On l_2 stabilization of linear systems with quantized control

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***Abstract*–This paper extends results from [5], where input-to-state stabilization (ISS) of linear systems with quantized feedback was considered. In this paper, we show that using the same scheme and under the same conditions as in [5] it is also possible to achieve (nonlinear gain) l_2 stabilization for linear systems. We also prove a new lemma on $\mathcal{K}_\infty$ functions that is interesting in its own right.**

I. Introduction

The subject of this paper is control of systems with quantized feedback. By quantized feedback we mean controllers that have values in a finite (or countable) set. Control of systems with quantized feedback is an emerging research area that brings together elements of control and information theory to provide novel insights into control over networks with bandwidth limitations. The area of Networked Control Systems (NCS) is relatively new, nevertheless, a number of important results have been published (e.g. stabilization for systems with quantized control/measurements was considered in [1], [2], [4]; robust stabilization and estimation was considered among others in [3], [5], [6], [7], [9]).

In this paper we explore nonlinear gain l_2 stability properties of linear time-invariant systems that are controlled by the controller proposed in [5]. In particular, we show that using the time-sampled scheme introduced in [5], it is possible to achieve nonlinear gain l_2 stability for the linear time-invariant systems with quantized feedback. Similar to [10], we use a concept of nonlinear gains to describe the robustness properties of the system with respect to external disturbances. This is consistent with the result in [6], where Martins shows that nonlinear gains are necessary when formulating properties of the disturbance attenuation for the linear discrete system with quantized feedback. Martins shows that linear (finite) l_p,

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$p \in [1, \infty]$, gains are not achievable when quantized control with finitely many levels is used. Our Theorem 1 in Section IV, on the other hand, shows what kind of nonlinear l_2 gains are achievable for linear systems with controllers from [5]. Moreover, we also state and prove Lemma 1 in Section II on properties of $\mathcal{K}_\infty$ functions that is of interest in its own right.

The remainder of the paper is organized as follows. In Section II we give definitions and lemmas that are used in the sequel. The closed loop system, switching rules and protocol are given in Section III. The main results are presented in Section IV. Section V offers conclusions. The proofs of technical lemmas are given in the appendix.

II. NOTATION AND PRELIMINARIES

In this section we introduce some notation and give the definitions that will make the discussed concepts precise. We denote the two-norm of the vector as follows: $|z| := \sqrt{\sum_{i=1}^{n}(z^i)^2}$, where $z = (z^1, z^2, \ldots, z^n)$, n is the dimension of the vector z. The sequence of vectors z_k for $k \in [k_1, k_2]$, is denoted as $z_{[k_1,k_2]}$. The two-norm of a sequence of vectors on a time-interval $[k_1, k_2]$ is denoted as $\|z_{[k_1,k_2]}\| := \sqrt{\sum_{k=k_1}^{k_2} |z_k|^2}$. $\|A\|$ denotes the induced two-norm of the matrix A. A quantizer is a piecewise constant function $q : \mathbb{R}^n \to Q$, where Q is a finite subset of $\mathbb{R}^n$. We use:

Assumption 1: [4] There exist strictly positive numbers $M > \Delta > 0$ and Δ_0, such that the following holds: 1. If $|z| \le M$ then $|z - q(z)| \le \Delta$; 2. If $|z| > M$ then $|q(z)| > M - \Delta$; 3. For all $|z| \le \Delta_0$ we have that $q(z) = 0$. ■

M is called the range of the quantizer; Δ is called the quantization error; Δ_0 is the dead-zone. The first condition gives a bound on the quantization error when the state is in the range of the quantizer, the second gives the possibility to detect saturation. The third condition is needed to preserve the origin as an equilibrium. We use the following one-parameter family of dynamic quantizers introduced in [4]:

$$q_\mu(x) = \mu q\left(\frac{x}{\mu}\right), \ \mu \ge 0, \tag{1}$$

where μ is an adjustable parameter, called "zoom" variable, that is updated at discrete instants of time. For each fixed μ the range of the quantizer is $M\mu$ and the quantization error is $\Delta\mu$. We use the following definition:

Definition 1: A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded. A function γ is subadditive if $\gamma(x + y) \le \gamma(x) + \gamma(y)$ for all $x, y \geq 0$. A function γ is superadditive if $\gamma(x) + \gamma(y) \leq \gamma(x+y)$ for all $x, y \geq 0$. We denote as $\mathcal{K}_+$ the class of functions that are of class $\mathcal{K}_\infty$ and are subadditive. Similarly, we denote by $\mathcal{K}^+$ the class of functions that are of class $\mathcal{K}_\infty$ and are superadditive.

A simple consequence of subadditivity of $\gamma \in \mathcal{K}_\infty$ is that for any positive integer N and any nonnegative numbers $a_0, \ldots, a_N$ we have $\gamma\left(\sum_{i=0}^N a_i\right) \leq \sum_{i=0}^N \gamma(a_i)$. Similarly, for superadditive function $\gamma \in \mathcal{K}_\infty$ we can write for any positive integer N and any nonnegative numbers $a_0, \ldots, a_N$ that $\sum_{i=0}^N \gamma(a_i) \leq \gamma\left(\sum_{i=0}^N a_i\right)$. To prove the main result in Section IV we state and prove the following technical lemma, that we believe is a new result and it is of interest in its own right.

Lemma 1: For any $\gamma \in \mathcal{K}_\infty$, there exist $\gamma_1 \in \mathcal{K}^+$ and $\gamma_2 \in \mathcal{K}_+$, such that $\gamma(s) \leq \gamma_1 \circ \gamma_2(s)$ for all $s \geq 0$.

III. Closed-loop system

In this section, we recall the plant model and quantized controller from [5] that are considered in the sequel. Consider the plant model

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^l$ are respectively state, control and disturbance. Matrix A is nonzero and non-Hurwitz. Define $t_k = kT$ for $k = 0, 1, 2, \ldots$, where $T > 0$ is a given sampling period. We shortly denote $x(t_k) = x_k$ and similarly for all other variables. Let $u(t) = u_k =$ const. for all $t \in [t_k, t_{k+1})$, $k \geq 0$. The discrete time plant model of the sampled-data plant (2) is more amenable to analysis:

$$x_{k+1} = \Phi x_k + \Gamma u_k + \omega_k, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{3}$$

where $\Phi = e^{AT}$; $\Gamma = \int_0^T e^{As} B\, ds$; $\omega_k := \int_{kT}^{k+1} e^{A((k+1)T-s)} Dd(s)ds$. To control the system (3) we use the quantized hybrid feedback, proposed in [5]. The controller dynamics is described by the following:

$$u_k = U(\Omega_k, \mu_k, x_k), \quad t \in [t_k, t_{k+1}) \tag{4}$$

$$U(\Omega_k, \mu_k, x_k) := \begin{cases} 0 & \text{if } \Omega_k = \Omega_{out} \\ Kq_k & \text{if } \Omega_k = \Omega_{in}, \end{cases} \tag{5}$$

(6)

where $q_k := q_{\mu_k}(x_k)$. The variable Ω determines the switching rule for the controller. It can take only two strictly positive values Ω_{out} and Ω_{in}, that will be defined later. If $\Omega_k = \Omega_{out}$

we say that zoom-out condition is triggered at time k. If $\Omega_k = \Omega_{in}$ we say that zoom-in condition is triggered at time k. During zoom-out stage the system is running in open loop $u_k = 0$, during zoom-in stage the certainty equivalence feedback $u_k = Kq_k$ is applied. The protocol dynamics is described by the following:

$$\mu_{k+1} = G(\Omega_k, \mu_k, x_k), \quad \mu_0 \in \mathbb{R}_{>0} \tag{7}$$

$$G(\Omega_k, \mu_k, x_k) := \begin{cases} \Omega_{out}(\mu_k + c) & \text{if } \Omega_k = \Omega_{out}, \quad c > 0 \\ \Omega_{in}\mu_k & \text{if } \Omega_k = \Omega_{in} \end{cases} \tag{8}$$

(9)

In our discussions we let $c = 1$. The adjustment policy for μ, called ”zooming protocol”, depends only on the quantized measurements of the state q_k, $k = 0, 1, 2, \ldots, N$. Geometrically, at each time instant $\mathbb{R}^n$ is divided into a finite number of quantization regions. Each region corresponds to a fixed value of the quantizer q_k. During zoom-out stage the value of adjustable parameter μ is increased at the rate faster than the growth of $|x_k|$ until the state can be adequately measured. During zoom-in stage the value of adjustable parameter μ is decreased in such way as to drive the state to the origin. The switching law dynamics is described by the following:

$$\Omega_k = H(\Omega_{k-1}, \mu_k, x_k), \quad \Omega_{-1} = \Omega_{out} \tag{10}$$

$$H(\Omega_{k-1}, \mu_k, x_k) := \begin{cases} \Omega_{out} & \text{if } |q_k| > l_{out}\mu_k \\ \Omega_{in} & \text{if } |q_k| < l_{in}\mu_k \\ \Omega_{k-1} & \text{if } |q_k| \in [l_{in}\mu_k, l_{out}\mu_k], \end{cases} \tag{11}$$

where $l_{out} = M - \Delta$, $l_{in} = \Delta_M - \Delta$ and $l_{out} > l_{in}$. The choice of the parameters M, Δ and Δ_M is given later. The hysteresis switching is used to switch between zoom-in and zoom-out stages.

Remark 1: We will analyze only the stability properties of the discrete-time system (3)-(11). It was shown in [8] and [11] how to use the underlying discrete-time model to conclude appropriate stability of the sampled-data system. ■

Remark 2: Note that it is not necessary to use $u = 0$ in (5) during zoom-out. We use this choice to simplify the analysis. For more detailed discussions on the flexibility in the controller and the zooming protocol design, refer to Liberzon and Nešić [5]. ■

We introduce some notation. For each $k \geq 0$ there are two possible cases: $\Omega_k = \Omega_{out}$ (in this case we say that zoom-out is triggered at time k) or $\Omega_k = \Omega_{in}$ (in this case we say that zoom-in is triggered at time k). Given an initial condition and a disturbance there is a sequence of zoom-out and zoom-in intervals. There may be infinitely many or finitely many such intervals. We introduce $k_i \in N$ such that $\Omega_k = \Omega_{out}$ if $k \in [k_{2i}, k_{2i+1}-1]$ and $\Omega_k = \Omega_{in}$ if $k \in [k_{2i+1}, k_{2i+1}-1]$; $i = 0, 1, \ldots, N$ (N may be infinity). For simplicity, we let the first interval always to be zoom-out: $\Omega_{-1} = \Omega_{out}$, keeping in mind that it may actually be an empty interval. To state our main result we need to consider the dynamics of the variable $\xi_k := \frac{x_k}{\mu_k}$, that governs the switching between zoom-ins and zoom-outs. For zoom-out stage we have:

$$|\xi_{k+1}| \leq \frac{\|\Phi\|}{\Omega_{out}}|\xi_k| + \frac{1}{\Omega_{out}}|\zeta_k| \quad k \in [k_{2i}, k_{2i+1}-1] \tag{12}$$

and for zoom-in stage we have:

$$\xi_{k+1} = \frac{1}{\Omega_{in}}(\Phi + \Gamma K)\xi_k + \frac{1}{\Omega_{in}}\Gamma K \nu_k + \frac{1}{\Omega_{in}}\zeta_k \quad k \in [k_{2i+1}, k_{2i+1}-1], \tag{13}$$

where $\nu_k := q(\xi_k) - \xi_k$ and $\zeta_k := \frac{w_k}{\mu_k}$. Under the assumption that $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur, there exists strictly positive numbers L_1, λ_1, γ, $\bar{\gamma}$ such that the solutions of the system (13) satisfy (more details can be found in [5]):

$$|\xi_k| \leq L_1 \exp(-\lambda_1 k)|\xi_0| + \gamma\|\nu\|_\infty + \bar{\gamma}\|\zeta_k\| \quad \forall k \geq 0. \tag{14}$$

IV. Main Results

The main contributions of this paper are presented in this section. The main purpose of our work is to use an appropriate notion of nonlinear gains to characterize a nonlinear gain l_2 stability property of the system and show that the scheme proposed by Liberzon and Nešić in [5] yields this property. We show that the closed loop system (3)-(11) possesses the following stability property:

Definition 2: The system (3)-(11) is said to be nonlinear gain (NG) l_2 stable if for every $\mu_0 > 0$ there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial conditions x_0, μ_0 and every disturbance w the following holds:

$$\|x_{[0,k]}\|^2 \leq \gamma_1(|x_0|) + \gamma_2\left(\sum_{i=0}^{k-1}\gamma_3(|w_i|)\right) \quad \forall k \geq 0.$$

Note, that γ_i, $i = 1, 2, 3$, are nonlinear functions that in general depend on μ_0. Definition 2 shows explicitly what we mean by nonlinear gains and is in a more general form than the definition of NG l_2 stability used by Martins in [6], where $\gamma_2(s) := \tilde{\gamma}_2(\sqrt{s})$ and $\gamma_3(s) := s^2$. Martins [6] showed that linear gains are not achievable with quantized feedback. The main contribution of our work is the following theorem, which presents conditions under which the system (3)-(11) is NG l_2 stable.

Theorem 1: Consider the system (3)-(11). Suppose that Assumption 1 holds and for a given sampling period $T > 0$: (i) K is such that $\Phi + \Gamma K$ is Schur; (ii) $\Omega_{out} > \|\Phi\|$; (iii) $\Omega_{in} \in (0, 1)$ is such that $\frac{1}{\Omega_{in}}(\Phi+\Gamma K)$ is Schur; (iv) M and Δ are such that $M > (2+L_1+\gamma)\Delta$, where L_1 and γ come from (14). Then, the system (3)-(11) is NG l_2 stable.

Theorem 1 shows that using the same controller design and the same conditions as in [5] it is possible to achieve NG l_2 stability of the system (3)-(11), which was not proven in [5] (only ISS was investigated in [5]). As opposed to the work by Martins [6] where he showed what kind of gains are not achievable for the discrete linear systems with quantized feedback, Theorem 1 shows what kind of gains are achievable with controllers from [5].

Remark 3: The controller design is the same as in [5] and is completed as follows. First, for the quantizer satisfying Assumption 1 we use a one-parameter family of dynamic quantizers (1). We design K such that $\Phi+\Gamma K$ is Schur, so that item (i) of Theorem 1 holds. We choose $\Omega_{out} > \|\Phi\|$ and $\Omega_{in} \in (0, 1)$ such that $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur, so that items (ii) and (iii) of Theorem 1 hold. Then, by Lemma III.2 in [5], there always exist strictly positive L_1 and γ such that the solutions of the system (13) satisfy (14). In the last step we choose M and Δ such that item (iv) of Theorem 1 holds. ■

The proof of Theorem 1 relies on Lemma 1 from Section I and Lemma 2 given below. Lemma 2 combines the results of Lemmas 3-6 in the appendix and shows a bound from w to x. Note that the bound in Lemma 2 depends on the switching times k_i, that in turn depend on x_0, μ_0 and w. Nevertheless, this bound implies NG l_2 stability via Lemma 1, as shown in the proof of Theorem 1 that is given below.

Lemma 2: Consider the system (3)-(11). Suppose that all conditions of Theorem 1 hold. Then for every $\mu_0 > 0$ there exist γ_1, $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$ such that for every initial conditions x_0, μ_0, any $k > 0$ and any disturbance w there exist switching times $k_i \in \mathbb{N}$, $i = 1, \ldots, N$ with $k_0 = 0$ and $k_N = k$, such that the following holds:

$$\|x_{[k_0,k_N]}\|^2 \leq \gamma_1(|x_{k_0}|) + \sum_{l=0}^{N-1} \varphi_1 \left(\sum_{j=k_l}^{k_{l+1}-1} \varphi_2(|w_j|) \right).$$

Proof of Theorem 1. For any $\mu_0 > 0$ there exist γ_1, φ_1 and φ_2 such that the bound from Lemma 2 holds. Then for any fixed initial conditions x_{k_0}, μ_{k_0} and any bounded disturbance w we can write:

$$\|x_{[k_0,k_N]}\|^2 \leq \gamma_1(|x_{k_0}|) + \sum_{l=0}^{N-1} \varphi_1 \left(\sum_{j=k_l}^{k_{l+1}-1} \varphi_2(|w_j|) \right)$$

$$\leq \gamma_1(|x_{k_0}|) + \sum_{l=0}^{N-1} \varphi_{11} \left(\varphi_{12} \left(\sum_{j=k_l}^{k_{l+1}-1} \varphi_3(|w_j|) \right) \right) \tag{15}$$

$$\leq \gamma_1(|x_{k_0}|) + \varphi_{11} \left(\sum_{l=0}^{N-1} \varphi_{12} \left(\sum_{j=k_l}^{k_{l+1}-1} \varphi_3(|w_j|) \right) \right) \tag{16}$$

$$\leq \gamma_1(|x_{k_0}|) + \varphi_{11} \left(\sum_{l=0}^{N-1} \sum_{j=k_l}^{k_{l+1}-1} \varphi_{12} \circ \varphi_3(|w_j|) \right) \tag{17}$$

$$= \gamma_1(|x_{k_0}|) + \gamma_2 \left(\sum_{j=k_0}^{k-1} \gamma_3(|w_j|) \right) \tag{18}$$

where the inequality (15) comes from Lemma 1 since $\varphi_1(s) \leq \varphi_{11} \circ \varphi_{12}(s), \forall s \geq 0$, $\varphi_{11} \in \mathcal{K}^+$ and $\varphi_{12} \in \mathcal{K}_+$. The inequality (16) is true since $\varphi_{11} \in \mathcal{K}^+$. The inequality (17) is true since $\varphi_{12} \in \mathcal{K}_+$. The last equality (18) comes from the fact that we denote $\gamma_2(s) := \varphi_{11}(s)$ and $\gamma_3(s) := \varphi_{12} \circ \varphi_3(s)$. This completes the proof. ■

Remark 4: Note that since $\varphi_{11} \in \mathcal{K}^+$, the nonlinear gain $\gamma_2 = \varphi_{11}$ has a form of a superadditive function (the example of a supperadditive function is a square function). Moreover, from the proof of Lemma 1 in the appendix, we have that $\varphi_{11}(s) := s\varphi_1(s)$ $\forall s \geq 1$, which is growing faster than a linear function. This is consistent with Martins results in [6]. Also, the gain γ_3 is a composition of $\mathcal{K}_\infty$ and $\mathcal{K}_+$ functions. ■

Note, that it was shown in [5] that it is possible to achieve global asymptotic stability in x when $c = 0$ in (8), but at the same time not have ISS. We show that $c > 0$ in (8) is in general also necessary for NG l_2 stability. Proposition 1 below shows the lack of robustness of the system (3) in NG l_2 sense when $c = 0$ in (8), which is similar to Proposition III.6 in [5]. Note, Proposition 1 employs a more general form of NG l_2 stability, than given in the Definition 2 (where γ_4 and γ_5 are identity functions). The proof of Proposition 1 follows almost the same steps as that of Proposition III.6 in [5] and is omitted.

Proposition 1: Consider the system (3)-(11). Let $c = 0$ and Φ be such that it has at least one eigenvalue $\lambda_m > 1$ and its corresponding eigenvector $\zeta_m = 1$. Then, for every fixed

$\mu_0 > 0$ there do not exist $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \mathcal{K}_\infty$, such that the following holds for every initial conditions x_0, μ_0 and every disturbance w:

$$\gamma_4(\sum_{i=0}^{k} \gamma_5(|x_i|)) \leq \gamma_1(|x_0|) + \gamma_2(\sum_{i=0}^{k} \gamma_3(|w_i|)) \quad \forall k \geq 0.$$

V. Conclusions

This paper presents results on nonlinear gain l_2 stability for linear time-invariant systems with quantized state measurements. Our main result (Theorem 1) shows that a particular type of NG l_2 stabilization is possible with the control scheme proposed in [5]. We state and prove Lemma 1 on properties of $\mathcal{K}_\infty$ functions that is of interest in its own right.

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VI. Appendix

Proof of Lemma 1. First note that there is no loss of generality in assuming that $\gamma(1) = 1$ since we can always scale γ as follows: $\gamma(s) = \gamma(1)\frac{1}{\gamma(1)}\gamma(s) =: \gamma(1)\gamma_1(s)$, and we can only concentrate on obtaining the bound on γ_1, with $\gamma_1(1) = 1$. Next we note that given any function $\tilde{\gamma} \in \mathcal{K}_\infty$ with $\tilde{\gamma}(1) = 1$, we can always write it as follows: $\tilde{\gamma}(s) = \tilde{\gamma}_1 \circ \tilde{\gamma}_2(s)$ where

$$\tilde{\gamma}_1(s) := \begin{cases} s, & s \in [0,1] \\ \tilde{\gamma}(s), & s \geq 1 \end{cases} \qquad \tilde{\gamma}_2(s) := \begin{cases} \tilde{\gamma}(s), & s \in [0,1] \\ s, & s \geq 1 \end{cases} \tag{19}$$

Note that since $\tilde{\gamma} \in \mathcal{K}_\infty$ and $\tilde{\gamma}(1) = 1$ we also have that the so constructed $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{K}_\infty$. Next we show that for any $\gamma \in \mathcal{K}_\infty$ there exists $\tilde{\gamma}$ such that $\gamma(s) \leq \tilde{\gamma}(s) \quad \forall s \geq 0$, and, moreover, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ constructed above are respectively superadditive and subadditive.

Construction of $\tilde{\gamma}_1$: Let $\tilde{\gamma}(s) := s\gamma(s)$, $s \geq 1$. Obviously, we have that $\gamma(s) \leq \tilde{\gamma}(s) = s\gamma(s)$, $\forall s \geq 1$. Moreover, $\tilde{\gamma}_1(s)$ defined in (19) is $\mathcal{K}_\infty$ since it is zero at zero, continuous (since $\tilde{\gamma}(1) = 1$), strictly increasing since γ is strictly increasing and unbounded since γ is. We just need to show superadditivity of $\tilde{\gamma}_1$. First, we show that if $\frac{\tilde{\gamma}_1(s)}{s}$ is a nondecreasing function of s, then $\tilde{\gamma}_1$ is supperadditive. Indeed, in this case we have for all x, y that:

$$\tilde{\gamma}_1(x+y) = x\frac{\tilde{\gamma}_1(x+y)}{x+y} + y\frac{\tilde{\gamma}_1(x+y)}{x+y} \geq x\frac{\tilde{\gamma}_1(x)}{x} + y\frac{\tilde{\gamma}_1(y)}{y} = \tilde{\gamma}_1(x) + \tilde{\gamma}_1(y) \ .$$

Using our construction of $\tilde{\gamma}_1$, we have that

$$\frac{\tilde{\gamma}_1(s)}{s} = \begin{cases} 1, & s \in [0,1] \\ \gamma(s), & s \geq 1 \end{cases}$$

and this is a nondecreasing function since γ is increasing and $\gamma(1) = 1$. Hence, we have that $\tilde{\gamma}_1 \in \mathcal{K}^+$.

Construction of $\tilde{\gamma}_2$: We introduce the following function: $q(s) := \sup_{t \in (s,1]} \frac{\gamma(t)}{t}$ for $s \in [0,1)$ and $q(1) = 1$. And define $\tilde{\gamma}(s) := s \cdot q(s)$ for all $s \in [0,1]$. First, it is easy to see that: $\tilde{\gamma}(s) \geq s\frac{\gamma(s)}{s} = \gamma(s)$, $\forall s \in [0,1]$. Next, we show that $\tilde{\gamma}$ is a class $\mathcal{K}$ function on the interval $[0,1]$. $\tilde{\gamma}$ is continuous on $(0,1]$ since it is a product of two continuous functions (note that q is continuous on $(0,1]$). Continuity at $s = 0$ can be shown using the below given analysis. Moreover, $\tilde{\gamma}(0) = 0$ as the following analysis shows. Either we have that $\lim_{s \to 0^+} q(s) = \frac{\gamma(s^*)}{s^*}$ for some fixed $s^* \in (0,1]$, in which case we have $\tilde{\gamma}(0) = \lim_{s \to 0^+} s\frac{\gamma(s^*)}{s^*} = 0$. Or we have that $\lim_{s \to 0^+} q(s) = \lim_{s \to 0^+} \frac{\gamma(s)}{s}$, in which case we have $\tilde{\gamma}(0) = \lim_{s \to 0^+} s\frac{\gamma(s)}{s} = \lim_{s \to 0^+} \gamma(s) = \gamma(0) = 0$. Now we need to show that $\tilde{\gamma}$ is strictly increasing. Note that for any s, either we have that $q(s)$ is strictly decreasing at s, in which case we have that $q(s) = \frac{\gamma(s)}{s}$ and hence $\tilde{\gamma}(s) = s\frac{\gamma(s)}{s} = \gamma(s)$, and, hence, $\tilde{\gamma}$ is strictly increasing or we have that $q(s)$ does not change as s increases, in which case from continuity of $q(s)$ we have that there exists $s^* \leq s$ such that $\tilde{\gamma}(s) = s\frac{\gamma(s^*)}{s^*}$, which is strictly increasing (linear) function of s (this also shows continuity at 0). Hence, $\tilde{\gamma}$ is a class $\mathcal{K}$ function on $[0,1]$. Moreover, since $\tilde{\gamma}(1)=1$, it is easy to see that the function $\tilde{\gamma}_2$ defined in (19) is of class $\mathcal{K}_\infty$. Note also that the function

$$\frac{\tilde{\gamma}_2(s)}{s} = \begin{cases} \frac{\tilde{\gamma}(s)}{s} = q(s), & s \in [0,1] \\ 1, & s \geq 1 \end{cases}$$

is nonincreasing since q is nonincreasing on $[0,1]$ and $q(1)=1$. This implies that $\tilde{\gamma}_2$ is subadditive. Indeed, we can write:

$$\tilde{\gamma}_2(x+y) = x\frac{\tilde{\gamma}_2(x+y)}{x+y} + y\frac{\tilde{\gamma}_2(x+y)}{x+y} \leq x\frac{\tilde{\gamma}_2(x)}{x} + y\frac{\tilde{\gamma}_2(y)}{y} = \tilde{\gamma}_2(x) + \tilde{\gamma}_2(y) \ .$$

Hence, $\tilde{\gamma}_2$ is of class $\mathcal{K}_+$, which completes the proof. ■

The following Lemmas 3-6 follows directly from Lemma IV.5 - IV.9 in [5]. Lemma 3 claims that the zoom-out interval is bounded and there exist a bound on x during the zoom-out interval. Lemma 4 claims that there exist a bound on μ at the end of each zoom-out. Lemma 5 claims that there exist a bound on x during zoom-in. And Lemma 6 claims that if zoom-in is followed by zoom-out, then x and μ are bounded by the functions of the disturbance only (initial conditions are forgotten).

Lemma 3: Consider the system (3)-(11). Suppose all conditions of Theorem 1 hold. Then there exist $\varphi_1, \varphi_2, \rho_1, \rho_2 \in \mathcal{K}_\infty$ such that for all $i = 0, 1, \ldots, N, x_{k_{2i}} \in \mathbb{R}^n, \mu_{k_{2i}} > 0, w_i \in \mathbb{R}^l$ the following holds:

$$k_{2i+1} - k_{2i} \leq 1 + \varphi_1(|x_{k_{2i}}|) + \varphi_2\left(\sum_{j=k_{2i}}^{k_{2i+1}-1} |w_j|^2\right),$$

$$\|x_{[k_{2i},k]}\| \leq \rho_1(|x_{k_{2i}}|) + \rho_2\left(\sum_{j=k_{2i}}^{k-1} |w_j|^2\right) \quad \forall k \in [k_{2i}, k_{2i+1}].$$

Lemma 4: Consider the system (3)-(11). Suppose all conditions of Theorem 1 hold. There exist $\mathcal{K}_\infty$ functions ρ, $\bar{\rho}$ and a continuous bounded function $\tilde{\rho}$ such that for all $i = 0, 1, \ldots, N$ and all $\mu_{k_{2i}} > 0$, $x_{k_{2i}} \in \mathbb{R}^n$, $w_i \in \mathbb{R}^l$ the following holds:

$$\mu_k^2 \leq \rho(|x_{k_{2i}}|) + \tilde{\rho}(\mu_{k_{2i}}) + \bar{\rho}\left(\sum_{j=k_{2i}}^{k-1} |w_j|^2\right) \quad \forall k \in [k_{2i}, k_{2i+1}].$$

Lemma 5: Consider the system (3)-(11). Suppose all conditions of Theorem 1 hold. Then there exist continuous nondecreasing functions $\rho_j^{in}(s,\mu) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$, $j = 1, 2$, with $\rho_j^{in}(0,\mu) = 0$ for all μ, such that for all $i = 0, 1, \ldots, N, x_{k_{2i+1}} \in \mathbb{R}^n, \mu_{k_{2i+1}} > 0, w_i \in \mathbb{R}^l$ the following holds:

$$\|x_{[k_{2i+1},k]}\| \leq \rho_1^{in}(|x_{k_{2i+1}}|^2, \mu_{k_{2i+1}}^2) + \rho_2^{in}(\sum_{j=k_{2i+1}}^{k-1} |w_j|^2, \mu_{k_{2i+1}}^2) \ \ \forall k \in [k_{2i+1}, k_{2i+2}].$$

Lemma 6: Consider the system (3)-(11). Suppose all conditions of Theorem 1 hold. There exists $\tilde{\gamma}_1 > 0$ such that for an arbitrary $i = 0, 1, \dots, N$, if $k_{2i+2} < +\infty$, then $i < N - 1$ and the following holds:

$$\max\left\{|x_{k_{2i+2}}|, \mu_{k_{2i+2}}\right\} \leq \tilde{\gamma}_1 \left(\sum_{j=k_{2i+1}}^{k_{2i+2}-1} |w_j|^2\right). \tag{20}$$

Now we combine Lemmas 3-6 to prove the next lemma.

Proof of Lemma 2: From Lemma 3 we have:

$$|x_k|^2 \leq \|x_{[k_0,k]}\| \leq \rho_1(|x_{k_0}|) + \rho_2(\sum_{j=k_0}^{k-1} |w_j|^2) \quad \forall k \in [k_0, k_1]. \tag{21}$$

By Lemma 3 the duration of the zoom-out interval is finite and there exist the time instant $k_1 < \infty$ such that the zoom-in is triggered. Then $[k_1, k_2]$ is a zoom-in interval, k_2 can be infinity, and from Lemma 5 we have:

$$|x_k|^2 \leq \|x_{[k_1,k]}\| \leq \rho_1^{in}(|x_{k_1}|^2, \mu_{k_1}^2) + \rho_2^{in}(\sum_{j=k_1}^{k-1} |w_j|^2, \mu_{k_1}^2) \quad \forall k \in [k_1, k_2]. \tag{22}$$

Substitute (21) with $k = k_1$ for $|x_{k_1}|^2$ in (22) and use triangular inequality[1]:

$$\|x_{[k_1,k]}\|^2 \leq \rho_1^{in}\left(\rho_1(|x_{k_0}|) + \rho_2(\sum_{j=k_0}^{k_1-1} |w_j|^2), \mu_{k_1}^2\right) + \rho_2^{in}\left(\sum_{j=k_1}^{k-1} |w_j|^2, \mu_{k_1}^2\right)$$

$$\leq \rho_1^{in}\left(2\rho_1(|x_{k_0}|), \mu_{k_1}^2\right) + \rho_1^{in}\left(2\rho_2(\sum_{j=k_0}^{k_1-1} |w_j|^2), \mu_{k_1}^2\right) + \rho_2^{in}\left(\sum_{j=k_1}^{k-1} |w_j|^2, \mu_{k_1}^2\right), \forall k \in [k_1, k_2].$$

From Lemma 4 we have: $\mu_{k_1}^2 \leq \rho(|x_{k_0}|) + \tilde{\rho}(\mu_{k_0}) + \bar{\rho}(\sum_{j=k_0}^{k_1-1} |w_j|^2)$. Substitute this for $\mu_{k_1}^2$ in the inequality above. Then for all $k \in [k_1, k_2]$ we have::

$$\begin{aligned}\|x_{[k_1,k]}\|^2 \leq\ & \rho_1^{in}\left(2\rho_1(|x_{k_0}|), \rho(|x_{k_0}|) + \tilde{\rho}(\mu_{k_0}) + \bar{\rho}(\sum_{j=k_0}^{k_1-1} |w_j|^2)\right)\\ &+\rho_1^{in}\left(2\rho_2(\sum_{j=k_0}^{k_1-1} |w_j|^2), \rho(|x_{k_0}|) + \tilde{\rho}(\mu_{k_0}) + \bar{\rho}(\sum_{j=k_0}^{k_1-1} |w_j|^2)\right)\\ &+\rho_2^{in}\left(\sum_{j=k_1}^{k-1} |w_j|^2, \rho(|x_{k_0}|) + \tilde{\rho}(\mu_{k_0}) + \bar{\rho}(\sum_{j=k_0}^{k_1-1} |w_j|^2)\right)\end{aligned}$$

[1]For $\mathcal{K}_\infty$ function ρ_1^{in} the following holds: $\rho_1^{in}(s_1 + s_2, \mu) \leq \rho_1^{in}(2s_1, \mu) + \rho_1^{in}(2s_2, \mu)$

$$\leq \rho_1^{in}(2\rho_1(|x_{k_0}|), 4\rho(|x_{k_0}|)) + \rho_1^{in}(2\rho(|x_{k_0}|), 4\tilde{\rho}(\mu_{k_0})) + \rho_1^{in}(2\rho_1(|x_{k_0}|), 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2))$$

$$+\rho_1^{in}(2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2), 4\rho(|x_{k_0}|))+\rho_1^{in}(2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2), 4\tilde{\rho}(\mu_{k_0}))+\rho_1^{in}(2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2), 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2))$$

$$+\rho_2^{in}(\sum_{j=k_1}^{k-1}|w_j|^2, 4\rho(|x_{k_0}|)) + \rho_2^{in}(\sum_{j=k_1}^{k-1}|w_j|^2, 4\tilde{\rho}(\mu_{k_0})) + \rho_2^{in}(\sum_{j=k_1}^{k-1}|w_j|^2, 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2)). \quad (23)$$

In the next step we use the property of $\mathcal{K}_\infty$ functions[2] applied to the functions ρ_1^{in}, ρ_2^{in} and apply a triangle inequality repeatedly in the inequality above:

$$\|x_{[k_1,k]}\|^2 \leq \rho_1^{in}(2\rho_1(|x_{k_0}|), 4\rho(|x_{k_0}|)) + \rho_1^{in}(2\rho(|x_{k_0}|), 4\tilde{\rho}(\mu_{k_0}))$$

$$+\rho_1^{in}(\phi\circ 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2), 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2)) + \rho_1^{in}(2\rho_1(|x_{k_0}|), \phi^{-1}\circ 2\rho_1(|x_{k_0}|))$$

$$+\rho_1^{in}(\phi\circ 4\rho(|x_{k_0}|), 4\rho(|x_{k_0}|) + \rho_1^{in}(2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2), \phi^{-1}\circ 2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2))$$

$$+\rho_1^{in}(\phi\circ 4\tilde{\rho}(\mu_{k_0}), 4\tilde{\rho}(\mu_{k_0})) + \rho_1^{in}(2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2), \phi^{-1}\circ 2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2))$$

$$+\rho_1^{in}(\phi\circ 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2), 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2)) + \rho_1^{in}(2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2), \phi^{-1}\circ 2\rho_2(\sum_{j=k_0}^{k_1-1}|w_j|^2))$$

$$+\rho_2^{in}(\phi\circ 4\rho(|x_{k_0}|), 4\rho(|x_{k_0}|)) + \rho_2^{in}(\sum_{j=k_1}^{k-1}|w_j|^2), \phi^{-1}(\sum_{j=k_1}^{k-1}|w_j|^2)) + \rho_2^{in}(\sum_{j=k_1}^{k-1}|w_j|^2, 4\tilde{\rho}(\mu_{k_0}))$$

$$+\rho_2^{in}(\phi\circ 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2), 2\bar{\rho}(\sum_{j=k_0}^{k_1-1}|w_j|^2)) + \rho_2^{in}(\sum_{j=k_1}^{k-1}|w_j|^2, \phi^{-1}(\sum_{j=k_1}^{k-1}|w_j|^2)) \quad (24)$$

$$\leq \bar{\gamma}_1(|x_{k_0}|, \mu_{k_0}) + \bar{\gamma}_2(\mu_{k_0}, \sum_{j=k_0}^{k_1-1}|w_j|^2)) + \bar{\gamma}_3(\mu_{k_0}, \sum_{j=k_1}^{k-1}|w_j|^2)\ \forall k\in[k_1,k_2], \quad (25)$$

where $\bar{\gamma}_1$ is the sum of all function in (24) with $(\mu_{k_0}, |x_{k_0}|)$ arguments, $\bar{\gamma}_2$ is the sum of all function in (24) with $(\mu_{k_0}, \sum_{j=k_0}^{k_1-1}|w_j|^2)$ arguments, and $\bar{\gamma}_3$ is the sum of all function in (24)

[2]If $\rho:\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ for any fixed y and for any fixed x is nondecreasing in y, then $\rho(x,y)\leq \rho(\phi(y),y)+\rho(x,\phi^{-1}(x))\quad \forall x,y\geq 0$, where ϕ is an arbitrary $\mathcal{K}_\infty$ function.

with $(\mu_{k_0}, \sum_{j=k_1}^{k-1} |w_j|^2)$ arguments. To have a bound for two intervals $[k_0, k_1]$, $[k_1, k_2]$, we add (21) with $k = k_1$ and (25) with $k = k_2$:

$$\|x_{[k_0,k_2]}\|^2 \leq \rho_1(|x_{k_0}|)+\rho_2(\sum_{j=k_0}^{k_1-1} |w_j|^2)+\bar{\gamma}_1(|x_{k_0}|, \mu_{k_0})+\bar{\gamma}_2(\mu_{k_0}, \sum_{j=k_0}^{k_1-1} |w_j|^2)+\bar{\gamma}_3(\mu_{k_0}, \sum_{j=k_1}^{k_2-1} |w_j|^2).$$

Define $\gamma_1(\mu, s) := \rho_1(s)+\bar{\gamma}_1(\mu, s)$ and $\bar{\varphi}_2(\mu, s) := \max\{\rho_2(s)+\bar{\gamma}_2(\mu, s), \bar{\gamma}_3(\mu, s)\}$. Then we have: $\|x_{[k_0,k_2]}\|^2 \leq \gamma_1(\mu_{k_0}, |x_{k_0}|) + \sum_{l=0}^{1} \bar{\varphi}_2(\mu_{k_0}, \sum_{j=k_l}^{k_{l+1}-1} |w_j|^2)$. If $k_2 = +\infty$ then the proof is complete. If $k_2 \leq +\infty$ then for zoom-out interval $[k_2, k_3]$ from Lemma 3 we have:

$$|x_k|^2 \leq \|x_{[k_2,k]}\| \leq \rho_1(|x_{k_2}|) + \rho_2\left(\sum_{j=k_2}^{k-1} |w_j|^2\right) \quad \forall k \in [k_2, k_3]. \tag{26}$$

Again, by Lemma 3 the duration of zoom-out interval is finite and there exist the time instant $k_3 < \infty$ such that the zoom-in is triggered. For the next zoom-in interval $[k_3, k_4]$ from Lemma 5 we have:

$$|x_k|^2 \leq \|x_{[k_3,k]}\| \leq \rho_1^{in}(|x_{k_3}|^2, \mu_{k_3}^2) + \rho_2^{in}(\sum_{j=k_3}^{k-1} |w_j|^2, \mu_{k_3}^2) \quad \forall k \in [k_3, k_4]. \tag{27}$$

From Lemma 4 for $k = k_3$ (end of second zoom-out interval) we have: $\mu_{k_3}^2 \leq \rho(|x_{k_2}|) + \tilde{\rho}(\mu_{k_2}) + \bar{\rho}\left(\sum_{j=k_2}^{k_3-1} |w_j|^2\right)$. Substitute this for $\mu_{k_3}^2$ and (26) with $k = k_3$ for $|x_{k_3}|^2$ into (27). Similarly to (25) we have for all $k \in [k_3, k_4]$:

$$\|x_{[k_3,k]}\|^2 \leq \bar{\gamma}_1(|x_{k_2}|, \mu_{k_2}) + \bar{\gamma}_2(\mu_{k_2}, \sum_{j=k_2}^{k_3-1} |w_j|^2) + \bar{\gamma}_3(\mu_{k_2}, \sum_{j=k_3}^{k-1} |w_j|^2). \tag{28}$$

To obtain a bound for all four intervals, add the bound for the first two intervals $[k_0, k_1]$, $[k_1, k_2]$ (25) with $k = k_2$, the bound for $[k_2, k_3]$ (26) with $k = k_3$ and the bound for $[k_3, k_4]$ (28) with $k = k_4$:

$$\begin{aligned}\|x_{[k_0,k_4]}\|^2 \leq\ & \bar{\gamma}_1(|x_{k_0}|, \mu_{k_0})+\bar{\gamma}_2(\mu_{k_0}, \sum_{j=k_0}^{k_1-1} |w_j|^2)+\bar{\gamma}_3(\mu_{k_0}, \sum_{j=k_1}^{k_2-1} |w_j|^2)+\rho_1(|x_{k_2}|)+\rho_2(\sum_{j=k_2}^{k_3-1} |w_j|^2) \\ & +\bar{\gamma}_1(|x_{k_2}|, \mu_{k_2}) + \bar{\gamma}_2(\mu_{k_2}, \sum_{j=k_2}^{k_3-1} |w_j|^2) + \bar{\gamma}_3(\mu_{k_2}, \sum_{j=k_3}^{k_4-1} |w_j|^2).\end{aligned} \tag{29}$$

From Lemma 6 we have: $\max\{|x_{k_2}|, \mu_{k_2}\} \leq \tilde{\gamma}_1\left(\sum_{j=k_1}^{k_2-1} |w_j|^2\right)$. We can see, that after the second zoom-in (which is followed by zoom-out), we forget about initial conditions x_{k_0}. The state depends only on the disturbance during this zoom-in interval. Substitute this inequality for $|x_{k_2}|$ and μ_{k_2} in (27). Then,

$$\|x_{[k_0,k_4]}\|^2 \leq \bar{\gamma}_1(|x_{k_0}|, \mu_{k_0}) + \bar{\gamma}_2(\mu_{k_0}, \sum_{j=k_0}^{k_1-1} |w_j|^2) + \bar{\gamma}_3(\mu_{k_0}, \sum_{j=k_1}^{k_2-1} |w_j|^2) + \rho_1 \circ \tilde{\gamma}_1(\sum_{j=k_1}^{k_2-1} |w_j|^2)$$

$$+\rho_2(\sum_{j=k_2}^{k_3-1}|w_j|^2)+\bar{\gamma}_1(\tilde{\gamma}_1(\sum_{j=k_1}^{k_2-1}|w_j|^2),\tilde{\gamma}_1(\sum_{j=k_1}^{k_2-1}|w_j|^2)+\bar{\gamma}_2(\tilde{\gamma}_1(\sum_{j=k_1}^{k_2-1}|w_j|^2),\sum_{j=k_2}^{k_3-1}|w_j|^2)$$

$$+\bar{\gamma}_3(\tilde{\gamma}_1(\sum_{j=k_1}^{k_2-1}|w_j|^2),\sum_{j=k_3}^{k_4-1}|w_j|^2) \tag{30}$$

$$\leq \bar{\gamma}_1(|x_{k_0}|,\mu_{k_0})+\bar{\gamma}_2(\mu_{k_0},\sum_{j=k_0}^{k_1-1}|w_j|^2)+\bar{\gamma}_3(\mu_{k_0},\sum_{j=k_1}^{k_2-1}|w_j|^2)+\bar{\gamma}_4(\mu_{k_0},\sum_{j=k_2}^{k_3-1}|w_j|^2)+\bar{\gamma}_5(\mu_{k_0},\sum_{j=k_3}^{k_4-1}|w_j|^2), \tag{31}$$

where $\bar{\gamma}_3$ is a sum of all function in (30) with $(\mu_{k_0},\sum_{j=k_1}^{k_2-1}|w_j|^2)$ arguments, $\bar{\gamma}_4$ is a sum of all function in (30) with $(\mu_{k_0},\sum_{j=k_2}^{k_3-1}|w_j|^2)$ arguments and $\bar{\gamma}_5$ is a sum of all function in (30) with $(\mu_{k_0},\sum_{j=k_3}^{k_4-1}|w_j|^2)$ arguments. Define φ_2 as the following:

$$\varphi_2(\mu,s):=\max\{\bar{\gamma}_2(\mu,s),\bar{\gamma}_3(\mu,s),\bar{\gamma}_4(\mu,s),\bar{\gamma}_5(\mu,s)\}. \tag{32}$$

Then we have: $\|x_{[k_0,k_4]}\|^2\leq\gamma_1(\mu_{k_0},|x_{k_0}|)+\sum_{l=0}^{3}\varphi_2(\mu_{k_0},\sum_{j=k_l}^{k_{l+1}-1}|w_j|^2)$. Note, that $\bar{\varphi}_2(\mu,s)\leq\varphi_2(\mu,s)$. If $k_4=+\infty$ then the proof is complete. If $k_4\leq+\infty$ then for N intervals (N may be ∞) (31) will be modified into the following:

$$\|x_{[k_0,k_N]}\|^2\leq\bar{\gamma}_1(|x_{k_0}|,\mu_{k_0})+\bar{\gamma}_2(\mu_{k_0},\sum_{j=k_0}^{k_1-1}|w_j|^2)+\bar{\gamma}_3(\mu_{k_0},\sum_{j=k_1}^{k_2-1}|w_j|^2)+\bar{\gamma}_4(\mu_{k_0},\sum_{j=k_2}^{k_3-1}|w_j|^2)$$

$$+\bar{\gamma}_5(\mu_{k_0},\sum_{j=k_3}^{k_4-1}|w_j|^2)+\cdots+\bar{\gamma}_4(\mu_{k_0},\sum_{j=k_{N-2}}^{k_{N-1}-1}|w_j|^2)+\bar{\gamma}_5(\mu_{k_0},\sum_{j=k_{N-1}}^{k_N-1}|w_j|^2)$$

For N intervals define φ_2 according to (32). Then we have: $\|x_{[k_0,k_N]}\|^2\leq\gamma_1(|x_{k_0}|,\mu_{k_0})+\sum_{l=0}^{N-1}\varphi_2(\mu_{k_0},\sum_{j=k_l}^{k_{l+1}-1}|w_j|^2)$. This completes the proof. ■