h_u(i, e(t_{s_i})) &
\end{array}
\qquad (11)
$$

where x_P and x_C are respectively states of the plant and the controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of most recently transmitted plant and controller output values via the network; e is the network induced error defined as

$$e(t) := \begin{pmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{pmatrix} = \begin{pmatrix} e_y \\ e_u \end{pmatrix} .$$

Note that if NCS has ℓ links, then the error vector can be partitioned as follows $e = [e_1^T \ e_2^T \ \dots \ e_\ell^T]^T$. The functions h_u and h_y are typically such that, if the jth link gets access to the network at some transmission time t_{s_i} we have that the corresponding part of the error vector has a jump. For several protocols, such as the RR and TOD protocols (see Examples 1 and 2), we typically assume that e_j is reset to zero at time $t_{s_i}^+$, that is $e_j(t_{s_i}^+) = 0$. However, we emphasize that this assumption is not needed in general and this will become clear in the next section.

We combine the controller and plant states into a vector $x := (x_P, x_C)$ and using the error vector defined earlier $e = (e_y, e_u)$ and the following definitions:

$$f(t,x,e,w) := \begin{pmatrix} f_P(t, x_P, g_C(t, x_C) + e_u, w) \\ f_C(t, x_C, g_P(t, x_P) + e_y, w) \end{pmatrix}; \qquad h(i,e) := \begin{pmatrix} h_y(i,e) \\ h_u(i,e) \end{pmatrix};$$

$$g(t,x,e,w) := \begin{pmatrix} \hat{f}_P(t, x_P, x_C, g_P(t, x_P) + e_y, g_C(t, x_C) + e_u, w) - \frac{\partial g_P}{\partial t}(t, x_P) - \frac{\partial g_P}{\partial x_P}(t, x_P) f_P(t, x_P, g_C(t, x_C) + e_u, w) \\ \hat{f}_C(t, x_P, x_C, g_P(t, x_P) + e_y, g_C(t, x_C) + e_u, w) - \frac{\partial g_C}{\partial t}(t, x_C) - \frac{\partial g_C}{\partial x_C}(t, x_C) f_C(t, x_C, g_P(t, x_P) + e_y, w) \end{pmatrix},$$

we can rewrite (11) as a system with jumps that is more amenable for analysis:

$$\begin{aligned} \dot{x} &= f(t,x,e,w) \qquad \forall t \in [t_{s_{i-1}}, t_{s_i}] && (12) \\ \dot{e} &= g(t,x,e,w) \qquad \forall t \in [t_{s_{i-1}}, t_{s_i}] && (13) \\ e(t_{s_i}^+) &= h(i, e(t_{s_i})) \ , && (14) \end{aligned}$$

where $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $w \in \mathbb{R}^{n_w}$. In order to write (13), we assumed that functions g_P and g_C in (11) are continuously differentiable (this assumption can be relaxed). The meaning of these equations is explained in the previous section (see the system (1)). We also assume that the system (12)-(14) satisfies all assumptions that we stated in the previous section for system (1).

We refer to (14) as a protocol. The protocol determines the algorithm by which access to the network is assigned to different nodes in the system. There are two types of protocols that are considered in the literature: *static* and *dynamic*. For NCS with ℓ nodes the static protocol assigns access to the network in a predetermined and cyclic manner; that is, there exists a number $M \geq \ell$ such that each node $j \in \{1, \dots, \ell\}$ is granted access to the network at transmission times $t_{s_{iM+k}}$ for all $i \in \mathbb{N}$ and some $k \in \{1, \dots, M\}$. Note that in this case access to network does not depend on the values of the error signal $e(t_{s_i})$ at transmission time t_{s_i}. On the other hand, dynamic protocols first take measurements $e(t_{s_i})$ of the error every transmission time t_{s_i} and then use the measurement $e(t_{s_i})$ to compute which node gets access to network based on an algorithm.

Results of this paper are applicable to a large class of static and dynamic protocols. However, for illustration purposes we will concentrate on two particular examples that have been studied in [26]. In particular, we will consider a static protocol of period ℓ and the TOD (Try-Once-Discard) protocol that is a dynamic protocol recently introduced in [26]. We next define $h(\cdot,\cdot)$ in (14) for these two important cases.

Example 1 (Round Robin (RR) protocol) We consider RR protocol as an illustrative example for static protocols. Let there be $\ell \geq 1$ nodes in NCS and let the protocol grant access to the network to the node $i \in \{1, \dots, \ell\}$ at $t_{s_{i+j\ell}}$, for all $j \in \mathbb{N}$, that is $e(t_{s_{i+j\ell}}^+) = 0, \forall j \in \mathbb{N}$. In this case we can write

$$h(i,e) = (I - \Delta(i))e \ , \qquad (15)$$

where $\Delta(i) = \text{diag}\{\Delta_1(i), \Delta_2(i), \ldots, \Delta_\ell(i)\}$. The square matrices $\Delta_k(i)$ have the dimension n_k, with $\sum_{k=1}^{\ell} n_k = n_e$ and $\Delta_k(i) = \delta_k(i) I_{n_k}$, where I_{n_k} are identity matrices of dimension n_k and

$$\delta_k(i) = \begin{cases} 1, & \text{if } i = k + j\ell,\ j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

■

Example 2 (TOD protocol) TOD protocol was recently introduced in [26] and it operates as follows. Suppose that there are ℓ nodes competing for access to the network. We can partition the error vector as $e = [e_1^T\ e_2^T\ \ldots e_\ell^T]^T$. The node i with the greatest weighted error at time t_{s_j} will be granted access to network at $t_{s_j}^+$ and hence we have that $e_i(t_{s_j}^+) = 0$. We assume that the weights are already incorporated into the model. If a data packet fails to win access to the network, it is discarded and new data is used at the next transmission time $t_{s_{j+1}}$. If two or more nodes have equal priority, a pre-specified ordering of the nodes is used to resolve the collision. This verbal description can be converted into the model of the form (14) where

$$h(e) = (I - \Psi(e))e \ ,$$

and $\Psi(e) := \text{diag}\{\psi_1(e) I_{n_1}, \psi_2(e) I_{n_2}, \ldots, \psi_\ell(e) I_{n_\ell}\}$. I_{n_j} are identity matrices of dimension n_j with $\sum_{j=1}^{\ell} n_j = n_e$ and

$$\psi_j(e) := \begin{cases} 1, & \text{if } j = \min\left(\arg\max_j |e_j|\right) \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2: We note that there is a strong interest in exploring more general classes of protocols than (14). In particular, the protocols of the form:

$$e(t_{s_i}^+) = h(i, x(t_{s_i}), e(t_{s_i})) \ , \tag{16}$$

and

$$\begin{aligned} \eta(i+1) &= g(i, \eta(i), x(t_{s_i}), e(t_{s_i})) \\ e(t_{s_i}^+) &= h(i, \eta(i), x(t_{s_i}), e(t_{s_i})) \end{aligned} \tag{17}$$

appear to be of interest. For instance, if we want to use some type of dynamic prediction algorithm in order to decide which link should be given access to the network then the model (17) can be used to represent such protocols (see [1]). In this paper we concentrate on the particular special case (14), but we believe that the approach we take can be extended to more general situations such as (16) and (17). ■

Remark 3: In all our examples we will assume that $\hat{f}_P = 0$ and $\hat{f}_C = 0$ in (11) in order to be able to compare our results with the existing results in the literature where this was also assumed (see, for instance [26]). ■

Remark 4: Our results apply to a more general situation when (13) is replaced by

$$\begin{aligned} \dot{e} &= g(t, x, e, w, \eta(j-1)) \\ \eta(j) &= q(j, \eta(j-1), e(t_{s_j}^+)) \ , \end{aligned}$$

where $\eta(\cdot)$ defines an *integration flag*. The integration flag can be used to cover multiple cases for how e evolves in between the transmission times. Two such situations are described below.

Discrete transmission: If a component $e_i(\cdot)$ of the error vector is set to zero by the scheduling protocol at some transmission time $t_{s_j}^+$, that is $e_i(t_{s_j}^+) = 0$ and the network is such that $e_i(t) \not\equiv 0$ for $t \in [t_{s_j}, t_{s_{j+1}})$, we do not need an integration flag to model this case.

Continuous transmission: If a component $e_i(\cdot)$ of the error vector is set to zero by the scheduling protocol at some transmission time $t_{s_j}^+$, that is $e_i(t_{s_j}^+) = 0$ and the network is such that $e_i(t) \equiv 0$ for $t \in [t_{s_j}, t_{s_{j+1}})$, then we can model this situation using an integrations flag. In particular, each component $g_i(t,x,e,w)$ would be multiplied by a component of the integration flag $\eta_i(j-1)$. This flag would be equal to one except when it corresponded to the component of e most recently set to zero in which case it would be zero. ■

IV. Lyapunov UGES protocols

In this section we define a large class of protocols and show that the RR protocol in Example 1 and the TOD protocol in Example 2 belong to this class. Hence, our presentation is unifying for seemingly different existing protocols. On the other hand, we will show that all protocols that belong to this large class will preserve important $\mathcal{L}_p$ stability properties of the system for sufficiently small values of MATI. Therefore, our results provide a *framework* for generating genuinely new classes of static and dynamic protocols whose performance is guaranteed for sufficiently small MATI.

Note that the protocol (14) is a mapping that specifies how errors at transmission times t_{s_j} are mapped to errors at times $t_{s_j}^+$. Note that the protocol does not relate errors at times t_{s_j} and $t_{s_{j+1}}$ and, hence, we can not say that (14) is a discrete-time system. However, we will find it very useful to introduce an auxiliary discrete-time system of the form:

$$e(i+1) = h(i, e(i)) \qquad i \in \mathbb{N} , \tag{18}$$

and refer to it as a *discrete-time system induced by the protocol (14)*. Sometimes we abuse the terminology and refer to (18) simply as a protocol. Central to this paper is the following class of protocols:

Definition 7: Let $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ be given and suppose that there exist $\rho \in [0,1)$ and $a_1, a_2 > 0$ such that the following conditions hold for the discrete-time system (18) for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{n_e}$:

$$a_1 |e| \ \leq \ W(i,e) \ \leq \ a_2 |e| \tag{19}$$

$$W(i+1, h(i,e)) \ \leq \ \rho W(i,e) \ . \tag{20}$$

Then, we say that the protocol (14) is *uniformly globally exponentially stable (UGES) with Lyapunov function W.* ■

Remark 5: Note that the system (18) and Examples 1 and 2 motivate the classification of protocols into *linear* and *nonlinear*. Indeed, it is easy to see that all static protocols induce a time-varying linear system (18), whereas for the TOD protocol we have that (18) is time-invariant and nonlinear. As far as we are aware, this classification of protocols has not been proposed in the literature yet. However, it is very important since it uncovers why analysing dynamic protocols (which are typically nonlinear) is harder than analysing static ones (which are linear). ■

Remark 6: Note that Definition 7 does not make any reference to the NCS (12), (13) and, hence, it captures intrinsic properties of the protocol itself. It appears that this is a novel approach to viewing protocols that has not been considered previously in the literature. The underlying tool that enabled us to extract this important definition is the model of NCS (12)-(14) that we use and, in particular, the model of the protocol itself (14). The unifying results of this paper illustrate the utility of our approach. Hence, we believe that using the model of NCS in the form (12)-(14) and viewing the system in the way we propose will be very useful for many other problems in this area. ■

It is a well known fact in the literature that the conditions (19), (20) are equivalent to uniform global exponential stability of the system (18). Indeed, using standard discrete-time converse Lyapunov theorems we can state:

Proposition 3: The following statements are equivalent:

1. There exist $a, K > 0$ such that for all $e(k_0) = e_\circ$ with $e_\circ \in \mathbb{R}^{n_e}$ the trajectories of (18) satisfy

$$|\phi(k, i, e(k_0))| \leq K e^{-a(k-k_0)} |e(k_0)| \qquad \forall\ k \geq k_0 \geq 0 \ .$$

2. The system is UGES with the Lyapunov function:

$$W(i,e) = \sqrt{\sum_{k=i}^{\infty} |\phi(k,i,e)|^2} \ ,$$

where $\phi(k,i,e)$ denotes the solution of (18) at time k starting at time i and initial condition e.

Remark 7: It is possible to define Lyapunov uniformly globally asymptotically stable (UGAS) protocols with Lyapunov function W by relaxing the conditions (19), (20) in the following way. Suppose there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ such that the following holds for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{n_e}$:

$$\alpha_1(|e|) \ \leq \ W(i,e) \ \leq \ \alpha_2(|e|) \tag{21}$$

$$W(i+1, h(i,e)) - W(i,e) \ \leq \ -\alpha_3(|e|) \ . \tag{22}$$

We analyse properties of NCS with UGAS protocols in a forthcoming paper [22]. ■

We next show that the protocols considered in Examples 1 and 2 are UGES with appropriate Lyapunov functions.

Proposition 4: The RR protocol in Example 1 is UGES with the Lyapunov function[3]

$$W(i,e) := \sqrt{\sum_{k=i}^{\infty} |\phi(k,i,e)|^2} \ .$$

In particular, we can take $a_1 = 1$, $a_2 = \sqrt{\ell}$ and $\rho = \sqrt{\frac{\ell-1}{\ell}}$. ■

Proof: Consider the system (14), where $h(i,e)$ comes from Example 1. Consider the Lyapunov function:

$$V(i,e) = \sum_{k=i}^{\infty} |\phi(k,i,e)|^2 \ , \tag{23}$$

where it is obvious that $|e|^2 = \phi^2(i,i,e) \leq V(i,e)$ for all $e \in \mathbb{R}^{n_e}, i \in \mathbb{N}$. Moreover, note that $\phi(t+i,i,e) = 0$ for all $i \in \mathbb{N}$ and all $t \geq \ell$. Using this and also the fact that $|h(i,e)| \leq |e|$, we can write that $V(i,e) \leq \ell\,|e|^2$ for all $e \in \mathbb{R}^{n_e}, i \in \mathbb{N}$. Furthermore, we have that

$$\begin{aligned}
V(i+1, h(i,e)) &= V(i,e) + \sum_{k=i+1}^{\infty} |\phi(k,i,e)|^2 - \sum_{k=i}^{\infty} |\phi(k,i,e)|^2 \\
&= V(i,e) + \sum_{k=i+1}^{\infty} |\phi(k,i,e)|^2 - \sum_{k=i+1}^{\infty} |\phi(k,i,e)|^2 - |e|^2 \\
&= V(i,e) - |e|^2 \\
&= \frac{\ell-1}{\ell} V(i,e) \ ,
\end{aligned} \tag{24}$$

for all $e \in \mathbb{R}^{n_e}$ and all $i \in \mathbb{N}$. Finally, with the definition $W(i,e) := \sqrt{V(i,e)}$ we can obtain by direct calculation that (19) and (20) hold with $a_1 = 1$, $a_2 = \sqrt{\ell}$, $\rho = \sqrt{\frac{\ell-1}{\ell}}$, which completes the proof. ■

Proposition 5: The TOD protocol in Example 2 is UGES with the Lyapunov function $W(i,e) := |e|$. In particular, we can take $a_1 = a_2 = 1$ and $\rho = \sqrt{\frac{\ell-1}{\ell}}$. ■

Proof: It is obvious that (19) holds with $a_1 = a_2 = 1$ since $W(i,e) = |e|$. Consider arbitrary e and suppose without loss of generality that $|e_1| \geq \max_{j \in [2,\ell]} |e_j|$ and $h_1(i,e) = 0$. Then, we can write:

$$|e_1|^2 \ = \ \max_{j \in [1,\ell]} |e_j|^2 \ \geq \ \frac{1}{\ell} \sum_{j=1}^{\ell} |e_j|^2 \ = \ \frac{1}{\ell} |e|^2 \ . \tag{25}$$

[3] This choice is motivated by discussion in Proposition 3.

Using the properties of TOD protocol, we can write:

$$|h(i,e)|^2 \;=\; \sum_{j=2}^{\ell} |h_j(i,e)|^2 \;=\; \sum_{j=2}^{\ell} |e_j|^2 \;=\; |e|^2 - |e_1|^2 \;. \tag{26}$$

Finally, using (25) and (26) we have

$$W(i+1, h(i,e)) = |h(i,e)| \;\leq\; \sqrt{|e|^2 - \frac{1}{\ell}|e|^2} \;=\; \sqrt{\frac{\ell-1}{\ell}}\,|e| \;=\; \sqrt{\frac{\ell-1}{\ell}} W(i,e),$$

which completes the proof. ∎

Remark 8: Note that both RR and TOD protocols are actually finite-time dead-beat stable and this is a stronger property than UGES. However, we are allowing for more general (UGES) protocols, such as those that cut an error in half at each transmission time, rather than setting the error to zero.

Furthermore, it is possible to obtain a range of other Lyapunov functions to show UGES of the RR or TOD protocols. For example, consider the TOD protocol with ℓ nodes and define the Lyapunov function to be a p-norm of the error vector e with $p \in [1, \infty)$, that is

$$W_p(i,e) := \left(\sum_{i=1}^{\ell} |e_i|^p \right)^{\frac{1}{p}} \;.$$

Using the result of Proposition 5 and the equivalence of different norms, we obtain that (19) holds. By carrying out similar analysis like in the proof of Proposition 5, we can also see that (20) holds with

$$\rho = \left(\frac{\ell - 1}{\ell} \right)^{\frac{1}{p}} \;.$$

This flexibility is very useful since some Lyapunov functions may turn out to give less conservative values for MATI that guarantee stability of the system (see results in the next two sections). However, for simplicity of presentation we will always use the Lyapunov functions from Propositions 4 and 5 when analysing the RR and TOD protocols respectively. ∎

V. $\mathcal{L}_p$ STABILITY PROPERTIES OF ERROR DYNAMICS WITH UGES PROTOCOLS

In this section we show that protocols that are UGES with Lyapunov function W induce $\mathcal{L}_p$ stability for the error dynamics subsystem (13) under relatively mild conditions on W and error dynamics (13). In particular, we show that it is enough that W is Lipschitz in e uniformly in i and g in (13) satisfies a linear bound in e, x, w uniformly in t. Results of this section are crucial in showing $\mathcal{L}_p$ stability of NCS that is addressed in the next section. Moreover, results of this section are proved for the important situation of data packet dropouts that was motivated in [30]. The result for the case without dropout is obtained as a corollary of the result with dropouts. However, we present the result without dropouts first since it is easier to understand and interpret.

Proposition 6: Consider the NCS (12)-(14) and suppose that:

1. The protocol (14) is UGES with Lyapunov function W that is locally Lipschitz in e, uniformly in i.
2. There exists $L \geq 0$ such that for every $i \in \mathbb{N}$, all t, x, w and almost all e we have that the following holds:

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq L W(i,e) + |\tilde{y}| \;, \tag{27}$$

where $\tilde{y} : \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \to \mathbb{R}$ is a continuous function of (x, w).

3. The MATI satisfies $\tau \in (\epsilon, \tau^*)$, where $\tau^* = \frac{1}{L}\ln\left(\frac{1}{\rho}\right)$, $\rho > 0$ comes from item 1 and $\epsilon \in (0, \tau^*)$ is arbitrary. In particular, if $L = 0$ we have

$$\tau^* = \lim_{L \to 0} \frac{1}{L}\ln\left(\frac{1}{\rho}\right) = \infty \;.$$

Then, for each $p \in [1, \infty]$ the system (13) is $\mathcal{L}_p$ stable from $\tilde{y}$ to W with the gain

$$\gamma = \frac{\exp(L\tau) - 1}{L(1 - \rho \exp(L\tau))} \ . \tag{28}$$

In particular, if $L = 0$, then

$$\gamma = \lim_{L \to 0} \frac{\exp(L\tau) - 1}{L(1 - \rho \exp(L\tau))} = \frac{\tau}{1 - \rho}$$

Moreover, the system is IOS from $\tilde{y}$ to W with the gain (28) and an exp-$\mathcal{KL}$ function. ■

Remark 9: When an output y is preconceived, the following bound may be easier to obtain than (27):

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq LW(i,e) + c\,|y| \ , \tag{29}$$

where $c > 0$. In this case, we usually redefine the output $\tilde{y} := cy$ and then it is obvious that (27) holds. This observation is used in several examples presented below. ■

Remark 10: Note that item 3 of Proposition 6 requires MATI to be sufficiently small. The bound on MATI in item 3 depends only on two parameters: (i) convergence properties of the protocol (the constant ρ that comes from item 1); (ii) the system properties with respect to the given protocol (the constant L in item 2). Moreover, the $\mathcal{L}_p$ stability gain γ in (28) depends on ρ, L and τ.

Similar bounds were obtained in [26], [25] to guarantee stability of the overall NCS with static and TOD protocols. However, there are several differences between our results and those in the cited references. First, we consider a more general case of systems with disturbances. Second, we split the analysis of the overall system into analysis of the error dynamics (13) and closed loop system dynamics (12), which is normally not done in the references. Hence, the cited references never prove the result similar to Proposition 6 that talks only about the properties of the error dynamics. However, since Proposition 6 uncovers the main mechanism that is used in proving stability of NCS, it is very important conceptually. Third, the result we state holds for a much larger class of protocols than those considered in the references. Fourth, we will prove in the next section that the bounds we obtain are much less conservative than the bounds obtained in the references when our results are applied to the special case of systems without disturbances. ■

Remark 11: All conditions of Proposition 6 are checkable. Indeed, we have already demonstrated in the previous section that item 1 of Proposition 6 can be verified for important cases such as the RR and TOD protocols. Item 3 is easy to establish via items 1 and 2. Item 2 is also not hard to verify in many cases as the following discussion illustrates. Typically, we want to establish $\mathcal{L}_p$ stability from (x, w) to e. In this case, note that item 2 follows from Remark 9, (19) and the following conditions:

(i) there exists $L_1 \geq 0$ such that for almost all $e \in \mathbb{R}^{n_e}$ and all $i \in \mathbb{N}$ we have:

$$\left| \frac{\partial W(i,e)}{\partial e} \right| \leq L_1, \tag{30}$$

(ii) there exists $L_2 \geq 0$ such that for all $i \in \mathbb{N}$, $t \in [t_{s_i}, t_{s_{i+1}}]$, and all x and w we have:

$$|g(t,x,e,w)| \leq L_2(|e| + |x| + |w|) \ . \tag{31}$$

Indeed, we only need to define $\tilde{y} := L_1 L_2 (x^T \ w^T)^T$ and (29) follows with $L = L_1 L_2$. We note that the condition (30) is equivalent to the requirement that W is globally Lipschitz in e uniformly in i (via the Lebourg's Lipschitz Mean Value Theorem [5, Theorem 2.3.7]). This property is satisfied for the $W's$ we have proposed for the RR and TOD protocols. The following two examples illustrate this remark in more detail and show that taking $L = L_1 L_2$ might be quite conservative. ■

Example 3: Consider the linear NCS system with the RR protocol:

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}e && (32)\\ \dot{e} &= A_{21}x + A_{22}e && (33)\\ e(t_{s_i}^{+}) &= h(i, e(t_{s_i})) \ , && (34) \end{aligned}$$

where $h(\cdot,\cdot)$ was defined in Example 1. Let the Lyapunov function $W(i,e)$ come from Proposition 4. Note that since the RR protocol is dead-beat stable in ℓ steps, the Lyapunov function W can be written in the following manner:

$$W(i,e) = \sqrt{\sum_{j=1}^{\ell} a_j^2(i)\,|e_j|^2} =: |D(i)e| \ ,$$

where $D(i) := \mathrm{diag}\{a_1(i)I_{n_1}, a_2(i)I_{n_2}, \dots, a_\ell(i)I_{n_\ell}\}$, I_{n_j} are identity matrices of the dimension n_j with $n_e = \sum_{j=1}^{\ell} n_j$ and $a_j(i)$ are time varying coefficients that satisfy the following: for any $i \in \mathbb{N}$ and any $j \in \{1,2,\dots,\ell\}$ there exists a *unique* $k \in \{1,2,\dots,\ell\}$ such that

$$a_j^2(i) = k \ .$$

We also introduce $a(i) := (a_1(i)\ a_2(i)\ \dots a_\ell(i))^T$. A simple consequence of the above property of $a_j(i)$ is that for all $i \in \mathbb{N}$ we have:

$$|a(i)|_\infty \ := \ \max_j |a_j(i)| \ = \ \sqrt{\ell} \ . \qquad (35)$$

Also, we will denote in this case $\tilde{y} := \sqrt{\ell}A_{21}x$ and show that item 2 of Proposition 6 holds. With this notation we can write for almost all e:

$$\begin{aligned} \left\langle \frac{\partial W(i,e)}{\partial e}, A_{21}x + A_{22}e \right\rangle &\leq |D(i)\dot{e}| \\ &= |D(i)(A_{22}e + A_{21}x)| \\ &\leq |D(i)A_{22}e| + |D(i)A_{21}x| \\ &= |D(i)A_{22}e| + |\tilde{y}| \ . \end{aligned} \qquad (36)$$

In the worst case we can further bound (36) as follows:

$$\begin{aligned} \left\langle \frac{\partial W(i,e)}{\partial e}, A_{21}x + A_{22}e \right\rangle &\leq |a(i)|_\infty |A_{22}|\,|e| + |\tilde{y}| \\ &\leq \sqrt{\ell}\,|A_{22}|\,|e| + |\tilde{y}| \\ &\leq \sqrt{\ell}\,|A_{22}|\,|D(i)e| + |\tilde{y}| \\ &=: LW + |\tilde{y}| \ , \end{aligned} \qquad (37)$$

where $L = \sqrt{\ell}\,|A_{22}|$ and the second last inequality follows from the fact that $a_j(i) \geq 1$ for all $j \in \{1,2,\dots,\ell\}$ and $i \in \mathbb{N}$. This shows that in general item 2 of Proposition 6 holds with $L = \sqrt{\ell}\,|A_{22}|$.

Moreover, in the special case when $D(i)A_{22} = A_{22}D(i)$ (e.g. A_{22} is block diagonal), we can obtain a tighter bound that is needed in the case study in Section IX. Indeed, we can bound (36) in the following way:

$$\begin{aligned}\left\langle \frac{\partial W(i,e)}{\partial e}, A_{21}x + A_{22}e \right\rangle &\leq |A_{22}D(i)e| + |\tilde{y}| \\ &\leq |A_{22}|\,|D(i)e| + |\tilde{y}| \\ &= LW + |\tilde{y}| \ .\end{aligned} \tag{38}$$

This shows that for the special case item 2 of Proposition 6 holds with $L = |A_{22}|$. ■

Example 4: Let $\lambda_{\max}(A)$ denote the maximum eigenvalue of a symmetric matrix A. Consider the system (32)-(34), where the protocol (34) is the TOD protocol from Example 2. We use $W(i,e) = |e|$ and $\tilde{y} = A_{21}x$. Then, noting that $e^Te = |e|^2 = W^2$, we can write for almost all e:

$$\begin{aligned}\left\langle \frac{\partial W(i,e)}{\partial e}, A_{21}x + A_{22}e \right\rangle &= \frac{1}{W}\left(e^TA_{22}e + e^TA_{21}x\right) \\ &= \frac{0.5}{W}\left(e^TA_{22}^Te + e^TA_{22}e\right) + \frac{1}{W}e^TA_{21}x \\ &\leq \max\left\{\frac{1}{2}\lambda_{\max}(A_{22}^T + A_{22}), 0\right\}\frac{e^Te}{W} + \frac{|e|}{W}\cdot|A_{21}x| \\ &= LW + |\tilde{y}| \ .\end{aligned} \tag{39}$$

This shows that item 2 of Proposition 6 holds with $L = \max\left\{\frac{1}{2}\lambda_{\max}(A_{22}^T + A_{22}), 0\right\}$. ■

VI. Relaxed UGES protocols

Now we introduce a more general problem of NCS with data packet dropouts. We state and prove a result similar to Proposition 6 for this more general case and Proposition 6 becomes a simple corollary of this more general result. Data packet dropout may occur in NCS for various reasons, such as malfunction of the node (sensor or smart actuator) or message collision. While most protocols are equipped with transmission retry mechanisms, they can only retry for a limited time. After this time has lapsed, the data is "dropped" and a new data collected. Dropping old data may be advantageous for the operation of NCS since the delays in the network can be potentially reduced in this way. However, NCS can cope with limited data loss and quantifying the effect that data loss has on stability and performance is an important issue in analysis of NCS.

We will model data packet dropout by slightly changing the condition (20). In particular, we use instead of (20) the following inequality

$$W(i+1, h(i,e)) \leq \rho_{i+1}W(i,e) \ , \tag{40}$$

which holds for all $e \in \mathbb{R}^{n_e}$ and $i \in \mathbb{N}$, where $\rho_i \geq 0, \forall i \in \mathbb{N}$. If there is a packet dropout at transmission time t_{s_i}, then typically we set $\rho_{i+1} = 1$, since due to the dropout we can not expect improvement in the error at time t_{s_i}. For example, the error may not change at all at transmission time t_{s_i} in which case $e(t_{s_i}^-) = e(t_{s_i}^+)$ and this corresponds to setting $\rho_{i+1} = 1$. In the ideal situation without dropouts, $\rho_i = \rho \in [0,1), \forall i$ and hence we recover (20). For the purpose of generality, we also allow situations where the ρ_i is different from 1 and ρ.

In what follows we use the following notation. Given $\rho_i \geq 0, L > 0$ and $\tau > 0$ we introduce $\lambda_i := \rho_i \exp(L\tau)$ and for any two integers i and j with $i \leq j$ we define $\lambda_{i\to j} := \lambda_i \cdots \lambda_j$. We use the convention $\lambda_{(i+1)\to i} = 1$. For nonnegative integers $k \leq m$ we define

$$\sigma_{f_{km}} := \sum_{s=k}^{m} \lambda_{(k+1)\to s}; \qquad \sigma_{b_{km}} := \sum_{s=k}^{m} \lambda_{(s+1)\to m}; \qquad \overline{\sigma}_{f_{km}} := \max_{\{i=0,\ldots,m-k-1\}} \sigma_{f_{k+i,m}} \ . \tag{41}$$

Note that $\sigma_{f_{km}} \geq 1$ and $\sigma_{b_{km}} \geq 1$. We also use $\varphi(t) := \exp(Lt)$ for $t \in [0,\tau]$ and $||\varphi[0,\tau]||_{\mathcal{L}_p}$ for $p \in [1,\infty)$ and $||\varphi[0,\tau]||_{\mathcal{L}_\infty}$, where the notation was explained in the Preliminaries section. Note that if $\lambda_i \equiv \rho\exp(L\tau) < 1$ then we can establish the upper bound

$$\max\left\{\sigma_{b_{km}}, \sigma_{f_{km}}, \sigma_{b_{km}}^{\frac{p-1}{p}}\overline{\sigma}_{f_{km}}^{\frac{1}{p}}\right\} \leq \frac{1}{1-\rho\exp(L\tau)} \ . \tag{42}$$

The main result of this section is presented next (the proof is given in Section X).

Proposition 7: Consider the NCS (12)-(14) with a given MATI $\tau > 0$ and suppose that the following conditions hold:

1. Inequalities (19) and (40) hold, where W is locally Lipschitz in e, uniformly in i.
2. There exists $L > 0$ such that for every $i \in \mathbb{N}$, for almost all e and all (t,x,w) we have that the following holds:

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq LW(i,e) + |\tilde{y}| \ , \tag{43}$$

where $\tilde{y} : \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \to \mathbb{R}$ is a continuous function of (x,w).

Let L, ρ and τ from items 1 and 2 generate $\sigma_{f_{km}}$, $\sigma_{b_{km}}$ and $\overline{\sigma}_{f_{km}}$. If for some $p \in [1,\infty)$ there exists $c_p > 0$ such that

$$c_p \geq \sigma_{b_{km}}^{\frac{p-1}{p}}\overline{\sigma}_{f_{km}}^{\frac{1}{p}} \ , \qquad \forall m \geq k \geq 0 \ , \tag{44}$$

then the system (13) is $\mathcal{L}_p$ stable from $\tilde{y}$ to W with gain $\gamma_p = c_p\frac{\exp(L\tau)-1}{L}$. If, on the other hand, there exists $c_\infty > 0$ such that

$$c_\infty \geq \sigma_{b_{km}} \ , \qquad \forall m \geq k \geq 0 \ , \tag{45}$$

then the system (13) is $\mathcal{L}_\infty$ stable from $\tilde{y}$ to W with gain $\gamma_\infty = c_\infty\frac{\exp(L\tau)-1}{L}$. Finally, if (45) holds and there exists $\sigma > 0$ such that

$$\sigma \geq \sigma_{f_{km}}, \qquad \forall m \geq k \geq 0 \ ,$$

then the system (13) is IOS stable from $\tilde{y}$ to W with gain γ_∞ and an exp-$\mathcal{KL}$ function.

■

Remark 12: Notice that the gain γ_p depends on three parameters: MATI (i.e., τ), stability properties of the protocol with dropouts (the sequence ρ_i) and maximum growth of W between consecutive transmission times (the constant L). In particular, the result allows $\rho_i \geq 1$ for some $i \in \mathbb{N}$ and this covers the case of *multiple packet dropouts*. Indeed, as long as bounds (44) or (45) hold (and this may happen in the case of multiple packet dropouts), we can conclude $\mathcal{L}_p$ stability from Proposition 7. Notice also that $||\varphi[0,\tau]||_{\mathcal{L}_1} = \frac{\exp(L\tau)-1}{L}$ is of order $O(\tau)$ and, hence, if there exists $\tau_1^* > 0$ such the quantity $\sigma_{b_{km}}^{\frac{p-1}{p}}\overline{\sigma}_{f_{km}}^{\frac{1}{p}}$ is bounded for all $0 \leq k \leq m$ and all $\tau \in (0,\tau_1^*)$, then given any $K_p, p \in [1,\infty]$ there exists $\tau_2^* \in (0,\tau_1^*)$ such that inequalities (44) and (45) hold for all $\tau \in (0,\tau_2^*)$. In other words, we can arbitrarily reduce the $\mathcal{L}_p$ gain from $\tilde{y}$ to W by reducing τ. This situation holds always for the case without dropouts and it is crucial in establishing results on $\mathcal{L}_p$ stability of NCS in the next section. ■

VII. $\mathcal{L}_p$ STABILITY PROPERTIES OF NCS WITH UGES PROTOCOLS

In this section we present the main results of this paper which show that under mild conditions UGES protocols induce $\mathcal{L}_p$ stability of NCS for sufficiently small values of MATI. The proofs follow directly from Theorem 1. In the next section we concentrate on the class of linear NCS without disturbances that were considered in [26] in order to establish that our main results (Theorem 4) yield, in general, much less conservative bound on MATI than the bound obtained in [26]. This clearly illustrates the improvements that can be obtained using our results even when there are no disturbances. The example that is presented in the next section illustrates further this important point.

We first present the case without dropouts.

Theorem 4: Consider NCS (12)-(14). Suppose that the following conditions hold:

1. Inequalities (19), (20) and (27) with $\tilde{y} := H(x) + w$ hold.
2. System (12) is $\mathcal{L}_p$ stable from (W, w) to $H(x)$ with gain γ for some $p \in [1, \infty]$.
3. MATI satisfies $\tau \in (\epsilon, \tau^*)$ where

$$\tau^* := \frac{1}{L} \ln \left(\frac{L + \gamma}{\rho L + \gamma} \right) , \tag{46}$$

$\epsilon \in (0, \tau^*)$ is arbitrary, $L \geq 0$ comes from (27) and $\rho > 0$ comes from (20). In particular, if $L = 0$ then

$$\tau^* = \lim_{L \to 0} \frac{1}{L} \ln \left(\frac{L + \gamma}{\rho L + \gamma} \right) = \frac{1 - \rho}{\gamma} .$$

Then, the NCS is $\mathcal{L}_p$ stable from w to $(H(x), W(i, e))$. ■

A direct corollary of Theorems 4 , 1 and 3 is for the case when $p \in [1, \infty)$:

Corollary 2: Suppose that for some $p \in [1, \infty)$:

1. All conditions of Theorem 4 are satisfied and consequently the NCS is $\mathcal{L}_p$ stable from w to $(H(x), W(i, e))$;
2. The NCS is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $(H(x), W(i, e))$ to (x, e);
3. The origin of the NCS is UGFTIS with linear gain.

Then, the NCS with $w \equiv 0$ is UGES.

Remark 13: Note that Proposition 6 requires MATI to satisfy $\tau < \frac{1}{L} \ln \left(\frac{1}{\rho} \right) =: \tau_1^*$. Hence, it appears that we should require in (46) that $\tau^* \leq \tau_1^*$. However, we did not write this since the bound τ^* in (46) is always smaller than τ_1^*. ■

We present the result for the case with dropouts.

Theorem 5: Consider NCS (12)-(14). Suppose that the following conditions hold for some given $\tau > 0$ and $p \in [1, \infty]$

1. All conditions of Proposition 7 hold with $\tilde{y} := H(x) + w$ so that the system (13) is $\mathcal{L}_p$ stable from $(H(x), w)$ to W with gain γ_1.
2. System (12) is $\mathcal{L}_p$ stable from (W, w) to $H(x)$ with gain γ_2.
3. The small gain condition holds, that is $\gamma_1 \gamma_2 < 1$.

Then, NCS (12)-(14) is $\mathcal{L}_p$ stable from w to $(H(x), W)$. ■

Remark 14: We note that conditions of Theorem 5 are relatively hard to check since we have taken a deterministic approach to analysis of stability of NCS with dropouts. More natural conditions would probably require stochastic analysis where dropouts are modeled by a random process with certain mean and variance and the conditions of the theorem are stated in probabilistic terms. This problem is an interesting topic for further research. ■

We can also state the following:

Theorem 6: Consider NCS (12)-(14). Suppose that the following conditions hold:

1. Inequalities (19), (20) and (27) with $\tilde{y} := H(x) + w$ hold.
2. System (12) is IOS stable from (W, w) to $H(x)$ with gain γ.
3. MATI satisfies $\tau \in (\epsilon, \tau^*)$ where

$$\tau^* := \frac{1}{L} \ln \left(\frac{L + \gamma}{\rho L + \gamma} \right) \tag{47}$$

$\epsilon \in (0, \tau^*)$ is arbitrary, $L \geq 0$ comes from (27) and $\rho > 0$ comes from (20).

Then, the NCS is IOS stable from w to $(H(x), W(i, e))$ with linear gain. ■

Theorem 7: Consider NCS (12)-(14). Suppose that the following conditions hold

1. All conditions of Proposition 7 hold with $\tilde{y} := H(x) + w$ so that the system (13) is IOS from $(H(x), w)$ to W with gain γ_1.

2. System (12) is IOS from (W, w) to $H(x)$ with gain γ_2
3. The small gain condition holds, that is $\gamma_1\gamma_2 < 1$.

Then, NCS (12)-(14) is IOS stable from w to $(H(x), W)$. ■

A consequence of Theorem 7 and Corollary 1 is

Corollary 3: Suppose that:

1. All conditions of Theorem 7 are satisfied and, hence, the NCS is IOS from w to $(H(x), W)$;
2. The NCS is IOSS from $(H(x), W)$ to (x, e).

Then, the NCS is ISS from w to (x, e) and, in particular, when $w \equiv 0$, the NCS is UGAS. If all the above properties hold with exp-$\mathcal{KL}$ functions and linear gains, then the NCS is exponentially stable when $w \equiv 0$. ■

Remark 15: Note that our results cover the case where the system (12) is only UGAS when $w \equiv 0$, and in this sense we generalize results of [25] that hold only for systems (12) that are UGES when $w \equiv 0$. However, we do require the system to have a linear gain. An example of a system that is UGAS in absence of disturbances but that has a linear gain is given by:

$$\dot{x} = -x^3 + w^3 \ ,$$

which can be seen by using the Lyapunov function $V(x) = \frac{1}{4}x^4$. By direct calculations we have $\dot{V} \leq -\frac{1}{2}x^6 - \frac{1}{2}|x|^3(|x|^3 - 2|w|^3)$, which implies that

$$|x| \geq 2^{\frac{1}{3}}\,|w| \implies \dot{V} \leq -\frac{1}{2}x^6 = -\frac{1}{2}(2V(x))^{\frac{3}{2}} \ .$$

Using [14, Theorem 5.2] we can conclude that the system is ISS with the linear gain. ■

A. Special case: a constant transmission interval

In this subsection we show that if the NCS has a constant transmission interval (that is $t_{s_{i+1}} - t_{s_i} = \tau = const., \forall i$) then the model (12)-(14) induces a discrete-time model that can be used for analysis of systems properties. In this sense, NCS with a constant transmission interval resemble standard sampled-data systems (see calculations below). Moreover, if both the system and the protocol are linear, then our results are not needed since the stability bound on MATI can be determined by an eigenvalue computation.

To make this observation more precise consider the system (12)-(14) and assume that there exists $\tau > 0$ such that $t_{s_{i+1}} - t_{s_i} = \tau, \forall i \in \mathbb{N}$. For simplicity, we assume that there are no exogenous disturbances, that is $w(\cdot) \equiv 0$. Consider now the sequence of transmission times $t_{s_i}^+$. By integrating (12) and (13) we can write the following equations:

$$x(t_{s_{i+1}}) = \bar{F}(t_{s_{i+1}}, t_{s_i}, x(t_{s_i}^+), e(t_{s_i}^+)) \tag{48}$$

$$e(t_{s_{i+1}}) = \bar{G}(t_{s_{i+1}}, t_{s_i}, x(t_{s_i}^+), e(t_{s_i}^+)) \ , \tag{49}$$

where $\bar{F}$ and $\bar{G}$ are respectively solutions of (12) and (13) at time $t_{s_{i+1}}$ starting from initial time t_{s_i} and initial state $(x(t_{s_i}^+), e(t_{s_i}^+))$. Note that since $t_{s_{i+1}} = t_{s_i} + \tau$, we can rewrite these equations as follows

$$x(t_{s_{i+1}}) = F(\tau + t_{s_i}, x(t_{s_i}^+), e(t_{s_i}^+)) \tag{50}$$

$$e(t_{s_{i+1}}) = G(\tau + t_{s_i}, x(t_{s_i}^+), e(t_{s_i}^+)) \tag{51}$$

$$, \tag{52}$$

with obvious definitions of F and G. Note also that $x(t_{s_i}^+) = x(t_{s_i}), \forall i \in \mathbb{N}$ and by using (14) we can write:

$$x(t_{s_{i+1}}^+) = F(\tau + t_{s_i}, x(t_{s_i}^+), e(t_{s_i}^+)) \tag{53}$$

$$e(t_{s_{i+1}}^+) = h(i+1, G(\tau + t_{s_i}, t_{s_i}, x(t_{s_i}^+), e(t_{s_i}^+))) \tag{54}$$

which is a discrete-time system that describes the NCS at transmission times $t_{s_i}^+$.

If the underlying continuous-time dynamics of NCS is nonlinear, then the first difficulty is that it is not possible to compute the model (50) and (51) exactly as this requires analytic solution of a nonlinear ordinary differential equation. On the other hand, a range of approximate discrete-time models that approximate (50), (51) can be found by using standard numerical integration methods such as Runge-Kutta. Tools for analysis of the exact discrete-time model (53), (54) via its approximate models are developed in [16], [17], [18], [19] and can be used in this context.

On the other hand, if the continuous-time NCS dynamics is linear time-invariant and we use a protocol with linear dynamics, such as the RR protocol, then (53), (54) takes the following form:

$$\begin{aligned} x(t_{s_{i+1}}^+) &= F_{11}x(t_{s_i}^+) + F_{12}e(t_{s_i}^+) && (55)\\ e(t_{s_{i+1}}^+) &= F_{21}(i)x(t_{s_i}^+) + F_{22}(i)e(t_{s_i}^+)\ , && (56)\end{aligned}$$

where $F_{21}(i), F_{22}(i)$ are periodically time varying matrices. Tools for analysis and design of periodically time-varying linear systems are well developed (see [6], [7]) and they can be used directly in analysis of NCS via its discrete-time model (55), (56). Moreover, in this case one can compute analytically the bound on MATI that preserves stability.

Finally, we remark that for the case when $w \not\equiv 0$ we can use results from [29] to conclude about L_p stability of the system from UGES of the underlying discrete-time model of the system with $w \equiv 0$.

VIII. Comparison with existing results

We now show that our results (when specialized to analysis of exponential stability for systems without disturbances) yield much less conservative bounds than those obtained in [25] for nonlinear NCS and in [26], [30] for linear NCS.

A. Summary of existing results

The following class of nonlinear systems was considered in [25]

$$\begin{aligned} \dot{x} &= f(t,x,e)\\ \dot{e} &= g(t,x,e)\ , \end{aligned} \qquad (57)$$

where also the shorthand notation is used:

$$\dot{z} = h(t,z)\ , \qquad (58)$$

with $z = (x^T\ e^T)^T$. Lipschitz constants for f, g and h are denoted respectively as k_f, k_g and k_h. The following class of linear systems was considered in [26], [30]:

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}e\\ \dot{e} &= A_{21}x + A_{22}e\ , \end{aligned} \qquad (59)$$

where also the shorthand notation is used:

$$\dot{z} = Az\ , \qquad (60)$$

with $z = (x^T\ e^T)^T$.

It is supposed in [25] that there exists a continuously differentiable Lyapunov function V such that the system (57) satisfies for all $x \in \mathbb{R}^{n_x}$:

$$\begin{aligned} c_1|x|^2 \ \le\ & V(t,x) \ \le\ c_2|x|^2 && (61)\\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x,0) \ \le\ & -c_3|x|^2 && (62)\\ \left|\frac{\partial V}{\partial x}\right| \ \le\ & c_4\,|x|\ , && (63)\end{aligned}$$

where c_1, c_2, c_3, c_4 are positive constants. A similar condition was used in [26], [30] for the linear system (59). Indeed, it was assumed that for some positive definite and symmetric matrix Q there exists a positive definite and symmetric matrix P that solves the Lyapunov matrix equation[4]:

$$A_{11}^T P + P A_{11} = -Q \; . \tag{64}$$

It is obvious that (64) implies that (61), (62), (63) are satisfied for the linear system (59) with $V(x) = x^T P x$ and

$$c_1 \;=\; \lambda_{\min}(P);\; c_2 \;=\; \lambda_{\max}(P);\; c_3 \;=\; \lambda_{\min}(Q)\;;\;\; c_4 \;=\; 2\lambda_{\max}(P). \tag{65}$$

Moreover, note that, for linear systems, we can (conservatively) let

$$k_h \;=\; k_f \;=\; k_g \;=\; |A| \, . \tag{66}$$

A bound on MATI that guarantees the stability of the linear system (59) with the RR and TOD protocols was obtained in [26], [30]. We denote bounds computed in [26], [30] respectively as τ_{old}^{RR} and τ_{old}^{TOD} for the RR and TOD protocols. Results in [26], [30] do not distinguish between different protocols and, in particular, it is obtained that $\tau_{old}^{RR} = \tau_{old}^{TOD}$. Similar results were obtained in [25] for nonlinear systems (57) with the RR and TOD protocols. Again, it is obtained for nonlinear systems[5] that $\tau_{old}^{RR} = \tau_{old}^{TOD}$. The bounds in [25], [26], [30] can be written in the following way:

$$\tau_{old}^{RR} = \tau_{old}^{TOD} \;=\; \frac{c_3}{M\ell(\ell+1)k_h k_f c_4} \; . \tag{67}$$

where the value of the constant M is different for the linear and nonlinear systems. Using results in [25] we have for nonlinear systems that

$$M = M_{NL} := 16\left(\frac{c_2}{c_1}\right)^{3/2}\left(\sqrt{\frac{c_2}{c_1}}+1\right) \; . \tag{68}$$

On the other hand, in [26], [30] the following is obtained for linear systems

$$M = M_L := 8\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}+1\right) \; , \tag{69}$$

and in the linear case the meaning of all constants in (67) is explained by (65) and (66).

Remark 16: We note that results in [26] provide the same MATI bound for both RR and TOD protocols, that is $\tau_{old}^{TOD} = \tau_{old}^{RR}$. Our bounds are, on the other hand, different in general. We point out that the real stability bounds for different protocols will be typically different and hence our result appears to be more natural. Example in the next section illustrates this point very well, where it is shown that the "real" stability bound[6] for the batch reactor with the TOD protocol is 0.089 *sec* and the stability bound for the same system with the RR protocol is 0.0657 *sec*. ■

B. Conservative estimates of our bounds on MATI

In this section we provide very conservative lower bounds (estimates) for the bound on MATI obtained using our main result. Since our bounds distinguish between different protocols, we first present the calculations for the RR protocol and then explain the changes needed to obtain the bound for the TOD protocol. Our calculations are presented only for the nonlinear system (57) since the result for the linear systems (59) is obtained as a special case of the nonlinear result by using (65) and (66).

[4]Results in [26] are only presented for the special case $Q = I$. The result with general Q is presented in [30].

[5]Note that we do not use different notation for MATI bounds for linear and nonlinear systems, although they are different in general. This is because it will be always clear from the context which bound we are talking about.

[6]In the case of constant transmission time intervals. The TOD bound is obtained via simulations of the system and the bound for RR protocol is determined using an eigenvalue computation (see Remark VII-A).

Repeating calculations from Example 3 and using the fact that $W(i,e) \geq |e|$ from Proposition 4, we have for the RR protocol that

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq \sqrt{\ell} k_h (|e| + |x|) \leq L W(i,e) + |\tilde{y}| \ , \tag{70}$$

where we denoted

$$L \ := \ \sqrt{\ell} k_h; \qquad \tilde{y} := Lx \ . \tag{71}$$

Using (62) and (63) we obtain:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,e) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x} \left(f(t,x,e) - f(t,x,0) \right) \\ &\leq -c_3 |x|^2 + c_4 k_f |x| \cdot |e| \ . \end{aligned} \tag{72}$$

By integrating, dividing by c_3 and using the Hölder inequality, we obtain:

$$\begin{aligned} \|x[t_0,t]\|_{\mathcal{L}_2}^2 &\leq \frac{V(x(t_0))}{c_3} + \frac{c_4 k_f}{c_3} \int_{t_0}^{t} |x(s)| \cdot |e(s)| \, ds \\ &\leq \frac{V(x(t_0))}{c_3} + \frac{c_4 k_f}{c_3} \|x[t_0,t]\|_{\mathcal{L}_2} \cdot \|e[t_0,t]\|_{\mathcal{L}_2} \ . \end{aligned} \tag{73}$$

This is a quadratic inequality in $\|x[t_0,t]\|_{\mathcal{L}_2}$. Note that for arbitrary quadratic inequality of the form $as^2 - bs - c \leq 0$ with $a, b, c \geq 0$ and $s \geq 0$ we have that

$$\begin{aligned} s &\leq \frac{b + \sqrt{b^2 + 4ac}}{2} \\ &\leq \frac{b + b + \sqrt{4ac}}{2} \\ &\leq b + \sqrt{ac} \ . \end{aligned} \tag{74}$$

Hence, using (73) and (74) with $s = \|x[t_0,t]\|_{\mathcal{L}_2}$, $a = 1$, $b = \frac{c_4 k_f}{c_3} \|e[t_0,t]\|_{\mathcal{L}_2}$ and $c = \frac{V(x(t_0))}{c_3}$, we can write that:

$$\|x[t_0,t]\|_{\mathcal{L}_2} \leq \sqrt{\frac{V(x(t_0))}{c_3}} + \frac{c_4 k_f}{c_3} \|e[t_0,t]\|_{\mathcal{L}_2} \ , \tag{75}$$

which shows that the gain from e to x is equal to $\gamma_{ex} = \frac{c_4 k_f}{c_3}$. Using (71) we have $|\tilde{y}| \leq L\,|x| = \sqrt{\ell} k_h \,|x|$ and since we also have that $|e| \leq W(i,e)$, this implies that the gain from W to $\tilde{y}$ is $\gamma = \gamma_{W\tilde{y}} \leq L \gamma_{ex} = \frac{\sqrt{\ell} c_4 k_f k_h}{c_3}$. Now we use our formula for MATI and the fact that $\gamma \leq L \frac{c_4 k_f}{c_3}$ to write:

$$\tau_{new}^{RR} = \frac{1}{L} \ln \left(\frac{L + \gamma}{\rho L + \gamma} \right) = \frac{1}{L} \ln \left(\frac{c_3 + c_4 k_f}{\rho c_3 + c_4 k_f} \right) \ . \tag{76}$$

Using the Mean Value Theorem and the expression for ρ from Proposition 4, as well as the fact that $\rho < 1$, we can write:

$$\begin{aligned} \ln \left(\frac{c_3 + c_4 k_f}{\rho c_3 + c_4 k_f} \right) &= \ln(c_3 + c_4 k_f) - \ln(\rho c_3 + c_4 k_f) \\ &\geq \frac{(1-\rho) c_3}{c_3 + c_4 k_f} \ . \end{aligned} \tag{77}$$

Using the Mean Value Theorem again we can write:

$$(1 - \rho) = \frac{\sqrt{\ell} - \sqrt{\ell - 1}}{\sqrt{\ell}} \geq \frac{1}{2\sqrt{\ell}} \frac{1}{\sqrt{\ell}} = \frac{1}{2\ell} \ . \tag{78}$$

Hence, from (76), (77) and (78) and the definition of L we obtain:

$$\tau_{new}^{RR} \geq \frac{(1-\rho) c_3}{L(c_3 + c_4 k_f)} \geq \frac{(1-\rho) c_3}{L c_4 k_f} \geq \frac{c_3}{2 L \ell c_4 k_f} = \frac{c_3}{2 \ell^{\frac{3}{2}} c_4 k_f k_h} \ . \tag{79}$$

Linear Systems [26], [30]	
RR Protocol	$\frac{\tau_{new}^{RR}}{\tau_{old}^{RR}} \geq 2\frac{\ell+1}{\sqrt{\ell}}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}+1\right)$
TOD Protocol	$\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \geq 2(\ell+1)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}+1\right)$
Nonlinear Systems [25]	
RR Protocol	$\frac{\tau_{new}^{RR}}{\tau_{old}^{RR}} \geq 8\frac{\ell+1}{\sqrt{\ell}}\left(\frac{c_2}{c_1}\right)^{\frac{3}{2}}\left(\sqrt{\frac{c_2}{c_1}}+1\right)$
TOD Protocol	$\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \geq 8(\ell+1)\left(\frac{c_2}{c_1}\right)^{\frac{3}{2}}\left(\sqrt{\frac{c_2}{c_1}}+1\right)$

TABLE VIII-B: SUMMARY OF COMPARISONS OF OUR RESULTS WITH EXISTING RESULTS.

Calculations for the TOD protocol follow exactly the same steps except that (70) holds with $L = k_h$ instead of $L = \sqrt{\ell}k_h$ as in (71) and we define $\tilde{y}$ using (71) with the new L. Note that the value of ρ is the same for the RR and TOD protocols (Propositions 4 and 5) and that we have also $|e| \leq W(i,e)$ and, hence, the only difference is in the value of L. With this small change and following the same calculations as above, we obtain that

$$\tau_{new}^{RR} \geq \frac{c_3}{2\ell c_4 k_f k_h} \ . \tag{80}$$

Finally, using (67), (68), (69), (79) and (80) we obtain the comparisons of our bounds to results in [25], [26], [30] that are summarized in Table VIII-B.

We now analyze results in Table VIII-B. Consider linear systems with the TOD protocol (the second row in the table). First, note that $\frac{\lambda_{max}(P)}{\lambda_{min}(P)} \geq 1$ and in the extreme case we have $\frac{\lambda_{max}(P)}{\lambda_{min}(P)} = 1$. Assume that this "best" case holds. We obtain that

$$\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \geq 4(\ell+1) \ ,$$

and, hence, for the simplest nontrivial case $\ell = 2$ we have that our bound is at least 12 times less conservative than the bound in [26]. Moreover, the larger the number of nodes ℓ, the less conservative our bound when compared to the one in [26]. In particular, as $\ell \to \infty$

$$\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \to \infty \ . \tag{81}$$

For instance, if $\ell = 10$ we have that our bound is at least 44 times larger than the bound in [26]. Suppose now that ℓ is fixed. In this case we have that the larger the ratio $\frac{\lambda_{max}(P)}{\lambda_{min}(P)}$, the less conservative our bound is compared to the bound in [26]. In particular, as $\frac{\lambda_{max}(P)}{\lambda_{min}(P)} \to \infty$ we have that (81) holds.

The formulas presented in Table VIII-B used a range of conservative bounds of τ_{new}^{TOD} and τ_{new}^{RR} that will not be needed in particular examples. Hence, we can expect that the ratio $\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}}$ is much larger than what the above discussion may suggest. Indeed, the example in the next section considers exactly this situation and we obtain that $\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} = 1000$.

Similar observations hold for nonlinear systems and/or the RR protocol cases from Table VIII-B. The only difference is that for the TOD protocol $\frac{\tau^{RR}_{new}}{\tau^{RR}_{old}}$ grows slower with ℓ because of the extra $\sqrt{\ell}$ term in the denominator (see the formulas in the second and fourth rows of Table VIII-B).

IX. Case study: a batch reactor

To further illustrate that our results improve considerably results of [26] even in the case when there are no disturbances, we consider the unstable batch reactor example considered in [26]. The linearized model of an unstable batch reactor is a two-input-two-output NCS that can be written as

$$\dot{x}_P \;=\; A_P x_P + B_P u; \qquad y \;=\; C_P x_P \;,$$

where

$$A_P = \begin{pmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{pmatrix}; \; B_P = \begin{pmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{pmatrix}; \; C_P = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This system is controlled by the PI controller whose transfer function is

$$K(s) = \begin{pmatrix} 0 & \frac{2s+2}{s} \\ \frac{-5s-8}{s} & 0 \end{pmatrix} ,$$

with its state space realization:

$$\dot{x}_C \;=\; A_C x_C + B_C y; \qquad u \;=\; C_C x_C + D_C y \;,$$

and

$$A_C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \qquad B_C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad C_C = -\begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix}; \qquad D_C = -\begin{pmatrix} 0 & 2 \\ -5 & 0 \end{pmatrix}.$$

Assuming only that the outputs are transmitted via the network, we have that $e = \hat{y} - y$ and using computations similar to those used to obtain the formula (1) in [26], the equations we need to consider take the form:

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} , \tag{82}$$

where

$$A_{11} = \begin{pmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{pmatrix}; \qquad A_{12} = \begin{pmatrix} B_P D_C \\ B_C \end{pmatrix}$$

$$A_{21} = -\begin{pmatrix} C_P & 0 \end{pmatrix} A_{11}; \qquad A_{22} = -\begin{pmatrix} C_P & 0 \end{pmatrix} A_{12} \;.$$

TOD protocol: We first compute[7] the $\mathcal{L}_2$ gain for the x subsystem from the input e to the "output" $\tilde{y} := A_{21}x$, which is $\gamma \approx 15.9222$. Note that since $W(e) = |e|$ the gain from W to $\tilde{y}$ is bounded by γ. Using Proposition 5 we see that the TOD protocol is UGES with Lyapunov function $W(e) = |e|$, where for two nodes ($\ell = 2$ since we have two outputs transmitted via the network) we have that $\rho = \sqrt{\frac{1}{2}} \approx 0.7071$. Furthermore, using Example 4 we have that the error subsystem satisfies the bound

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq LW + |\tilde{y}| \;, \tag{83}$$

[7] Computation of $\mathcal{L}_p$ gains was done using the μ-synthesis toolbox in MATLAB.

TOD protocol		
Definition	Notation	Value [sec]
simulation based actual bound on MATI (with random t_{s_j}) obtained in [26]	τ^{TOD}_{rand}	0.06
simulation based actual bound on MATI (with equidistant t_{s_j})	τ^{TOD}_{const}	0.089
theoretical bound on MATI computed in [26]	τ^{TOD}_{old}	10^{-5}
theoretical bound via Theorem 4	τ^{TOD}_{new}	0.01
RR protocol		
Definition	Notation	Value [sec]
analytically computed actual bound on MATI (with equidistant t_{s_j})	τ^{RR}_{const}	0.0657
theoretical bound on MATI computed in [26]	τ^{RR}_{old}	10^{-5}
theoretical bound via Theorem 4	τ^{RR}_{new}	0.0082

TABLE IX: SUMMARY OF BOUNDS ON MATI THAT GUARANTEE STABILITY.

where $L = \lambda_{\max}\left(0.5(A_{22}^T + A_{22})\right) = 15.73$ and $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix. Finally, using our Theorem 4 and Corollary 2 we obtain that the NCS is UGES for all MATI $\tau \in (0, \tau^{TOD}_{new})$ where

$$\tau^{TOD}_{new} = \frac{1}{L}\ln\left(\frac{L+\gamma}{\rho L+\gamma}\right) \approx 0.01 \ .$$

The theoretical bound on MATI obtained using results in [26] for which NCS remains stable is $\tau \in (0, \tau^{TOD}_{old})$, where $\tau^{TOD}_{old} \approx 10^{-5}$ *sec* (see pg. 443). The simulations with randomly generated transmission times t_{s_j} reported in [26] showed that the system remains stable for the values of MATI $\tau \in (0, \tau^{TOD}_{rand})$, where $\tau^{TOD}_{rand} \approx 0.06$ *sec*. Note that

$$\frac{\tau^{TOD}_{new}}{\tau^{TOD}_{old}} \approx 1000 \text{ and } \frac{\tau^{TOD}_{rand}}{\tau^{TOD}_{new}} \approx 6 \ .$$

Hence, our result yields a theoretical bound on MATI that is about 1000 times less conservative than the bound obtained in [26]! This is only about 6 times more conservative that the real bound on MATI observed in simulations. We have also carried out simulations for the case when transmission times t_{s_j} are equidistant and we obtained a slightly larger value for the MATI for which stability is preserved $\tau^{TOD}_{const} \approx 0.089$ *sec* and, hence, our result is slightly more conservative for this situation.

Finally, we remark that if we assume that both outputs and inputs are transmitted via the network, then we can use equations (1) in [27] as the appropriate model of NCS. Using the same computations as above for the new model, we obtain MATI $\tau^* = 0.0089$ *sec*.

RR protocol: Suppose now that RR protocol is used for the same system. In this case, it turns out that the Lyapunov function from Proposition 4 takes the form:

$$W(i,e) = \sqrt{\sum_{k=i}^{\infty} \phi^2(k,i,e)} = \sqrt{a_1^2(i)e_1^2 + a_2^2(i)e_2^2} \ ,$$

where $a_j^2(i) = 1$ or 2 for $j = 1,2$ and all $i \in \mathbb{N}$. This implies that $\max\{a_1(i), a_2(i)\} \leq \sqrt{2}$ (see Example 3). Moreover, the error system has the diagonal form:

$$\begin{aligned} \dot{e}_1 &= 15.73 e_1 + f_1 \\ \dot{e}_2 &= 11.358 e_2 + f_2 \ , \end{aligned}$$

where $(f_1\ f_2)^T := A_{21}x$. In this case, we let $\tilde{y} := \sqrt{2}A_{21}x$ (note the difference with the $\tilde{y}$ defined in the case of TOD protocol). As a result, we can show that for almost all e we have

$$\left\langle \frac{\partial W(i,e)}{\partial e}, g(t,x,e,w) \right\rangle \leq LW + |\tilde{y}| \ , \tag{84}$$

where $L \approx 15.73$ is the same as in the previous example (see the equation (38) and the surrounding discussion). The slight difference between the definition of $\tilde{y}$ in (83) and (84) gives the gain from W to the newly defined $\tilde{y}$ equal to $\gamma_1 = \gamma\sqrt{2}$, where $\gamma \approx 15.2222$ was computed in the previous example as the gain from W to $A_{21}x$. Finally, using our Theorem 4 and Corollary 2 we obtain that the NCS is UGES for all MATI $\tau \in (0, \tau^{RR}_{new})$ where

$$\tau^{RR}_{new} = \frac{1}{L}\ln\left(\frac{L+\gamma_1}{\rho L + \gamma_1}\right) \approx 0.0082\ sec\ .$$

Assume now that the transmission times are equidistant. Using the procedure described in Remark VII-A it was computed for this situation that $\tau^{RR}_{const} = 0.0657\ sec$. Hence, we have that

$$\frac{\tau^{RR}_{const}}{\tau^{RR}_{new}} \approx 8\ ,$$

which means that our theoretical bound is only about 8 times more conservative than the real bound obtained from analytic calculations.

X. Proofs of main results

Proof of Proposition 7:

Case $p \in [1, \infty)$:

Let $t_0 \geq 0$ be given and let $k \in \mathbb{N}$ be such that $t_0 \in [t_{s_k}, t_{s_{k+1}})$. In the analysis that follows we denote t_0 as t_{s_k} to simplify the notation in the proof.

Since (27) holds for all (t,x,w) and almost all e, then for all integers i and almost all t we have[8]:

$$\frac{d}{dt}W(i,e(t)) \leq LW(i,e(t)) + |\tilde{y}|\ . \tag{85}$$

Applying standard comparison lemmas to the above inequality we get that for all i and all $\theta \in [t_{s_i}, t_{s_{i+1}}]$ we have

$$W(i,e(\theta)) \leq \exp(L(\theta - t_{s_i}))W(i,e(t^+_{s_i})) + \int_{t_{s_i}}^{\theta} \exp(L(\theta - s))|\tilde{y}(s)|ds\ . \tag{86}$$

Define

$$\eta := (\exp(-L\tau)||\varphi[0,\tau]||_{\mathcal{L}_1})^{\frac{p-1}{p}}\ . \tag{87}$$

Using (86) with $\theta = t_{s_{i+1}}$ and Hölder's inequality, we can write (see calculations in [14, pp. 265–266]):

$$\begin{aligned} W(i,e(t_{s_{i+1}})) &\leq \exp(L(t_{s_{i+1}} - t_{s_i}))W(i,e(t^+_{s_i})) + \int_{t_{s_i}}^{t_{s_{i+1}}} \exp(L(t_{s_{i+1}} - s))|\tilde{y}(s)|ds \\ &\leq \exp(L\tau)W(i,e(t^+_{s_i})) + ||\varphi[0,\tau]||_{\mathcal{L}_1}^{\frac{p-1}{p}} \left(\int_{t_{s_i}}^{t_{s_{i+1}}} \exp(L(t_{s_{i+1}} - s))|\tilde{y}(s)|^p ds\right)^{\frac{1}{p}} \\ &\leq \exp(L\tau)W(i,e(t^+_{s_i})) + \exp(L\tau)^{\frac{1}{p}} \cdot ||\varphi[0,\tau]||_{\mathcal{L}_1}^{\frac{p-1}{p}} \cdot \left\|y[t_{s_i}, t_{s_{i+1}}]\right\|_{\mathcal{L}_p}\ . \end{aligned} \tag{88}$$

[8]This follows from the discussion in [23, pp.99-100]. Indeed, since W is locally Lipschitz we have that $\dot{W}(i,e(t))$ exists and is equal for almost all t to the one-sided directional derivative. Moreover, it is known that the Clarke generalized directional derivative $W^\circ(e; g(t,x,e,w))$ upper bounds the one-sided directional derivative. Finally, from item 5 in [23, pg.100] we have that if $\langle\frac{\partial W(i,e)}{\partial e}, g(t,x,e,w)\rangle \leq LW(i,e) + |\tilde{y}|$ holds for almost all e and all t,x,w, then we have that for all i and almost all t we have that $\dot{W}(i,e(t)) \leq W^\circ(e; g(t,x,e,w)) \leq LW(i,e(t)) + |\tilde{y}(t)|$.

Using (88), $L \geq 0$, definition $\lambda_i := \rho_i \exp(L\tau)$ and (40), we get that, for each nonnegative integer i,

$$\begin{aligned} W(i+1, e(t_{s_{i+1}}^+)) &\leq \lambda_{i+1} \left(W(i, e(t_{s_i}^+)) + \exp(L\tau)^{-1} \exp(L\tau)^{1/p} ||\varphi[0,\tau]||_{\mathcal{L}_1}^{\frac{p-1}{p}} ||\tilde{y}[t_{s_i}, t_{s_{i+1}}||_{\mathcal{L}_p} \right) \\ &= \lambda_{i+1} \left(W(i, e(t_{s_i}^+)) + \eta ||\tilde{y}[t_{s_i}, t_{s_{i+1}}||_{\mathcal{L}_p} \right) . \end{aligned} \tag{89}$$

It follows by iterating that, for each nonnegative integer k and each integer $i \geq k$,

$$W(i, e(t_{s_i}^+)) \leq \lambda_{(k+1)\to i} W(k, e(t_{s_k}^+)) + \eta \sum_{j=k}^{i-1} \lambda_{(j+1)\to i} ||\tilde{y}[t_{s_j}, t_{s_{j+1}}]||_{\mathcal{L}_p} . \tag{90}$$

Combining (86) and (90) we see that $W(\cdot, e(\cdot))$ is bounded by two terms, one involving $W(k, e(t_{s_k}^+))$ and one involving a summation of terms involving $\tilde{y}(\cdot)$. From the triangle inequality for signal norms, we can compute a bound on the $\mathcal{L}_p$ norm of $W(\cdot, e(\cdot))$ by computing bounds on the $\mathcal{L}_p$ norms of the two pieces separately.

Term involving $W(k, e(t_{s_k}^+))$: To handle the term involving $W(k, e(t_{s_k}^+))$ we set $\tilde{y} \equiv 0$ and we get, for each integer $i \geq k$ and all $\theta \in [t_{s_i}, t_{s_{i+1}}]$,

$$W(i, e(\theta)) \leq \exp(L(\theta - t_{s_i})) \lambda_{(k+1)\to i} W(k, e(t_{s_k}^+)) \tag{91}$$

and, for all $i \in \{k, \ldots, m\}$,

$$W(i, e(\theta))^p \leq \varphi(\theta - t_{s_i})^p \sigma_{f_{km}}^{p-1} \lambda_{(k+1)\to i} W(k, e(t_{s_k}^+))^p . \tag{92}$$

Integrating and summing we get for all $t \in [t_{s_k}, t_{s_m}]$,

$$||W[t_{s_k}, t]||_{\mathcal{L}_p}^p \leq ||\varphi[0,\tau]||_{\mathcal{L}_p}^p \sigma_{f_{km}}^{p-1} \sigma_{f_{km}} W(k, e(t_{s_k}^+))^p \tag{93}$$

and thus

$$||W[t_{s_k}, t]||_{\mathcal{L}_p} \leq \sigma_{f_{km}} ||\varphi[0,\tau]||_{\mathcal{L}_p} W(k, e(t_{s_k}^+)) . \tag{94}$$

Terms involving $\tilde{y}(\cdot)$: To handle the terms involving $\tilde{y}(\cdot)$ we set $W(k, e(t_{s_k}^+)) = 0$ and we get, for each integer $i \geq k$ and all $\theta \in [t_{s_i}, t_{s_{i+1}}]$,

$$W(i, e(\theta)) \leq \exp(L(\theta - t_{s_i})) \eta \sum_{j=k}^{i-1} \lambda_{(j+1)\to i} ||\tilde{y}[t_{s_j}, t_{s_{j+1}}]||_{\mathcal{L}_p} + \int_{t_{s_i}}^{\theta} \exp(L(\theta - s)) |\tilde{y}(s)| ds . \tag{95}$$

Case: $\sigma_{b_{km}} = 1$ In this case for each $i \in \{k, \ldots, m\}$ and $j \in \{k, \ldots, i-1\}$ we have $\lambda_{(j+1)\to i} = 0$ and so we have, for all $\theta \in [t_{s_i}, t_{s_{i+1}}]$,

$$W(i, e(\theta))^p \leq \left(\int_{t_{s_i}}^{\theta} \exp(L(\theta - s)) |\tilde{y}(s)| ds \right)^p . \tag{96}$$

Integrating, using [14, p. 266] on each time interval, and summing we get

$$||W[t_{s_k}, t]||_{\mathcal{L}_p}^p \leq ||\varphi[0,\tau]||_{\mathcal{L}_1}^p ||\tilde{y}[t_{s_k}, t]||_{\mathcal{L}_p}^p \tag{97}$$

and so, since $\sigma_{b_{km}} = 1$ and $\overline{\sigma}_{f_{km}} \geq 1$,

$$||W[t_{s_k}, t]||_{\mathcal{L}_p} \leq \sigma_{b_{km}}^{\frac{p-1}{p}} \overline{\sigma}_{f_{km}}^{\frac{1}{p}} ||\varphi[0,\tau]||_{\mathcal{L}_1} ||\tilde{y}[t_{s_k}, t]||_{\mathcal{L}_p} . \tag{98}$$

Case: $\sigma_{b_{km}} > 1$ We will use the two facts about the (convex) function $s \mapsto s^p$ given in the appendix. Using Fact 2 in the appendix, it follows from (95) and $L \geq 0$ that

$$\begin{aligned} W(i, e(\theta))^p \leq & \left(\frac{\sigma_{b_{km}}}{\sigma_{b_{km}} - 1} \exp(L\tau) \right)^{p-1} \exp(L(\theta - t_{s_i})) \eta^p \left(\sum_{j=k}^{i-1} \lambda_{(j+1)\to i} ||\tilde{y}[t_{s_j}, t_{s_{j+1}}]||_{\mathcal{L}_p} \right)^p \\ & + \sigma_{b_{km}}^{p-1} \left(\int_{t_{s_i}}^{\theta} \exp(L(\theta - s)) |\tilde{y}(s)| ds \right)^p . \end{aligned} \tag{99}$$

Like above when going from (96) to (97), when we integrate and sum the last term in (99) we get the bound

$$\sigma_{b_{km}}^{p-1}||\varphi[0,\tau]||_{\mathcal{L}_1}^p||\tilde{y}[t_{s_k},t]||_{\mathcal{L}_p}^p \ .$$

For the first term, integrating and using the definition of η in (87) we get the bound

$$\left(\frac{\sigma_{b_{km}}}{\sigma_{b_{km}}-1}\right)^{p-1}||\varphi[0,\tau]||_{\mathcal{L}_1}^p\left(\sum_{j=k}^{i-1}\lambda_{(j+1)\to i}||\tilde{y}[t_{s_j},t_{s_{j+1}}]||_{\mathcal{L}_p}\right)^p \ . \tag{100}$$

Using Fact 1 in the appendix, we have

$$\left(\sum_{j=k}^{i-1}\lambda_{(j+1)\to i}||\tilde{y}[t_{s_j},t_{s_{j+1}}]||_{\mathcal{L}_p}\right)^p \leq (\sigma_{b_{km}}-1)^{p-1}\sum_{j=k}^{i-1}\lambda_{(j+1)\to i}||\tilde{y}[t_{s_j},t_{s_{j+1}}]||_{\mathcal{L}_p}^p \ . \tag{101}$$

Now summing we get

$$\sum_{i=k}^{m-1}\sum_{j=k}^{i-1}\lambda_{(j+1)\to i}||\tilde{y}[t_{s_j},t_{s_{j+1}}]||_{\mathcal{L}_p}^p \leq \sum_{j=k}^{m-1}\sum_{i=j+1}^{m}\lambda_{(j+1)\to i}||\tilde{y}[t_{s_j},t_{s_{j+1}}]||_{\mathcal{L}_p}^p \leq (\overline{\sigma}_{f_{k,m}}-1)||\tilde{y}[t_{s_k},t]||_{\mathcal{L}_p}^p \ . \tag{102}$$

Combining (100)-(102) it follows that integrating and summing the first term in (99) we get the bound

$$\sigma_{b_{km}}^{p-1}(\overline{\sigma}_{f_{km}}-1)||\varphi[0,\tau]||_{\mathcal{L}_1}^p||\tilde{y}[t_{s_k},t]||_{\mathcal{L}_p}^p \ .$$

It now follows that

$$||W[t_{s_k},t]||_{\mathcal{L}_p}^p \leq \left(\sigma_{b_{km}}^{p-1}(\overline{\sigma}_{f_{km}}-1)+\sigma_{b_{km}}^{p-1}\right)||\varphi[0,\tau]||_{\mathcal{L}_1}^p||\tilde{y}[t_{s_k},t]||_{\mathcal{L}_p}^p = \left(\sigma_{b_{km}}^{\frac{p-1}{p}}\overline{\sigma}_{f_{km}}^{\frac{1}{p}}||\varphi[0,\tau]||_{\mathcal{L}_1}||\tilde{y}[t_{s_k},t]||_{\mathcal{L}_p}\right)^p \ . \tag{103}$$

This establishes the result for $p \in [1,\infty)$.

Case $p = \infty$:

This proof is carried out in a very similar way as the previous case but it is stated here for completeness. First, notice that (90) holds, where the $\eta := \exp(-L\tau)\,||\varphi[0,\tau]||_{\mathcal{L}_1}$. Combining (86) and (90) we again obtain that $W(\cdot,e(\cdot))$ is bounded by two terms, one involving $W(k,e(t_{s_k}^+))$ and another involving $\tilde{y}(\cdot)$.

Term involving $W(k,e(t_{s_k}^+))$: We again set $\tilde{y}\equiv 0$ and then we see that for each integer $i \geq k$ and all $\theta \in [t_{s_i},t_{s_{i+1}}]$ we have that (91) holds. By taking supremum and summing all the terms we obtain:

$$||W[t_{s_k},t]||_{\mathcal{L}_\infty} \leq \sigma_{f_{km}}\,||\varphi[0,\tau]||_{\mathcal{L}_\infty}\,W(k,e(t_{s_k}^+)) \ . \tag{104}$$

Terms involving $\tilde{y}(\cdot)$: We again set $W(k,e(t_{s_k}^+)) = 0$ and get, for each integer $i \geq k$ and all $\theta \in [t_{s_i},t_{s_{i+1}}]$,

$$\begin{aligned} W(i,e(\theta)) &\leq \exp(L(\theta-t_{s_i}))\eta\sum_{j=k}^{i-1}\lambda_{(j+1)\to i}||\tilde{y}[t_{s_j},t_{s_{j+1}}]||_{\mathcal{L}_\infty}+\int_{t_{s_i}}^{\theta}\exp(L(\theta-s))|\tilde{y}(s)|ds \qquad (105)\\ &\leq ||\varphi[0,\tau]||_{\mathcal{L}_1}\left(\sum_{j=k}^{i-1}\lambda_{(j+1)\to i}||\tilde{y}[t_{s_j},t_{s_{j+1}}]||_{\mathcal{L}_\infty}+||\tilde{y}[t_{s_i},t]||_{\mathcal{L}_\infty}\right) \ . \qquad (106)\end{aligned}$$

By over bounding and summing the term in brackets we obtain

$$||W[t_{s_k},t]||_{\mathcal{L}_\infty} \leq \sigma_{b_{km}}\,||\varphi[0,\tau]||_{\mathcal{L}_1}\,||\tilde{y}[t_{s_k},t]||_{\mathcal{L}_\infty} \ , \tag{107}$$

which completes the proof by combining (104) and (107).

Finally, we prove the statement about the $\mathcal{KL}$ function. For this purpose for any $a \in \mathbb{R}$ we define $\lceil a \rceil := \min\{i : i \in \mathbb{Z},\ a \leq i\}$. The following claim is needed to prove the result:

Claim 1: If $\sigma > 0$ is such that, for all $m \geq k$,

$$\sigma_{f_{km}} = \sum_{s=k}^{m} \lambda_{(k+1)\to s} \leq \sigma \tag{108}$$

then, for all $i \geq k$,

$$\lambda_{(k+1)\to i} \leq (\lambda^*)^{i-k}\, 2\sigma \tag{109}$$

where

$$\lambda^* = \left(\frac{1}{2}\right)^{1/\lceil 2\sigma^2 \rceil} < 1 \ . \tag{110}$$

Proof of the claim. We first note that, according to (108), for all $\ell_2 \geq \ell_1$,

$$\lambda_{(\ell_1+1)\to \ell_2} \leq \sigma \ . \tag{111}$$

For future reference we define

$$T^* := \lceil 2\sigma^2 \rceil \ . \tag{112}$$

Then it also follows from (108) that, for each k, there exists $j \in \{k, \ldots, k + T^* - 1\}$ such that

$$\lambda_{(k+1)\to j} \leq \frac{\sigma}{T^*} \ . \tag{113}$$

(If not then take $m = k + T^* - 1$ and note that there would be T^* terms in the summation in (108) and each term in the summation would exceed $\frac{\sigma}{T^*}$ so that the sum would exceed σ.) We conclude that, for each k,

$$\lambda_{(k+1)\to i} \leq \sigma \qquad \forall i \in \{k, \ldots, k + T^* - 1\} \tag{114}$$

and

$$\lambda_{(k+1)\to(k+T^*-1)} = \lambda_{(k+1)\to j} \cdot \lambda_{(j+1)\to(k+T^*-1)} \leq \frac{\sigma}{T^*}\sigma \leq \frac{1}{2} \ . \tag{115}$$

By induction, for each integer k and each integer ℓ,

$$\lambda_{(k+1)\to i} \leq \left(\frac{1}{2}\right)^{\ell-1} \sigma \qquad \forall i \in \{k + (\ell-1)T^*, k + \ell T^* - 1\} \tag{116}$$

and

$$\lambda_{(k+1)\to(k+\ell T^*-1)} \leq \left(\frac{1}{2}\right)^{\ell} \ . \tag{117}$$

Finally, using the relationship between i and ℓ in (116), which is that

$$\ell \geq \frac{i-k}{T^*} \tag{118}$$

the inequality (116) can be rewritten as:

$$\lambda_{k+1\to i} \leq \left(\frac{1}{2}\right)^{\frac{i-k}{T^*}} 2\sigma \tag{119}$$

which establishes the claim. ■

Now we complete the proof. Let $t \in [t_{s_i}, t_{s_{i+1}}]$ and $i \geq k \geq 0$ be arbitrary. Then, we have that $t - t_{s_k} \leq (i - k + 1)\tau$. Using the claim and (91) and since $\lambda^* < 1$, we can write that

$$\begin{aligned}
W(i, e(t)) &\leq \exp(L\tau)\lambda_{(k+1)\to i} W(k, e(t_{s_k})) \\
&\leq \exp(L\tau)\frac{1}{\lambda^*}(\lambda^*)^{i-k+1}\, 2\sigma W(k, e(t_{s_k})) \\
&\leq \frac{\exp(L\tau)2\sigma}{\lambda^*}(\lambda^*)^{\frac{t-t_{s_k}}{\tau}} W(k, e(t_{s_k})) \\
&= \frac{\exp(L\tau)2\sigma}{\lambda^*}\exp\left(\frac{-c(t-t_{s_k})}{\tau}\right) W(k, e(t_{s_k})) \ ,
\end{aligned}$$

where $c := -\ln(\lambda^*)$, which completes the proof. ■

Proof of Proposition 6: It follows directly from proof of Proposition 7 and the bound (42). ■

XI. Conclusions

We presented results on $\mathcal{L}_p$ stability for Networked Control Systems that apply to a general class of scheduling protocols. Our results generalize those in [25], [26] since they apply to general nonlinear systems with disturbances that may have data packet dropouts, as well as a more general class of protocols than those considered in the references. In particular, we show that the static and TOD protocols considered in [25], [26] belong to the class of protocols that we considered. Moreover, our result can serve as a framework for generating new protocols that have good properties. Our proof technique, which is based on the small gain theorem, and the model of NCS that we use appear to be novel. We illustrated via an example considered in [26] that our results produce several orders of magnitude less conservative results than those obtained in [26].

References

[1] O.Beldiman, G.C.Walsh and L.Bushnell, “Predictors for networked control systems”, *Proc. Americ. Contr. Conf.*, Chicago, 2000, pp. 2347-2351.

[2] R.W.Brockett and D.Liberzon, *Quantized stabilization of linear systems*, IEEE Trans. Automat. Contr., 45 (2000), pp. 1279–1289.

[3] R.W.Brockett, *Minimum attention control*, Proc. Conf. Decis. Contr., San Diego, CA, 1997, pp. 2682–2632.

[4] T. Chen and B.Francis, “Input-output stability of sampled-data systems”, *IEEE Trans. Automat. Contr.*, vol. 36, No. 1 (1991), pp. 50-58.

[5] F.H. Clarke, *Optimization and nonsmooth analysis*, Classics in Applied Mathematics, vol. 5, SIAM, Philadelphia, 1990.

[6] G.E. Dullerud and S. Lall, “A synchronous hybrid systems with jumps analysis: analysis and synthesis problems”, *Syst. Contr. Lett.*, vol. 37 (1999), pp. 61–89.

[7] G.E. Dullerud and S. Lall, “A new approach for analysis and synthesis of time varying systems”, *IEEE Trans. Automat. Contr.*, vol. 44 (1999), pp. 1486–1497.

[8] N.Eilia and S.K.Mitter, *Stability of linear systems with limited information*, IEEE Trans. Automat.. Contr., 46 (2001), pp. 1384–1400.

[9] A.F. Filippov. Differential Equations with Discontinuous Righthand Sides. *Kluwer Academic Publishers*, 1988.

[10] A.Hassibi, S.P.Boyd and J.P.How, “Control of asynchronous dynamical systems with rate constraints on events”, Proc. *IEEE Conf. Decis. Contr.*, Phoenix, AZ, USA, (1999), pp. 1345–1351.

[11] H.Ishii and B.A.Francis, *Stabilizing a linear system by switching control with dwell time*, IEEE Trans. Automt. Contr., 47 (2002), pp. 1962–1973.

[12] A.Isidori, *Nonlinear control systems II*, Springer: New York, 1999.

[13] Z. P. Jiang, A. R. Teel and L. Praly, “Small gain theorem for ISS systems and applications”, *Math. Contr. Sign. Syst.*, 7 (1994), pp. 95–120.

[14] H.K.Khalil, *Nonlinear Systems, 2nd Edition.* Prentice-Hall: New Jersey, 1996.

[15] G. N. Nair, Robin J. Evans, “Stabilization with data-rate-limited feedback: tightest attainable bounds”, *Syst. Contr. Lett.*, vol. 41, No. 1 (2000) pp. 49–56.

[16] D. Nešić, A. R. Teel and P. V. Kokotović, “Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations”, *Sys. Contr. Lett.*, vol. 38, No. 4-5 (1999), 259-270.

[17] D. Nešić and D.S. Laila, “A note on input-to-state stabilization for nonlinear sampled-data systems”, IEEE Trans. Automat. Contr., 47 (2001), pp. 1153–1158.

[18] D. Nešić and D. Angeli, “Integral versions of ISS for sampled-data nonlinear systems via their approximate discrete-time models”, IEEE Trans. Automt. Contr.,47 (2002), pp. 2033–2038.

[19] D. Nešić and A.R. Teel, ”A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models”, accepted in IEEE Trans. Automat. Contr., 32 pages, 2003.

[20] E.D.Sontag, “The ISS philosophy as a unifying framework for stability-like behaviour”, In *Nonlinear Control in the Year 2000* (A. Isidori, F.Lamnabhi-Lagarrigue and W. Respondek, eds.), Springer-Verlag, Berlin, 2000, pp. 443–468.

[21] A.R.Teel, E.Panteley and A.Loria, “Integral characterizations of uniform asymptotic and exponential stability with applications”, Math. Contr. Sig. Sys., vol. 15 (2002), pp. 177–201.

[22] A.R. Teel and D. Nešić, “Input to state stability of NCS”, in preparation, 2003.

[23] A.R.Teel and L.Praly, "On assigning the derivative of a disturbance attenuation control Lyapunov function", Math. Contr. Sign. Sys., vol. 13 (2000), 95–124.

[24] G. C. Walsh and H. Ye, "Scheduling of Networked Control Systems", *IEEE Contr. Magazine*, vol. 21, No. 1 (2001), pp. 57–65.

[25] G. C. Walsh, O. Beldiman and L. G. Bushnell, "Asymptotic behaviour of nonlinear networked control systems", *IEEE Trans. Automat. Contr.*, vol. 46, No. 7 (2001), 1093–1097.

[26] G. C. Walsh, H. Ye and L. G. Bushnell, "Stability analysis of networked control systems", *IEEE Trans. Contr. Syst. Techn.*, vol. 10, No. 3 (2002), pp. 438–446.

[27] G. C. Walsh, O. Beldiman and L. G. Bushnell, "Error encoding algorithms for networked control systems", *Automatica*, vol. 38 (2002), pp. 261–267.

[28] H. Ye, G. C. Walsh and L. G. Bushnell, "Real-time mixed-traffic wireless networks", *IEEE Trans. Indust. Electr.*, vol. 48, No. 5 (2001), pp. 883–890.

[29] L. Zaccarian, A.R. Teel and D. Nešić, "On finite gain L_p stability of sampled-data nonlinear systems", to appear in *Syst. Contr. Lett.*, 2003.

[30] W. Zhang, M.S. Branicky and S.M. Phillips, "Stability of networked control systems", *IEEE Contr. Syst. Magazine*, Vol. 21, No. 1 (2001), pp. 84–99.

XII. Appendix

Fact 1: If $a_j \geq 0$ for all $j \in \{1, \ldots, i\}$ and $\sum_{j=1}^{i} a_j \leq \sigma$ then, for all z such that $z_j \geq 0$ for all $j \in \{1, \ldots, i\}$,

$$\left(\sum_{j=1}^{i} a_j z_j \right)^p \leq \sigma^{p-1} \sum_{j=1}^{i} a_j z_j^p \ . \tag{120}$$

■

Proof. From Hölder's inequality we have that for $\alpha_n \geq 0$ and $\beta_n \geq 0$:

$$\sum_{n=1}^{i} \alpha_n \beta_n \leq \left(\sum_{n=1}^{i} \alpha_n^p \right)^{1/p} \left(\sum_{n=1}^{i} \beta_n^{p'} \right)^{1/p'} \qquad (1/p + 1/p' = 1, 1 \leq p \leq \infty).$$

Fact 1 follows by letting $\alpha_j = a_j^{1/p} z_j$ and $\beta_j = a_j^{1/p'}$. ■

Fact 2: If $\mu > 1$ then, for all $x_1 \geq 0$, $x_2 \geq 0$,

$$(x_1 + x_2)^p \leq \left(\frac{\mu}{\mu - 1} \right)^{p-1} x_1^p + \mu^{p-1} x_2^p \ . \tag{121}$$

■

Proof: Follows directly from Fact 1 by letting $i = 2$, $z_1 = \frac{\mu}{\mu-1} x_1$, $z_2 = \mu x_2$, $a_1 = \frac{\mu-1}{\mu}$, $a_2 = \frac{1}{\mu}$. ■