On uniform boundedness of parameterized discrete-time systems with decaying inputs: applications to cascades*

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Abstract

A framework for controller design of sampled-data nonlinear systems via their approximate discrete-time models has been established recently. Within this framework naturally arises the need to investigate stability properties of parameterized discrete-time systems. Further results that guarantee appropriate stability of the parameterized family of discrete-time systems that is used within this framework have been also established for systems with *cascaded* structure. A fundamental condition that is required in this framework is uniform boundedness of solutions of the cascade. However, this is difficult to check in general. In this paper we provide a range of sufficient conditions for uniform boundedness that are easier to check. These results further contribute to the toolbox for controller design of sampled-data nonlinear systems via their approximate discrete-time models.

1 Introduction

The class of sampled-data nonlinear models is strongly motivated by the prevalence of computer controlled systems and the fact that nonlinearities can often not be neglected in controller design. Recently, a framework for controller design for this class of systems via their approximate discrete-time models has been proposed in [13, 15]. Within this framework naturally arises the need to investigate stability properties of parameterized families of discrete-time systems. Consequently, stability of parameterized families of discrete-time systems in cascaded form has been investigated in [10]. The conditions presented in this reference are necessary and sufficient for stability of parameterized cascades but they are often difficult to check. In this paper, we provide a range of simpler-to-check sufficient conditions for stability of parameterized cascades to be used within this framework.

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In particular, we study here discrete-time parameterized systems with inputs of the general form

$$x(k+1) \;=\; f_T(k, x(k), z(k)) \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$ and T is the sampling period. In the system (1), z is seen as a generic input which may be an exogenous signal, the state of another system or possibly a control input. In the case when it corresponds to the state of another dynamical system, say,

$$z(k+1) = g_T(k, z(k)) \ , \tag{2}$$

we call the system (1), (2) a *cascaded* system. Such systems have attracted the attention of the control research community for many years now and for many reasons. From a theoretical viewpoint, it probably originated in geometric nonlinear control where it was shown that many systems can be transformed into a cascade via a local change of coordinates (see, for example, [6, Lemma 1.6.1]).

A natural but fundamental question that arises is the following: assuming that (2) and the zero-input system

$$x(k+1) \;=\; f_T(k, x(k), 0) \ , \tag{3}$$

are uniformly asymptotically stable (UGAS) –see next section for precise definitions– under which conditions is the cascade UGAS? This apparently simple question largely triggered by the puzzling "peaking phenomenon" (cf. [23]) led to a range of significant results mainly in the context of continuous-time systems (see e.g. [20] and references therein). Among the most significant achievements stands the statement that a necessary and sufficient condition for UGAS of the cascade is that the solutions of the system with inputs be uniformly globally bounded. This property requires and implies the so-called "converging-input bounded state" property for the system (1) when z is regarded as input. In the continuous time context it dates back to [21, 19] for autonomous systems and more recently it was proved that the same holds for non autonomous systems in [17].

In the discrete-time context there is a considerable lag in this research direction. Some notable exceptions are however, the recent paper [7] where significant results within the framework of the so-called input-to-state stability have been established. See also [11] for other results for non ISS cascades. Nevertheless, these results apply only to autonomous, non-parameterized discrete time systems i.e., for a fixed sampling period T, and can be used only when the exact discrete-time of the system is known.

Very recently, as far as we know the first results on UGAS of parameterized non-autonomous cascades were established in [9, 10]. More precisely, it was shown that if (2) and (3) are UGAS then, the cascade (1), (2) is also UGAS if and only if (1) is uniformly globally bounded (UGB). In essence, this is the same result which has been known for years in the continuous-time context. Interestingly, as we will see later and it is thoroughly discussed in [9, 11] there exist important technical differences between the continuous and the discrete-time contexts.

The necessary condition of UGB imposed in [9, 10] is in general hard to check. This paper establishes a range of easier-to-check sufficient conditions for UGB to hold.

Our results can be classified into two types: (i) integral conditions; (ii) conditions involving forward completeness. For the first case, we present conditions to verify UGB via the property of integral input-to-state stability (iISS) and integral input-to-state neutral practical stability. In essence, integral conditions impose a minimal convergence rate of the solutions of the subsystem (2) seen as "inputs" to the top subsystem (1). These results follow closely results in [5] and [17]. Moreover, we make strong connections between iISS results of [5] and the integrability condition results of [17]. Conditions involving forward completeness establish a relation between the convergence rate of the top system and the growth rate of the interconnection terms in (1), i.e., the terms of f_T which depend on z. The first of this is a result involving *dead-beat stability* property for (3) and the second can be seen as a result which parallels [17, Theorem 4] in which systems $\dot{x} = f(t,x) + g(t,x,z)$ are studied under the assumption that $f(t,\cdot)$ and $g(t,\cdot,z)$ have similar growth rates. As discussed in the latter reference, this case excludes input-to-state stable (ISS) systems. Again, we provide a unified proof for these two seemingly unrelated results.

Thus, the results we present here contribute together with those in [10] to what we may call cascaded-based control design (see the latter reference for an illustrative application) for sampled-data systems via their approximate discrete-time models within the framework that was established in [13, 14]. In particular, the definitions that we use are strongly motivated by results in the latter references.

Our results generalize the time-invariant discrete-time results in [7] as well as [11] and parallel similar continuous-time results from [5, 17]. However, as it will become clear below, the results presented here are not a simple translation of their counterparts in continuous-time. Indeed, the properties we consider, the conditions we impose and the proofs we establish here are notably different and the sufficient conditions we impose are tailored specifically for discrete-time parameterized systems.

The rest of the paper is organized as follows. In next section we present some mathematical preliminaries and formulate the precise problem that we address. In Section 3 we present our main results. We conclude with some remarks in Section 4.

2 Mathematical preliminaries and problem setting

For the system (1), (2) we use the notation $\xi := [x^T\ z^T]^T$ to denote the state of the overall system. In our main results we regard z in the system (1) as an exogenous input that is not necessarily generated by the subsystem (2) in order to obtain more general results that can be used in establishing stability of cascades. Hence, we refer to the subsystem (1) as the *system with input* z. The solution of the system (1) with input z at time k that starts at initial time instant $k_\circ$ from the initial state $x(k_\circ) = x_\circ$ and under the action of the input sequence $\omega^z_{[k_\circ,k)} := \{z(k_\circ), \ldots z(k-1)\}$ is denoted as $\phi^x_T(k, k_\circ, x_\circ, \omega^z_{[k_\circ,k)})$. We also use $\omega^z := \omega^z_{[k_\circ,\infty)}$. Note that the solution of the system (3) is the same as the solution for system (1) with input z when $z(j) \equiv 0, \forall j \in [k_\circ, k]$ and hence for solutions of (3) we use the notation $\phi^x_T(k, k_\circ, x_\circ, 0)$. Similarly we use notation $\phi^\xi_T(k, k_\circ, \xi_\circ)$, $\phi^x_T(k, k_\circ, \xi_\circ)$ and $\phi^z_T(k, k_\circ, z_\circ)$ to denote solutions of the overall system (1), (2) and its x and z components respectively.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for all $t > 0$, $\beta(\cdot, t) \in \mathcal{K}$, for all $s > 0$, $\beta(s, \cdot)$ is decreasing to zero. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{N}$ if $\gamma(\cdot)$ is continuous and nondecreasing.

For an arbitrary $r \in \mathbb{R}$ we use the notation $\lfloor r \rfloor := \max_{z \in \mathbb{Z}, z \leq r} z$. Given strictly positive real numbers L, T we use the following notation:

$$\ell_{L,T} := \left\lfloor \frac{L}{T} \right\rfloor \ . \tag{4}$$

Our main results are targeted at establishing the following form of asymptotic stability, which is motivated by the framework of [13, 14] for sampled-data systems.

Definition 1 *The family of the parameterized time-varying systems*

$$y(k+1) = F_T(k, y(k)) \tag{5}$$

is uniformly globally asymptotically stable (UGAS) if there exists $\beta \in \mathcal{KL}$ and $T^* > 0$ such that for all $k_\circ \geq 0$, $y(k_\circ) = y_\circ$, $y_\circ \in \mathbb{R}^n$ and $T \in (0, T^*)$ the following holds:

$$|\phi_T^y(k, k_\circ, y_\circ)| \leq \beta(\, |y_\circ| \, , (k - k_\circ)T \,) \tag{6}$$

for all $k \geq k_\circ$. □

For this, we know from [9, 10] that a necessary and sufficient condition is that the system with inputs (1) have the UGB property (cf. Theorem 1). We define the latter as follows.

Definition 2 *The system (5) is uniformly globally bounded (UGB), if there exist $\kappa \in \mathcal{K}_\infty$, c and $T^* > 0$ such that for all $k_\circ \geq 0$, $y(k_\circ) = y_\circ$, $y_\circ \in \mathbb{R}^n$ and $T \in (0, T^*)$ it holds that*

$$|\phi_T^y(k, k_\circ, y_\circ)| \leq \kappa(|y_\circ|) + c \tag{7}$$

for all $k \geq k_\circ$. □

More precisely, in [9, 10] the following was established.

Theorem 1 *Suppose that the solutions of the system (1) with input z satisfy the following:*

Assumption 1 *There exists $T^* > 0$ such that for any strictly positive reals η, ϵ and L there exists $\mu > 0$ such that for all $T \in (0, T^*)$, all $z(\cdot)$ with $\|\omega^z\| \leq \mu$, $k_\circ \geq 0$ and all $x(k_\circ) = x_\circ$ with $|x_\circ| \leq \eta$, we have that*

$$\left| \phi_T^x(k, k_\circ, x_\circ, \omega^z_{[k_\circ, k)}) - \phi_T^x(k, k_\circ, x_\circ, 0) \right| \leq \epsilon \ , \quad \forall k \in [k_\circ, k_\circ + \ell_{L,T}] \ . \tag{8}$$

Then, the system (1), (2) is UGAS if and only if the following conditions hold:

1. *The system (3) is UGAS;*
2. *The system (2) is UGAS;*
3. *The system (1), (2) satisfies the property UGB.*

□

A "semiglobal-practical" version of Theorem 1 can also be stated (see [10] for details).

The first assumption, which we called in [10], uniform semi(global) continuity, is not restrictive. It is a technical condition motivated from numerical analysis methods and is satisfied for instance when $f_T(k,x,z)$ satisfies a particular type of Lipschitz property in x and z that is uniform in k but not in the parameter T. (see [10] for further details).

In contrast to this, the UGB assumption is in general hard to check. Our main results, which are presented next, are focussed on establishing sufficient conditions for UGB to hold. Semiglobal versions of this property (useful to establish semiglobal practical uniform asymptotic stability) can also be established by carefully restricting the domain of attraction and making appropriate changes to the definitions. These are omitted here for simplicity and space reasons.

3 Main results

As mentioned above, the sufficient conditions that we establish for UGB can roughly be classified into conditions involving integral input-to-state stability and conditions involving the property of uniform forward completeness.

All our results can be easily modified to be applicable to non-parameterized discrete-time systems and they are briefly commented on. The interested reader should refer to [11] where precise statements with proofs of these results can be found.

3.1 Conditions involving integral ISS

In this section we prove a result that is a discrete-time version of [5, Theorem 1]. Moreover, we show that the result involving integrability conditions in [17, Theorem 5] is closely related and can be recovered using the iISS framework. For the purposes of this paper we have modified the definitions of integral input-to-state stability properties given in [1] to be applicable to parameterized discrete-time systems.

Definition 3 (iISS) *The system (1) with input z is Integral Input-to-State Stable (iISS) with gain μ if there exist α, $\mu \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ and $T^* > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$ with $x_\circ \in \mathbb{R}^{n_x}$, all inputs $z(\cdot)$ and $T \in (0, T^*)$*

$$\alpha(\left|\phi_T^x(k, k_\circ, x_\circ, \omega^z_{[k_\circ,k)})\right|) \leq \beta(|x_\circ|, (k-k_\circ)T) + \sum_{i=k_\circ}^{k-1} T\mu(\left|\omega^z_{[k_\circ,k)}\right|),$$

for all $k \geq k_\circ$. □

Definition 4 (iISNpS) *The system (1) is Integral Input-to-State Neutrally practically Stable (iISNpS) with gain μ and input z if there exist α, γ_1, $\mu \in \mathcal{K}_\infty$, $\gamma_2 \in \mathcal{N}$ and $T^* > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$ with $x_\circ \in \mathbb{R}^{n_x}$, all inputs $z(\cdot)$ and $T \in (0, T^*)$*

$$\alpha(\left|\phi_T^x(k, k_\circ, x_\circ, \omega^z_{[k_\circ,k)})\right|) \leq \gamma_1(|x_\circ|) + \gamma_2\left(\sum_{i=k_\circ}^{k-1} T\mu(\left|\omega^z_{[k_\circ,k)}\right|)\right), \tag{9}$$

for all $k \geq k_\circ$. □

Necessary and sufficient conditions for iISS of non-parameterized discrete-time time-invariant systems can be found in [1]. Integral input to state stability properties of parameterized systems were investigated in [2]. The following Proposition captures a similar result to that contained in [5, Theorem 1]. This result shows that in order to have UGAS of the system (1), (2), there is a trade-off between the rate of convergence of trajectories of the system (2) and the shape of the iISNpS gain of the system (1).

Proposition 1 *Suppose that there exist $\alpha, \gamma_1, \mu, \sigma \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$, $\gamma_2 \in \mathcal{N}$ and λ, c, $T^* > 0$ such that:*

1. *The system (1) is iISNpS with gain μ and input z;*
2. *The system (2) is UGAS with $\beta(r,t) := \sigma(\kappa(r)e^{-\lambda t})$;*
3. *The following condition holds:*

$$\int_0^1 \frac{\mu \circ \sigma(s)}{s} ds \le c < \infty . \tag{10}$$

Then, the cascade (1), (2) is UGB. □

Proof. The proof follows closely that of [5, Theorem 1]. By assumption, there exist α, γ_1, $\mu \in \mathcal{K}_\infty$, $\gamma_2 \in \mathcal{N}$, $\kappa \in \mathcal{K}$ and λ, $T^* > 0$ such that for all $k_\circ \ge 0$, $\xi(k_\circ) = \xi_\circ$ with $\xi_\circ \in \mathbb{R}^n$ and $k \ge k_\circ$

$$\alpha(|\phi_T^x(k, k_\circ, \xi_\circ)|) \le \gamma_1(|x_\circ|) + \gamma_2\left(T \sum_{k=k_\circ}^{\infty} \mu \circ \sigma\left(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)T}\right)\right) . \tag{11}$$

The sum on the right hand side of the inequality above satisfies

$$T \sum_{k=k_\circ}^{\infty} \mu \circ \sigma\left(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)T}\right) = T\mu \circ \sigma \circ \kappa(|z_\circ|) + T \sum_{k=k_\circ+1}^{\infty} \mu \circ \sigma\left(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)T}\right)$$

and since $\mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)T})$ is monotonically decreasing in $(k - k_\circ)T$, the last term on the right hand side of this equation satisfies

$$T \sum_{k=k_\circ+1}^{\infty} \mu \circ \sigma\left(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)T}\right) \le \int_{t_\circ:=k_\circ T}^{\infty} \mu \circ \sigma\left(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(t-t_\circ)}\right) dt .$$

Define as in [5], $s := \kappa(|z_\circ|)\mathrm{e}^{-\lambda(t-t_\circ)}$ then

$$\int_{t_\circ:=k_\circ T}^{\infty} \mu \circ \sigma\left(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(t-t_\circ)}\right) dt = \int_0^{\kappa(|z_\circ|)} \frac{\mu \circ \sigma(s)}{\lambda\, s} ds .$$

From item 3 of the proposition we have that

$$\kappa_1(s) := \frac{1}{\lambda} \int_0^s \frac{\mu \circ \sigma(t)}{t} dt$$

exists for all $s \geq 0$ and it is of class $\mathcal{K}$ because $\kappa_1(0) = 0$ and $\frac{\mu\circ\sigma(t)}{t} > 0$ for all $t > 0$. Putting all these bounds together and using item 2 of the Proposition we obtain that

$$\alpha(|\phi_T^x(k, k_\circ, \xi_\circ)|) \leq \gamma_1(|x_\circ|) + \gamma_2\left(T^*\mu \circ \sigma \circ \kappa(|z_\circ|) + \kappa_1 \circ \kappa(|z_\circ|)\right)$$

for all $k \geq k_\circ \geq 0$ and since the system (2) is UGAS, the solutions of (1), (2) are UGB. $\square$

We remark that the assumption in item 2 of Proposition 1 is not restrictive since such bound exists for any UGAS system. Indeed, it was shown in [22] that given any $\beta \in \mathcal{KL}$, there exist $\sigma \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$ such that $\beta(r,t) \leq \sigma(\kappa(r)e^{-t})\ \forall\, r,t \geq 0$.

The condition in item 3 was first used in [5, Theorem 1] and it is very related to the summability condition in [17, Theorem 5]. Item 3 is restrictive and it shows a tradeoff between the iISNpS gain μ of (1) and the convergence rate of (2). As a matter of fact, and this should be clear from the above proof, this condition can be regarded as a sufficient condition for *summability* of the solutions of the subsystem (2) that are weighted by the gain μ. More precisely, it is a sufficient condition for

$$T\sum_{k=k_\circ}^{\infty} \mu\left(|\phi_T^z(k, k_\circ, z_\circ)|\right)$$

to be uniformly bounded by a function of the initial states. Then, it is evident from Definitions 3 and 4 that the system (1) is UGAS (resp. UGB) if it is iISS (resp. iISNpS) and the quantity above is uniformly bounded. The above given condition was used in [17, Theorem 5]. Another observation to keep in mind is that item 3 holds if the subsystem (2) is uniformly locally exponentially stable and μ is locally linear. See [10, 9] for further discussions and a physical example illustrating these observations.

Items 2 and 3 of the Proposition rely on the ability to compute a $\mathcal{KL}$ bound for the solutions of (2). This may often be done from a Lyapunov function (when available) for the system (2) and comparison arguments; for instance, see [13, 15]. Sufficient Lyapunov type conditions for iISS of parameterized systems can be found in [2] and are given below:

Proposition 2 *If there exist $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$, a positive definite function $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $T^* > 0$ such that for all $T \in (0, T^*)$ there exists a continuous function $V_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, $k \geq 0$ and $T \in (0, T^*)$ we have:*

$$\alpha_1(|x|) \leq V_T(k,x) \leq \alpha_2(|x|) \tag{12}$$

$$V_T(k+1, f_T(k,x,z)) - V_T(k,x) \leq -T\alpha_3(|x|) + T\gamma(|z|)\ , \tag{13}$$

then there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}$ such that the system (1) with input z is iISS with gain μ. $\square$

Obviously, iISS implies iISNpS and hence, the above given Proposition provides sufficient conditions for iISNpS as well. However, since iISNpS is a weaker notion than iISS, we state and prove below weaker sufficient Lyapunov conditions for iISNpS.

Proposition 3 *Suppose that there exist $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \varphi \in \mathcal{K}_\infty$, $c, T^* > 0$ and for each $T \in (0, T^*)$ there exists $V_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}$, $k \geq 0$ and*

$T \in (0, T^*)$ *we have that :*

$$
\begin{aligned}
\tilde{\alpha}_1(|x|) \ &\leq\ V_T(k,x) \ \leq\ \tilde{\alpha}_2(|x|) + c & (14)\\
V_T(k+1, f_T(k,x,z)) - V_T(k+1, f_T(k,x,0)) \ &\leq\ T\tilde{\gamma}_1(|z|)\varphi(V_T(k,x)) + T\tilde{\gamma}_2(|z|) & (15)\\
V(k+1, f_T(k,x,0)) - V_T(k,x) \ &\leq\ 0 & (16)\\
\int_1^\infty \frac{ds}{\varphi(s)} \ &=\ \infty\ . & (17)
\end{aligned}
$$

Then, the system (1) with input z is iISNpS with gain $\mu(s) = \tilde{\gamma}_1(s) + \frac{\tilde{\gamma}_2(s)}{\varphi(1)}$. □

Proof. First, note that using (15), (16) we can write

$$
\begin{aligned}
V_T(k+1, f_T(k,x,z)) - V_T(k,x) \ &=\ V_T(k+1, f_T(k,x,0)) - V_T(k,x)\\
&\quad + V_T(k+1, f_T(k,x,z)) - V_T(k+1, f_T(k,x,0))\\
&\leq\ T\tilde{\gamma}_1(|z|)\varphi(V_T(k,x)) + T\tilde{\gamma}_2(|z|).
\end{aligned}
$$

The proof follows by direct application of the technical Lemmas 1 and 2 which are presented in Appendix A.

The proposition above makes clear other interesting links with conditions used in the literature in the context of continuous-time systems to prove UGB. Note that the condition (17) restricts the growth of φ. In particular, it holds when $\varphi(s) = s$; this situation was considered for instance in [17] with $\tilde{\gamma}_1(s) \equiv 0$. Earlier results using similar conditions are found in [12] and in the context of forward completeness already in [18].

If we consider differentiable V_T's and assume that $f_T(k,x,z) := F_T(k,x) + TG_T(k,x,z)$, with $G_T(k,x,0) = 0$ then we may interpret condition (15) as follows. By the mean value theorem, there exists $\eta = F_T(k,x) + \theta G_T(k,x,z), \theta \in (0,1)$ such that

$$
|V_T(k+1, f_T(k,x,z)) - V_T(k+1, f_T(k,x,0))| = T\left|\frac{\partial V_T}{\partial x}(\eta)\right| |G_T(k,x,z)|\ .
$$

If we assume moreover that: (**A**) $\left|\frac{\partial V_T}{\partial x}(\eta)\right| |x| \leq cV_T(k,x)\kappa_1(|z|)$ for all x and $|G_T(k,x,z)| \leq \kappa_2(|z|)\,|x|$ then one can show that (15) holds. The *linear* growth restriction on the interconnection term $G_T(k,\cdot,z)$ in the assumption (**A**) has been exhaustively used to establish UGB in the context of continuous-time systems[1] (see e.g. [20, 16] and references therein) and it finds its original motivation in the so called peaking phenomenon introduced in [8]. It is interesting that the linear growth condition is not required for time-invariant non parameterized discrete-time systems as the following example illustrates:

Example 1 *Consider the system:*

$$
\begin{aligned}
x_{k+1} &= ax_k + x_k^p y_k\\
y_{k+1} &= 0\ , & (18)
\end{aligned}
$$

[1]For simplicity we strengthen here the condition (**A**) to hold on the whole state space but in the cited references the authors require this to hold only for "large" states. Moreover, the first part of assumption (**A**) in continuous-time is usually stated as $\left|\frac{\partial V}{\partial x}(x)\right| |x| \leq cV(x)$. However, in discrete-time this assumption is too restrictive since η depends on z.

where $p \geq 0$ is arbitrary and $a \in (0,1)$. We claim that the discrete-time system is GAS for any $p \geq 0$. We prove this by constructing a Lyapunov function for the system. Let $\epsilon > 0$ be such that $a + \epsilon - 1 < 0$ and define the Lyapunov function

$$V(x,y) := |ax + x^p y| + \epsilon |x| + |y| \ . \tag{19}$$

This function is obviously positive definite and radially unbounded for any value of $p \geq 0$. Finally, the first difference of the Lyapunov function is:

$$\begin{aligned} \Delta V &= |a(ax + x^p y)| + \epsilon |ax + x^p y| - |ax + x^p y| - \epsilon |x| - |y| \\ &= (a + \epsilon - 1) |ax + x^p y| - \epsilon |x| - |y| \ , \end{aligned} \tag{20}$$

which is negative definite for any $p \geq 0$ since $a + \epsilon - 1 < 0$ and $\epsilon > 0$.

We close this section with another interesting observation which was proved in [1] for time-invariant discrete-time systems

$$x(k+1) = f(x(k), u(k)) \, . \tag{21}$$

The system (21) is iISS if and only if the zero input system $x(k+1) = f(x(k), 0)$ is GAS.

This result is not true for continuous-time systems. Actually, it was shown in [4] for continuous-time systems with inputs

$$\dot{x} = f(x, u) \ , \tag{22}$$

that if the system $\dot{x} = f(x, 0)$ is GAS and moreover (22) is forward complete, this still does not imply that (22) is iISS. These results and Proposition 1 indicate that one can expect large differences between continuous-time and discrete-time cascade results. Indeed, following results of [1] and Proposition 1 for time-invariant *non parameterized* cascades

$$x(k+1) = f_1(x(k), z(k)) \tag{23}$$

$$z(k+1) = f_2(z(k)) \tag{24}$$

we can state the following:

> If $x(k+1) := f_1(x(k), 0)$ is GAS, then there exists a GAS subsystem (24) so that the cascade (23), (24) is GAS.

We will show in the next section that even a stronger statement is true for discrete-time non parameterized time-varying cascades if the bottom system is dead-beat stable. However, the above statement is not true in continuous-time even for time-invariant systems, as shown by [11, Example 2].

3.2 Conditions involving uniform forward completeness

Sometimes it is possible to relax the UGB assumption in our main results and replace it with a particular type of forward completeness. This section contains several results that follow this approach. In particular, we present a unified proof for two important situations: (i) the bottom system is dead-beat stable; (ii) certain conditions involving growth rate restrictions on the function f_T hold. The first of these results is very important in discrete-time since dead-beat behaviour can often be achieved. The second result was considered in [17] and it uses conditions that restrict the growth rate of f_T. The definitions that we use to prove the main result are stated in sufficient generality in order to prove the two results in a unified manner (see Proposition 4).

Definition 5 *The system (1), (2) is uniformly forward complete (UFC) if there exist $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ and $T^*, c > 0$ such that for all $k_\circ \geq 0$, $\xi(k_\circ) = \xi_\circ$, with $\xi_\circ \in \mathbb{R}^n$, and $T \in (0, T^*)$ we have:*

$$\left|\phi_T^\xi(k, k_\circ, \xi_\circ)\right| \leq \sigma_1(|\xi_\circ|) + \sigma_2(T(k - k_\circ)) + c \ , \tag{25}$$

for all $k \geq k_\circ$. □

Clearly, UFC is a weaker condition than UGB, since the bound in (25) grows unbounded as $(k - k_\circ)T \to \infty$.

UFC as defined above is very similar to the property of forward completeness for continuous-time systems. Indeed, Definition 5 is inspired from [3]. It is important to note that a crucial feature of the UFC property is the particular dependence of the bound on the parameter T.

The following stability property is crucial in unifying the results of this section.

Definition 6 *The system (2) is practically dead-beat stable (PDBS) with offset C if there exist $\sigma, L \in \mathcal{K}_\infty$, $C \geq 0$ and $T^* > 0$ such that for all $k_\circ \geq 0$, $z(k_\circ) = z_\circ$, with $z_\circ \in \mathbb{R}^{n_z}$, and $T \in (0, T^*)$ we have:*

$$\begin{array}{rcll} |\phi_T^z(k, k_\circ, z_\circ)| & \leq & \sigma(|z_\circ|), & \forall k \in [k_\circ, k_\circ + \ell_{L,T}] \\ |\phi_T^z(k, k_\circ, z_\circ)| & \leq & C \ , & \forall k \geq k_\circ + \ell_{L,T} \end{array} \tag{26}$$

where $L := L(|z_\circ|)$ and $\ell_{L,T}$ is defined by (4). □

Obviously, if the system (2) is UGAS, then it is PDBS with offset C for any $C > 0$. However, the system may be UGAS but not PBDS with offset $C = 0$. Our motivation to use Definition 6 is to unify proofs for the cases when the system (2) is PDBS with offset $C = 0$ and when it is UGAS. To state our main result in this section we introduce one more definition.

Definition 7 *The system (1) with input z has the property of bounded input bounded state (BIBS) with input bound C if there exists $C, d \geq 0$, $T^* > 0$ and $\sigma \in \mathcal{K}_\infty$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, with $x_\circ \in \mathbb{R}^{n_x}$, $T \in (0, T^*)$ and $|z(k)| \leq C, \forall k \geq k_\circ \geq 0$ we have:*

$$\left|\phi_T^x(k, k_\circ, x_\circ, \omega^z_{[k_\circ, k)})\right| \ = \ \sigma(|x_\circ|) + d, \qquad k \geq k_\circ \ . \tag{27}$$

□

The following proposition establishes the conditions when the UFC property implies the stronger UGB property needed in Theorem 1.

Proposition 4 *Suppose that there exists $C \geq 0$ such that the following conditions hold:*

1. *The system (1), (2) is UFC.*

2. The system (2) is PDBS with offset C.

3. The system (1) with input z is BIBS with input bound C.

Then, the system (1), (2) is UGB. □

Proof. Let $C \geq 0$ come from conditions of the Proposition. Let $\sigma_1, \sigma_2, c, T_1^*$ come from item 1, L, T_2^* come from item 2 and $\sigma_3 \in \mathcal{K}_\infty, T_3^*$, come from item 3. We show that (1), (2) is UGB with $T^* := \min\{T_1^*, T_2^*, T_3^*\}$,

$$\kappa(s) := \max\{\sigma_3\left(2\sigma_1(s) + 2\sigma_2 \circ L(s)\right), \sigma_1(s) + \sigma_2 \circ L(s)\}$$

and $c_1 := \max\{\sigma_3(2c) + d + C, c\}$. Let $T \in (0, T^*)$ and $\xi_\circ \in \mathbb{R}^n$ be arbitrary. First, consider the solutions of the system (1), (2) on the time interval $k \in [k_\circ, k_\circ + \ell_{L,T}]$. Then, using item 1 and the fact that $T\ell_{L,T} \leq L(|z_\circ|) \leq L(|\xi_\circ|)$ we can write

$$\begin{aligned} \left|\phi_T^\xi(k, k_\circ, \xi_\circ)\right| &\leq \sigma_1(|\xi_\circ|) + \sigma_2(T \cdot \ell_{L,T}) + c \\ &\leq \sigma_1(|\xi_\circ|) + \sigma_2 \circ L(|\xi_\circ|) + c\ , \qquad k \in [k_\circ, k_\circ + \ell_{L,T}]\ . \end{aligned} \tag{28}$$

Now consider the solutions on the interval $k \geq k_\circ + \ell_{L,T}$. On this time interval we have from item 2 that $|z(k)| \leq C$. Denote $x_1 := \phi_T^x(k_\circ + \ell_{L,T}, k_\circ, \xi_\circ)$. Suppose that we use 1-norm for vectors. In this case, with an abuse of notation, we have that $|\xi| = |x| + |z|$. From items 2 and 3, we have that

$$\begin{aligned} \left|\phi_T^\xi(k, k_\circ, \xi_\circ)\right| &= |\phi_T^x(k, k_\circ, \xi_\circ)| + |\phi_T^z(k, k_\circ, \xi_\circ)| \\ &\leq |\phi_T^x(k, k_\circ + \ell_{L,T}, x_1)| + C \\ &\leq \sigma_3(|x_1|) + d + C \\ &\leq \sigma_3\left(\sigma_1(|\xi_\circ|) + \sigma_2 \circ L(|\xi_\circ|) + c\right) + d + C \\ &\leq \kappa(|\xi_\circ|) + c_1\ , \qquad k \geq k_\circ + \ell_{L,T}\ . \end{aligned} \tag{29}$$

Combining (28) and (29) and in view of the compatibility of norms, the result follows.

Two special cases of the above Proposition that are easier to check are given below. Corollary 1 establishes that if the solutions of (1) (that is, driven by the inputs $z(\cdot)$) do not explode in finite time then UGB follows observing that there always exists a sufficiently large (but finite) instant k such that the solutions of (2) enter (and remain) in a ball of radius $C \geq 0$. In particular, if $C = 0$ then we impose that (2) be deadbeat stable. The second corollary concerns the case when $C > 0$ and as it will become clear later, in this case it is sufficient to impose a growth-rate restriction on the interconnection terms of (1).

Another interesting special case of our results is when the bottom subsystem (2) satisfies a stronger dead-beat stability property.

Corollary 1 *Suppose that the following conditions hold:*

1. The system (1), (2) is UFC.

2. The system (2) is PDBS with offset $C = 0$.

3. The system (3) is UGAS.

Then, the system (1), (2) is UGB. □

Proof. It follows directly from the proof of Proposition 4 by noting that UGAS of system (3) trivially implies BIBS with input bound $C = 0$ of the system (1). Indeed, we can take $\sigma(s) = \beta_x(s, 0)$ and $d = 0$.

Remark 1 *Note that since non-parameterized discrete-time cascades are forward complete if the right hand side of (23), (24) is defined everywhere, using a similar argument like in the proof of Proposition 4 we can show that the following is true (for more details see [11]):*

> *If (24) is UGDS, then for any system (23) such that $x(k+1) = f_1(x(k), 0)$ is UGAS we have that the cascade (23), (24) is UGAS.*

Example 1 illustrates the above statement. While the above statement is true even for forward complete continuous-time systems, we are not aware of a reference where this result is explicitly stated. This is probably because finite time dead-beat stability is a less common property for continuous-time systems. □

The next result imposes a growth rate restrictions on f_T and it is similar to [17, Theorem 4].

Corollary 2 *Suppose that the following conditions hold:*

1. *The system (1), (2) is UFC.*
2. *The system (2) is UGAS.*
3. *There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, $\lambda, c, T^* > 0$ and for all $T \in (0, T^*)$ there exist functions $V_T, W_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, $T \in (0, T^*)$ and $k \geq 0$ we have*

$$\begin{aligned}
\alpha_1(|x|) &\leq V_T(k, x) \leq \alpha_2(|x|) + c && (30)\\
V_T(k+1, f_T(k, x, 0)) - V_T(k, x) &\leq -TW_T(k, x) && (31)\\
V_T(k+1, f_T(k, x, z)) - V_T(k+1, f_T(k, x, 0)) &\leq T\lambda W_T(k, x)\sigma(|z|) \; . && (32)
\end{aligned}$$

Then, the system (1), (2) is UGB. □

Proof. Let $\alpha_1, \alpha_2, \sigma, \lambda, c, T^*$ come from item 3. Since the system (2) is UGAS, it is also PDBS with any offset $C > 0$. Let $C = \sigma^{-1}(1/\lambda)$. Inequalities (31) and (32) in item 3 imply:

$$\begin{aligned}
V_T(k+1, f_T(k, x, z)) - V_T(k, x) &\leq V_T(k+1, f_T(k, x, 0)) - V_T(k, x)\\
&\quad + V_T(k+1, f_T(k, x, z)) - V_T(k+1, f_T(k, x, 0))\\
&\leq -TW_T(k, x) + T\lambda W_T(k, x)\sigma(|z|)\\
&= -TW_T(k, x)(1 - \lambda\sigma(|z|)) \; .
\end{aligned}$$

If $|z| \leq C$, then

$$V_T(k+1, f_T(k, x, z)) - V_T(k, x) \leq 0 \; .$$

Hence, if $|z(k)| \le C, \forall k \ge k_\circ \ge 0$, we have

$$V_T(k, \phi_T^x(k, k_\circ, x_\circ, \omega^z_{[k_\circ,k)})) \le V_T(k_\circ, x_\circ), \qquad \forall k \ge k_\circ \ge 0 \ ,$$

and using (30) it follows that

$$\left|\phi_T^x(k, k_\circ, x_\circ, \omega^z_{[k_\circ,k)})\right| \le \alpha_1^{-1}(\alpha_2(|x_\circ|) + c) \ , \qquad \forall k \ge k_\circ \ge 0.$$

Hence, the system (1) with input z is BIBS with the input bound C. The conclusion follows from Proposition 4

We present next a sufficient condition for UFC stated in terms of a radially unbounded Lyapunov function (similar conditions for continuous-time systems were used in [3, 18, 17]). The proof follows directly from Lemmas 1 and 2 from the appendix by letting $\tilde{\gamma}_1(s) \equiv 1$, $\tilde{\gamma}_2(s) \equiv \tilde{c}_2$.

Proposition 5 *Suppose that there exist $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$, $\tilde{c}_1, \tilde{c}_2, T^* > 0$ and for any $T \in (0, T^*)$ there exists $V_T : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that for all $\xi \in \mathbb{R}^n$, $k \ge 0$ and $T \in (0, T^*)$ we have*

$$\begin{aligned} \tilde{\alpha}_1(|\xi|) \ \le \ V_T(k, x, z) \ &\le \ \tilde{\alpha}_2(|\xi|) + \tilde{c}_1 && (33) \\ V_T(k+1, f_T(k, x, z), g_T(k, z)) - V_T(k, x, z) \ &\le \ T\tilde{\alpha}_3(V_T(k, x, z)) + T\tilde{c}_2 \ , && (34) \\ \int_1^\infty \frac{ds}{\tilde{\alpha}_3(s)} \ &= \ \infty \ . && (35) \end{aligned}$$

Then, the system (1), (2) is UFC. □

The condition (35), which is clearly similar to (17), deserves special attention. It shall not be surprising that this condition for UFC is actually very tight (though not necessary) since as mentioned before, UFC is very close to the property forward completeness for continuous-time systems. As already noted the condition (35) restricts the growth of $\tilde{\alpha}_3$. In particular, this condition holds if $\tilde{\alpha}_3(s) = O(s)$ as $s \to \infty$. For example, if all conditions of Proposition 5 hold with (34) replaced by

$$V_T(k+1, f_T(k, x, z), g_T(k, z)) - V_T(k, x, z) \le Tc_1 V_T(k, x, z) + Tc_2 \ ,$$

for some $c_1, c_2 \ge 0$, then the system (1), (2) is UFC.

4 Conclusions

It has been established in [10] that a necessary and sufficient condition for UGAS of cascaded discrete-time parameterized systems is that the systems solutions be uniformly globally bounded. This condition is in general difficult to check. In this paper we have presented a range of sufficient conditions guaranteeing UGB. Our results are formulated in terms of the well-studied (in the continuous-time context) notions of integral input-to-state stability and forward completeness. Our results contribute further to cascaded-based control design for sampled-data systems via their approximate discrete-time models.

References

[1] D. Angeli. Intrinsic robustness of global asymptotic stability. *Syst. & Contr. Letters*, 38:297–307, 1999.

[2] D. Angeli and D. Nešić . Power characterizations of ISS and integral ISS. *IEEE Trans. on Automat. Contr.*, 46:1298–1303, 2001.

[3] D. Angeli and E. Sontag. Forward completeness, unboundedness observability, and their Lyapunov characterizations. *Syst. & Contr. Letters*, 38:209–217, 1999.

[4] D. Angeli, E. Sontag, and Y. Wang. A characterization of integral input to state stability. *IEEE Trans. on Automat. Contr.*, 45:1082–1097, 2000.

[5] M. Arcak, D. Angeli, and E. Sontag. A unifying integral ISS framework for stability of nonlinear cascades. *SIAM J. on Contr. and Opt.*, 40:888–1904, 2002.

[6] A. Isidori. *Nonlinear control systems.* Springer Verlag, 3rd. edition, 1995.

[7] Z-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37(6):857–869, 2001.

[8] P. V. Kokotović and H. J. Sussmann. A positive real condition for global stabilization of nonlinear systems. *Syst. & Contr. Letters*, 13(4):125–133, 1989.

[9] A. Loría and D. Nešić . On necessary and sufficient conditions for UGAS of discrete-time parameterized systems. Technical report, LSS, 2002.

[10] A. Loría and D. Nešić . On uniform asymptotic stability of time-varying parameterized discrete-time cascades. Technical report, LSS, 2002. Submitted to IEEE Trans. Automat. Contr.

[11] A. Loría and D. Nešić . Stability of time-varying discrete-time cascades. In *Proc. 15th. IFAC World Congress*, Barcelona, Spain, 2002. Paper no. 652.

[12] F. Mazenc and L. Praly. Adding integrators, saturated controls and global asymptotic stabilization of feedforward systems. *IEEE Trans. on Automat. Contr.*, 41(11):1559–1579, 1996.

[13] D. Nešić , A. Teel, and P. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Syst. & Contr. Letters*, 38:259–270, 1999.

[14] D. Nešić , A. Teel, and E. Sontag. Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems. *Syst. & Contr. Letters*, 38:49–60, 1999.

[15] D. Nešić and A. R. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Trans. on Automat. Contr.*, 2002. Accepted subject to minor modifications.

[16] E. Panteley and A. Loría . On global uniform asymptotic stability of non linear time-varying non autonomous systems in cascade. *Syst. & Contr. Letters*, 33(2):131–138, 1998.

[17] E. Panteley and A. Loría . Growth rate conditions for stability of cascaded time-varying systems. *Automatica*, 37(3):453–460, 2001.

[18] G. Sansone and R. Conti. *Nonlinear Differential equations.* Pergamon Press, Oxford, England, 1964.

[19] P. Seibert and R. Suárez. Global stabilization of nonlinear cascaded systems. *Syst. & Contr. Letters*, 14:347–352, 1990.

[20] R. Sepulchre, M. Janković, and P. Kokotović. *Constructive nonlinear control.* Springer Verlag, 1997.

[21] E. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. on Automat. Contr.*, 34(4):435–443, 1989.

[22] E. D. Sontag. Comments on integral variants of ISS. *Syst. & Contr. Letters*, 34:93–100, 1998.

[23] H. J. Sussman and P. V. Kokotović. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Trans. on Automat. Contr.*, 36(4):424–439, 1991.

A Some technical lemmas

Lemma 1 *Consider the system:*

$$y(k+1) = F_T(k, y(k), u(k)) \ . \tag{36}$$

Suppose that there exist $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3 \in \mathcal{K}_\infty$, $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{N}$, $\tilde{c}, T^* > 0$ *and for any* $T \in (0, T^*)$ *there exists* $V_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that for all* $y \in \mathbb{R}^n$, $k \geq 0$ *and* $T \in (0, T^*)$ *we have that*

$$\begin{aligned} \tilde{\alpha}_1(|y|) \ \leq \ V_T(k, y) \quad &\leq \quad \tilde{\alpha}_2(|y|) + \tilde{c} && (37)\\ V_T(k+1, F_T(k, y, u)) - V_T(k, y) \quad &\leq \quad T\tilde{\gamma}_1(|u|)\tilde{\alpha}_3(V_T(k, x)) + T\tilde{\gamma}_2(|u|) \ , && (38)\\ \int_1^\infty \frac{ds}{\tilde{\alpha}_3(s)} \quad &= \quad \infty \ . && (39) \end{aligned}$$

Then, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $c > 0$ *and for each* $T \in (0, T^*)$ *there exists* $W_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that all* $y \in \mathbb{R}^n$, $k \geq 0$ *and* $T \in (0, T^*)$ *we have that*

$$\begin{aligned} \alpha_1(|y|) \ \leq \ W_T(k, y) \quad &\leq \quad \alpha_2(|y|) + c && (40)\\ W_T(k+1, F_T(k, y, u)) - W_T(k, y) \quad &\leq \quad T\kappa(|u|) \ , && (41) \end{aligned}$$

where $\kappa(s) := \left(\tilde{\gamma}_1(s) + \tilde{\gamma}_2(s)\frac{1}{\tilde{\alpha}_3(1)}\right)$. □

Proof. Let $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\gamma}_1, \tilde{\gamma}_2$ come from the conditions of the lemma. Define:

$$q(s) := \begin{cases} \frac{1}{\tilde{\alpha}_3(1)}, & s \leq 1 \\ \frac{1}{\tilde{\alpha}_3(s)}, & s > 1 \end{cases} \ .$$

Let $\rho \in \mathcal{K}_\infty$ be defined as

$$\rho(s) := \int_0^s q(\tau) d\tau \ .$$

Note that $\rho \in \mathcal{K}_\infty$ and ρ is differentiable with $\frac{d\rho}{ds}(s) = q(s)$. Also, $q(\cdot)$ is non-increasing since $\tilde{\alpha}_3 \in \mathcal{K}_\infty$.

Introduce $W_T(k, y) := \rho(V_T(k, y))$. Then, it is obvious that (40) holds with $\alpha_1(s) := \rho \circ \tilde{\alpha}_1(s)$, $\alpha_2(s) = \rho \circ 2\tilde{\alpha}_2(s)$ and $c = \rho(2\tilde{c})$. We now show that (41) holds.

We consider two cases. If $V_T(k+1, F_T(k, y, u)) \leq V_T(k, y)$, then since $\rho \in \mathcal{K}_\infty$, we have:

$$W_T(k+1, F_T(k, y, u)) - W_T(k, y) \leq 0 \leq T \left(\tilde{\gamma}_1(|u|) + \tilde{\gamma}_2(|u|) \frac{1}{\tilde{\alpha}_3(1)} \right) \ . \tag{42}$$

On the other hand, if $V_T(k+1, F_T(k, y, u)) > V_T(k, y)$, then using the Mean Value Theorem we can write

$$\begin{aligned} W_T(k+1, F_T(k, y, u)) - W_T(k, y) &= q(V^*)[V_T(k+1, F_T(k, y, u)) - V_T(k, y)] \\ &\leq Tq(V^*)[\tilde{\gamma}_1(|u|)\tilde{\alpha}_3(V_T(k, y)) + \tilde{\gamma}_2(|u|)] \end{aligned} \tag{43}$$

where $V_T(k, y) < V^* < V_T(k+1, F_T(k, y, u))$. Since q is a non increasing function, we have

$$q(V_T(k, y))) \geq q(V^*) \ ,$$

and hence

$$W_T(k+1, F_T(k, y, u)) - W_T(k, y) \ \ \leq \ \ Tq(V_T(k, y))[\tilde{\gamma}_1(|u|)\tilde{\alpha}_3(V_T(k, y)) + \tilde{\gamma}_2(|u|)] \ .$$

If $V_T(k, y) \leq 1$, then we have:

$$\begin{aligned} W_T(k+1, F_T(k, y, u)) - W_T(k, y) &\leq T \frac{\tilde{\gamma}_1(|u|)\tilde{\alpha}_3(V_T(k, y)) + \tilde{\gamma}_2(|u|)}{\tilde{\alpha}_3(1)} \\ &\leq T \frac{\tilde{\gamma}_1(|u|)\tilde{\alpha}_3(1) + \tilde{\gamma}_2(|u|)}{\tilde{\alpha}_3(1)} \\ &= T \left(\tilde{\gamma}_1(|u|) + \tilde{\gamma}_2(|u|) \frac{1}{\tilde{\alpha}_3(1)} \right) \ . \end{aligned} \tag{44}$$

If $V_T(k, y) \geq 1$, then

$$\begin{aligned} W_T(k+1, F_T(k, y, u)) - W_T(k, y) &\leq T \frac{\tilde{\gamma}_1(|u|)\tilde{\alpha}_3(V_T(k, y)) + \tilde{\gamma}_2(|u|)}{\tilde{\alpha}_3(V_T(k, y))} \\ &\leq T \left(\tilde{\gamma}_1(|u|) + \frac{\tilde{\gamma}_2(|u|)}{\tilde{\alpha}_3(V_T(k, y))} \right) \\ &= T \left(\tilde{\gamma}_1(|u|) + \tilde{\gamma}_2(|u|) \frac{1}{\tilde{\alpha}_3(1)} \right) \ . \end{aligned} \tag{45}$$

The proof is completed by combining (42), (44) and (45).

Lemma 2 *Consider the system (36). Suppose that there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\kappa \in \mathcal{N}$, $c > 0$ *and for each* $T \in (0, T^*)$ *there exists* $W_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that all* $y \in \mathbb{R}^n$, $k \geq 0$ *and* $T \in (0, T^*)$ *we have that (40) and (41) hold. Then, the solutions of the system (36) satisfy (9) with* $\gamma_1(s) = \alpha_2(s)$, $\alpha = \alpha_1$, $\gamma_2(s) = s + c$ *and* $\mu(s) = \kappa(s)$. *In particular, if* $\kappa(s) \in \mathcal{K}_\infty$, *then the system (36) is iISNpS. If* $\kappa(0) = C$ *for some* $C \geq 0$, *and* $u(k) \equiv 0$, *then the system (36) is UFC.* □

Proof. Consider arbitrary $T \in (0, T^*)$, $k_\circ \geq 0$, $y(k_\circ) = y_\circ$ and $u(k)$. Denote $w(k) := W_T(k, \phi_T^y(k, k_\circ, y_\circ, \omega^z_{[k_\circ,k)}))$. Then, using (41) we can write

$$w(k+1) \leq w(k) + T\kappa(|u(k)|), \qquad \forall k \geq k_\circ \geq 0 \ .$$

By induction, we it follows that:

$$w(k) \leq w(k_\circ) + T \sum_{t=k_\circ}^{k-1} \kappa(|u(k)|), \qquad \forall k \geq k_\circ \geq 0 \ .$$

Finally, using (40) we obtain

$$\alpha_1\left(\left|\phi_T^y(k, k_\circ, y_\circ, \omega^z_{[k_\circ,k)})\right|\right) \ \leq \ \alpha_2(|y_\circ|) + c + T \sum_{t=k_\circ}^{k-1} \kappa(|u(k)|), \qquad \forall k \geq k_\circ \geq 0 \ , \quad (46)$$

which shows that the bound (9) holds. Hence, if $\kappa \in \mathcal{K}_\infty$ the system (36) is iISNpS with the above defined functions. If on the other hand $\kappa(0) = C$, and $u(k) \equiv 0$, then we have:

$$\alpha_1(|\phi_T^y(k, k_\circ, y_\circ, 0)|) \ \leq \ \alpha_2(|y_\circ|) + c + CT(k - k_\circ), \qquad \forall k \geq k_\circ \geq 0 \ , \qquad (47)$$

which shows that the input-free system (36) is UFC.