Input-to-State Stability of Networked Control Systems ⋆

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Abstract

A new class of Lyapunov uniformly globally asymptotically stable (UGAS) protocols in networked control systems (NCS) is considered. It is shown that if the controller is designed without taking into account the network so that it yields input-to-state stability (ISS) with respect to external disturbances (not necessarily with respect to the error that will come from the network implementation), then the same controller will achieve semi-global practical ISS for the NCS when implemented via the network with a Lyapunov UGAS protocol. Moreover, the ISS gain is preserved. The adjustable parameter with respect to which semi-global practical ISS is achieved is the maximal allowable transfer interval (MATI) between transmission times.

Key words: Disturbances, Networked Control Systems, Nonlinear, Stability.

1 Introduction

In networked control systems (NCS), one or more dynamical systems are controlled by feedback over a communication network. The transmission capacity of the communication network is limited. This limits the number of bits or packets per second which can be transported via the network and, consequently, restricts the achievable performance. This area has grown rapidly in the last few years with the emergence of applications ranging from microelectromechanical chips and Internet congestion protocols to "drive-by-wire" systems.

NCS are currently receiving considerable attention in the literature as illustrated by recent articles [16,17,11,19] and references listed therein. The area of NCS is still in its infancy and existing results can be improved in at least two directions. First, most existing literature considers only stabilization of *linear* NCS whereas *nonlinear* NCS have received little attention (with few exceptions, such as [11,16]). Second, most results treat NCS without disturbances and we are aware only of limited results on stability of NCS with disturbances, such as the $\mathcal{L}_\infty$ to root-mean-square stability of a class of NCS considered in [5]; $\mathcal{L}_p$ stability of NCS considered in [11]; results on input-output stability of linear jump parameter systems in [3] that can be exploited for certain NCS with static protocols. Also, in some cases it is possible to use tools for linear sampled-data systems [4] for analysis and design of certain classes of linear NCS. In this paper we consider input-to-state stability (ISS) of *nonlinear NCS with disturbances.*

We follow the method proposed in [16,17], in which one first designs the controller without taking into account the network and then in the second step one determines a design parameter called the maximum allowable transfer interval (MATI) so that the closed loop remains stable when some control and sensor signals are transmitted via the network. This approach was shown to produce stabilizing controllers for linear NCS in [16] and nonlinear NCS in [17].

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Moreover, $\mathcal{L}_p$ stability of nonlinear systems with a large class of uniformly globally exponentially stable (UGES) protocols was investigated in [11]. It was shown in [11] that several common static and dynamic protocols investigated in [16,17] belong to the class of UGES protocols.

We consider Lyapunov uniformly globally asymptotically stable (UGAS) protocols that generalize uniformly globally exponentially stable (UGES) protocols considered in [11]. We show that if the controller is designed without taking into account the network so that it yields input-to-state stability (ISS) of the closed loop system (see [15]), then the same controller will achieve semi-global practical ISS of NCS when implemented via the network with a Lyapunov UGAS protocol. The parameter that can be adjusted in the protocol and that is used to achieve semi-global practical ISS is MATI (see [16,17]).

2 Preliminaries

$\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the sets of real and natural numbers. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative reals. Given $t \in \mathbb{R}$ and a piecewise continuous function $f : \mathbb{R} \to \mathbb{R}^n$, we use the notation $f(t^+) := \lim_{s\to t, s>t} f(s)$. All vector norms, denoted as $|\cdot|$, are Euclidean norms unless otherwise stated. Given a measurable, locally essentially bounded signal $\varphi : [t_\circ, \infty) \to \mathbb{R}^n$ we denote its $\mathcal{L}_\infty$ norm as follows: $\|\varphi\|_\infty := \operatorname{ess\,sup}_{s\geq t_\circ} |\varphi(s)|$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{G}$ if it is continuous, zero at zero and nondecreasing. It is of class $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing. A function is $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. γ is of class $\mathcal{L}$ if it is continuous and decreasing to zero. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if for each $s > 0$ the function $\beta(s, \cdot)$ is of class $\mathcal{L}$ and for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$. In a similar way we define functions of class $\mathcal{KK}$ and $\mathcal{KLL}$. To shorten notation we often use $(x, y) := (x^T\ y^T)^T$.

3 Definition of Networked Control Systems

We consider general nonlinear NCS with disturbances of the following form (see also [17,11]):

$$\begin{aligned}
\dot{x}_P &= f_P(t, x_P, \hat{u}, w),\ t \in [t_{s_{i-1}}, t_{s_i}] \\
y &= g_P(t, x_P) \\
\dot{x}_C &= f_C(t, x_C, \hat{y}, w),\ t \in [t_{s_{i-1}}, t_{s_i}] \\
u &= g_C(t, x_C) \\
\dot{\hat{y}} &= \hat{f}_P(t, x_P, x_C, \hat{y}, \hat{u}, w),\ t \in [t_{s_{i-1}}, t_{s_i}] \\
\dot{\hat{u}} &= \hat{f}_C(t, x_P, x_C, \hat{y}, \hat{u}, w),\ t \in [t_{s_{i-1}}, t_{s_i}] \\
\hat{y}(t_{s_i}^+) &= \hat{y}(t_{s_i}) + h_u(i, e(t_{s_i})) \\
\hat{u}(t_{s_i}^+) &= \hat{u}(t_{s_i}) + h_y(i, e(t_{s_i}))
\end{aligned} \tag{1}$$

where the sequence $t_{s_i}, i \in \mathbb{N}$ of monotonically increasing transmission times satisfy $\epsilon \leq t_{s_{i+1}} - t_{s_i} \leq \tau$ for all $i \in \mathbb{N}$ and some fixed $\epsilon, \tau > 0$. We adopt terminology from [17] and refer to τ as the *maximum allowable transmission interval* (MATI). The number ϵ ensures that our model does not have any Zeno solutions where infinitely fast switching may occur. x_P and x_C are respectively states of the plant and the controller; y is the plant output and u is the controller output; w is an exogenous disturbance input; $\hat{y}$ and $\hat{u}$ are the vectors of most recently transmitted plant and controller output values via the network; e is the network induced error defined as $e(t) := (\hat{y}(t) - y(t), \hat{u}(t) - u(t)) = (e_y, e_u)$. Note that if NCS has ℓ links, then the error vector can be partitioned as follows $e = [e_1^T\ e_2^T\ \ldots\ e_\ell^T]^T$. At each transmission time t_{s_j}, the protocol gives access to the network to one of the "nodes" e_i, $i \in \{1, 2, \ldots, \ell\}$ and this causes the vector $e_i(\cdot)$ to undergo a "jump" at t_{s_j}. We combine the controller and plant states into a vector $x := (x_P, x_C)$ and using the error vector defined earlier $e = (e_y, e_u)$ and the following definitions: $f(t, x, e, w) := (f_P(t, x_P, g_C(t, x_C) + e_u, w), f_C(t, x_C, g_P(t, x_P) + e_y, w))$; $h(i, e) := (h_y(i, e), h_u(i, e))$; $g(t, x, e, w) := (g_1, g_2)$, where $g_1 := \hat{f}_P(t, x_P, x_C, g_P(t, x_P) + e_y, g_C(t, x_C) + e_u, w) - \frac{\partial g_P}{\partial t}(t, x_P) - \frac{\partial g_P}{\partial x_P}(t, x_P) f_P(t, x_P, g_C(t, x_C) + e_u, w)$, $g_2 := \hat{f}_C(t, x_P, x_C, g_P(t, x_P) + e_y, g_C(t, x_C) + e_u, w) - \frac{\partial g_C}{\partial t}(t, x_C) - \frac{\partial g_C}{\partial x_C}(t, x_C) f_C(t, x_C, g_P(t, x_P) + e_y, w)$. We can rewrite (1) as a system with jumps that is more amenable for analysis:

$$\dot{x} = f(t, x, e, w) \qquad \forall t \in [t_{s_{i-1}}, t_{s_i}] \tag{2}$$

$$\dot{e} = g(t, x, e, w) \qquad \forall t \in [t_{s_{i-1}}, t_{s_i}] \tag{3}$$

$$e(t_{s_i}^+) = h(i, e(t_{s_i}))\ , \tag{4}$$

where $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $w \in \mathbb{R}^{n_w}$. In order to write (3), we assumed that functions g_P and g_C in (1) are continuously differentiable. The explanation of how trajectories of the system (2), (3), (4) are defined can be found in [11]. We use the following assumption that holds if f is locally Lipschitz in x, w and e, uniformly in t.

Assumption 1 *There exist $L \in \mathcal{K}$ and $M \in \mathcal{KK}$ such that, for each $c > 0$,* $\max\{|x|, |\bar{x}|, |w|, |e|\} \leq c$ *implies* $|f(t, x, w, e) - f(t, \bar{x}, w, 0)| \leq L(c + 1)|x - \bar{x}| + M(c + 1, |e|)$. ■

We refer to (4) as a protocol. The protocol determines the algorithm that assigns access to the network to different nodes in the system. It was shown in [11] that static protocols and the so called try-once-discard (TOD) dynamic protocol introduced in [16,17] can be modelled in this manner. The functions h_u and h_y are typically such that, if the jth link gets access to the network at some transmission time t_{s_i} we have that the corresponding part of the error vector has a jump. For some protocols, such as the TOD protocol, we typically assume that e_j is reset to zero at time $t_{s_i}^+$, that is $e_j(t_{s_i}^+) = 0$. However, we emphasize that this assumption is not needed in general (see [11] for more details).

Remark 1 *It is typically assumed in the literature (see, for instance [16,17,19]) that $\hat{f}_P = 0$ and $\hat{f}_C = 0$ in (1) but we state our results for more general forms of these functions. This more general model was considered for the first time in [11].*

4 Lyapunov UGAS protocols

In this section we introduce a class of Lyapunov UGAS protocols that generalize Lyapunov UGES protocols introduced in [11]. We show an important property that Lyapunov UGAS protocols possess under relatively weak conditions. In particular, we show for these protocols that for uniformly bounded plant state $x(\cdot)$ and the disturbance $w(\cdot)$, the state of the error dynamics $e(\cdot)$ satisfies a semi-global-practical stability bound in the MATI. This technical result is instrumental in establishing ISS properties of the NCS in the next section. Note that the equation (4) that describes the operation of the protocol is not a discrete-time system since this equation does not provide a relationship between error signals at consecutive transmission times t_{s_i} and $t_{s_{i+1}}$ for any $i \in \mathbb{N}$. However, we find it convenient to introduce an auxiliary discrete-time system induced by the protocol (4):

$$e(i+1) = h(i, e(i)) \ . \tag{5}$$

This idea was proposed for the first time in [11]. Moreover, in [11] we introduced the class of Lyapunov uniformly globally exponentially stable (UGES) protocols. It was shown in [11] how one can model token ring and the Try-Once-Discard Protocol from [17] using (5) and also that they are Lyapunov UGES. In this paper we generalize this class of protocols and we consider Lyapunov UGAS protocols [1] :

[1] The difference between Lyapunov UGAS protocols defined here and Lyapunov UGES protocols defined in [11] is that the functions α_1 and α_2 were required to be linear for Lyapunov UGES protocols.

Definition 1 *Let a function $W : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a real number $\lambda \in [0, 1)$ be such that for all $e \in \mathbb{R}^n$ and all $i \in \mathbb{N}$ the following holds:*

$$\alpha_1(|e|) \leq W(i, e) \leq \alpha_2(|e|) \tag{6}$$

$$W(i+1, h(i, e)) \leq \lambda W(i, e) \ . \tag{7}$$

Then, we say that the protocol (4) is Lyapunov UGAS with (W, α_1, α_2, λ). ■

Assumption 2 *The protocol (4) is Lyapunov UGAS with (W, α_1, α_2, λ), where $W(i, e)$ is continuous in e, uniformly in i.* ■

We note that results of [11] required that W is Lipschitz in e, uniformly in i. Hence, in this paper besides considering a more general class of Lyapunov UGAS protocols, we also relax the uniform Lipschitz property of W to uniform continuity. It is sometimes easier to prove (see Example 1) that instead of (7) we have that the following inequality holds:

$$W(i+1, h(i, e)) \leq W(i, e) - \alpha_3(W(i, e)) \ , \tag{8}$$

for some positive definite α_3. The following proposition shows that this is enough for our purposes.

Proposition 1 *Suppose that (6) and (8) hold, where α_3 is a continuous, positive definite function and $W(i, e)$ is continuous in e, uniformly in i. Then, there exists a smooth function $\rho \in \mathcal{K}_\infty$ such that $U(i, e) := \rho(W(i, e))$ satisfies all conditions of Assumption 2.* ■

Sketch of proof: Lyapunov UGAS with U can be shown to hold in a similar way as in the continuous-time literature (For example, see [14] and [8, Theorem 3.6.10] for the case when α_3 is a class $\mathcal{K}$ function. Also, the result that uses a similar transformation to go from a positive definite function α_3 to a class $\mathcal{K}$ function $\hat{\alpha}_3$ is given in [15, pp. 440]). Moreover, given any $\rho \in \mathcal{K}_\infty$ (which is by definition continuous) we have that $U(i, e)$ is continuous in e, uniformly in i, since W has the same property. ■

In some cases it is possible that (5) is Lyapunov UGAS in an appropriate sense but it may be hard to explicitly construct W satisfying Assumption 2 (see Example 2). The following propositions is useful in such situations and it makes use of converse Lyapunov theorems proved in [7] for difference inclusions with upper semi-continuous right hand sides (see also [2] for similar results for time-invariant systems with Lipschitz right hand sides).

Proposition 2 *Suppose that*[2] *for each* $e \in \mathbb{R}^{n_e}$ *the function* $h(\cdot, e)$ *is periodic in* i. *Then, there exists* W *satisfying Assumption 2 if and only if the origin of the difference inclusion* $e^+ \in H(i,e)$, *where* $H(i,e) := cl \bigcap_{|v|\le\delta,\delta>0}\{z : z \in h(i, e+v)\}$, *is stable and globally attractive.*

It was shown in [11] that token ring and try once discard (TOD) protocols are Lyapunov UGES (see [11,17]). We present next two examples of Lyapunov UGAS protocols that are not Lyapunov UGES. The first example behaves for large e in the same way as TOD protocol but for small e the error jumps are smaller because we transmit less information. The second example is a modified token ring protocol that for large error e behaves exactly in the same way as token ring but for small e it transmits less frequently.

Example 1 (Modified TOD Protocol) *Consider the protocol (5), where* $h(e) = (I - \Psi(e))e$ *and* $\Psi(e) := diag\{\psi_1(e)I_{n_1}, \ldots, \psi_\ell(e)I_{n_\ell}\}$, *where* $\psi_j(e) = sat(|e_j|)$ *if* $j = \min(arg\max_j |e_j|)$ *and* $\psi_j(e) = 0$ *otherwise. This protocol behaves like TOD for large* $|e|$ *and for small* $|e|$ *it makes the error jumps smaller (e.g. because it is transmitting less information). Using* $W(e) = |e|$, *which is continuous, we can show that the inequality (8) holds and via Proposition 1 we conclude that there exists* $U(e) := \rho(W)$ *for some* $\rho \in \mathcal{K}_\infty$ *such that the protocol satisfies Assumption 2 with* U *and some* $\alpha_1, \alpha_2, \lambda$. *The protocol is not UGES since convergence is slower for smaller* e.

Example 2 (Modified Token Ring Protocol) *Define for* $x \in \mathbb{R}_{\ge 0}$ *the following function* $\lfloor x \rfloor = \min\{z : x \le z, z \in \mathbb{N}\}$. *Also, let* $sat(s) := \min\{s, 1\}$ *for all* $s \ge 0$. *Consider the protocol (5) where* $h(i,e) = (I - \Delta(i,e))e$ *and* $\Delta(i,e) = diag\{\delta_1(i,e)I_{n_1}, \ldots, \delta_\ell(i,e)I_{n_\ell}\}$, $\sum_{i=1}^{\ell} n_i = n_e$ *and*

$$\delta_k(i,e) = \begin{cases} 1 & \text{if } |e| > 0, i = \left\lfloor \frac{1}{sat(|e|)} \right\rfloor (k + j\ell), j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

This protocol behaves exactly like the token ring for $|e| > \frac{1}{2}$ *and for small* $|e|$ *it transmits less frequently (e.g. for* $|e| \in \left(\frac{1}{11}, \frac{1}{10}\right]$ *the protocol transmits at a frequency that is* 10 *times smaller than that of the token ring protocol).* $\Delta(\cdot,\cdot)$ *is positive semi-definite, it has a norm less than* 1 *and for every* $\delta > 0$ *there exist* $L := \left\lfloor \frac{1}{sat(\delta)} \right\rfloor \cdot \ell$ *such that for all* $k_0 \in \mathbb{N}$ *we have*

$$|e| \ge \delta \implies \sum_{i=k_0}^{k_0+L} \Delta(i,e) \ge I \; . \tag{9}$$

Stability of the corresponding difference inclusion[3] *follows immediately using the Lyapunov function* $|e|$. *Global attractivity can be established using the uniform* δ*-PE concept in [9] and [10] (see [12] for related tools in discrete-time).*

[2] In this proposition we use the usual definitions of stability and global attractivity for the origin of a time varying system.

Note that we often abuse the terminology and refer either to (4) or (5) as the protocol. For instance, in the above definition we say that (4) is Lyapunov UGAS with $(W, \alpha_1, \alpha_2, \lambda)$ when this data can be used to show UGAS of (5). Our results are stated for arbitrary Lyapunov UGAS protocol in the sense of Definition 1. All proofs are given in the Appendix.

Proposition 3 *Suppose that: (1)* $W(i,e)$ *is continuous in* e, *uniformly in* i; *(2)* $g(t,x,e,w)$ *is bounded on compact sets, uniformly in* t. *Then, there exists* $\tau_1^* \in \mathcal{KL}$ *such that for each pair of strictly positive real numbers* (ε, c) *the following holds: if* $[t_a, t_b] \subseteq [t_{s_k}, t_{s_{k+1}}] \subseteq [t_{s_k}, t_{s_k} + \tau_1^*(\varepsilon, c)]$ *and* $\max\{||x||_\infty, ||w||_\infty, W(k, e(t_a))\} \le c$, *then the following holds:*

$$W(k, e(t_b)) \le W(k, e(t_a)) + \varepsilon \; . \tag{10}$$

The main result of this section is presented next. It states that any protocol satisfying Assumption 2 yields semi-global practical uniform asymptotic stability (in the MATI) of the error dynamics (3). Note that this stability property is uniform with respect to initial times $t_\circ$, as well as disturbances w.

Theorem 1 *Let* $W : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* $\lambda \in [0,1)$ *be given. Suppose that the following holds for system (2)-(4):*
(1) Assumption 2 holds;
(2) $g(t,x,e,w)$ *is bounded on compact sets, uniformly in* t.
Then, there exist $\beta_e \in \mathcal{KL}$, $\gamma_e \in \mathcal{K}_\infty$ *and* $\tau^* \in \mathcal{KL}$ *such that, for each pair of strictly positive real numbers* (ε, c) *the following holds: if* $\tau \le \tau^*(\varepsilon, c)$ *and* $\max\{||x||_\infty, ||w||_\infty, |e(t_\circ)|\} \le c$, *then the following holds*

$$|e(t)| \le \max\left\{\beta_e\left(|e(t_\circ)|, \frac{t - t_\circ}{\tau}\right), \gamma_e(\varepsilon)\right\} \; , \tag{11}$$

[3] Note that in this example $H(i,e)$ defined in Proposition 2 is set valued at points where $1/|e|$ is an integer.

for all $t \geq t_\circ \geq 0$. ■

5 Main Result

Our main result (Theorem 2) is stated in this section. The result states that under appropriate conditions any controller that achieves ISS of the closed loop system in the absence of a network will also achieve semi-global-practical ISS of NCS in the MATI. The result is true for any Lyapunov UGAS protocol in the sense of Definition 1. In particular, we use the properties of the following auxiliary system:

$$\dot{\bar{x}} = f(t, \bar{x}, w, 0) \ , \tag{12}$$

which is the model of the closed loop system when there is no network (i.e. $e(\cdot) \equiv 0$).

Theorem 2 *Suppose that:*
(1) Assumption 1 holds.
(2) All conditions of Theorem 1 hold.
(3) There exist $\beta \in \mathcal{KL}$ *(continuous) and* $\gamma \in \mathcal{G}$ *such that, for each* $t_\circ \geq 0$, *the solutions of (12) satisfy*

$$|\bar{x}(t)| \leq \max\{\beta(|\bar{x}(t_\circ)|, t - t_\circ), \gamma(||w||_\infty)\} \tag{13}$$

for all $t \geq t_\circ \geq 0$.
Then, there exist $\beta_e, \tau^* \in \mathcal{KL}$ *such that, for each pair of strictly positive numbers* (ε, c) *and each* $t_\circ \geq 0$, *the following holds: if* $\tau \leq \tau^*(\varepsilon, c)$ *and* $\max\{|x(t_\circ)|, ||w||_\infty, |e(t_\circ)|\} \leq c$

$$|x(t)| \leq \max\{\beta(|x(t_\circ)|, t - t_\circ), \gamma(||w||_\infty)\} + \varepsilon$$
$$|e(t)| \leq \max\left\{\beta_e\left(|e(t_\circ)|, \frac{t - t_\circ}{\tau}\right), \varepsilon\right\} \ \forall t \geq t_\circ \geq 0$$

■

Note that the result of Theorem 2 is meant to be qualitative and it shows that the ISS gain is preserved semi-globally practically for sufficiently small MATI. It would be important in future work to reveal more quantitatively useful formulas that indicate how the ISS gain degrades as MATI increases.

Remark 2 *Theorem 2 suggests that the ISS controller design can be carried in two steps. In the first step the control designer ignores the network and designs the controller to achieve ISS of the closed loop system. In the second step the control designer needs to choose sufficiently small MATI that will achieve appropriate ISS stability bounds on an appropriate bounded set of initial states and disturbances.* ■

The proof technique that we use to prove Theorem 2 is similar to the one exploited in [13]. The proof makes use of ISS of the auxiliary system (12) to show that we can achieve semi-global practical ISS in MATI of the NCS (2)-(4). The main technical step in establishing this result is presented below and its proof is presented in the appendix. This result states that the solutions of the auxiliary system (12) and the actual NCS (2)-(4) can be made arbitrarily close on arbitrarily long time intervals if the MATI is chosen sufficiently small.

Lemma 1 *Consider system (2)-(4) and suppose that all conditions of Theorem 2 hold. Then, there exists* $\tau^* \in \mathcal{KLL}$ *such that, for each strictly positive triple* (ρ, T, c), *each* $t_\circ \geq 0$ *and each* $|x(t_\circ)| \leq c$, *there exists* $\bar{x}(t_\circ) \in \mathbb{R}^n$ *such that the following holds: if* $\tau \leq \tau^*(\rho, T, c)$ *and* $\max\{||x||_\infty, ||w||_\infty, ||e||_\infty\} \leq c$, *then*

$$|x(t) - \bar{x}(t)| \leq \rho \quad \forall t \in [t_\circ, t_\circ + T] \ . \tag{14}$$

Remark 3 *The conclusion of Lemma 1 may hold when Assumption 1 is weakened. For example, in the time invariant case, continuity of* f *is sufficient.* ■

The next proposition follows directly using the proof of [13, Theorem 1] and its proof is omitted. This proposition establishes under conditions of Lemma 1 that an ISS stability bound holds for (2).

Proposition 4 *Under the conclusion of Lemma 1 there exists* $\tau^* \in \mathcal{KL}$ *such that, for each pair of strictly positive numbers* (ε, c) *and each* $t_\circ \geq 0$, *the following holds: if* $\tau \leq \tau^*(\varepsilon, c)$ *and* $\max\{|x(t_\circ)|, ||w||_\infty, ||e||_\infty\} \leq c$, *then*

$$|x(t)| \leq \max\{\beta(|x(t_\circ)|, t - t_\circ), \gamma(||w||_\infty)\} + \varepsilon,$$

for all $t \geq t_\circ \geq 0$. ■

Sketch of proof of Theorem 2: The proof of the main result follows by combining Lemma 1, Theorem 1 and Proposition 4 and using causality to remove the assumptions on $||x||_\infty$ and $||e||_\infty$. This proof technique very similar to the proof of the ISS small gain theorem in [6] and for space reasons it is omitted. ■

References

[1] D. Angeli, E.D. Sontag and Y. Wang. A characterization of integral input-to-state stability. *IEEE Transactions on Automatic Control*, vol. 45, no. 6, June 2000, pp. 1082–1097.

[2] P. Diamond and P. Kloeden, “Spacial discretization of mappings”, Computers Math. Applic. Vol. 25, No. 6, pp. 85-94, 1993.

[3] G.E. Dullerud and S. Lall, “A synchronous hybrid systems with jumps analysis: analysis and synthesis problems”, *Syst. Contr. Lett.*, vol. 37 (1999), pp. 61–89.

[4] G.E. Dullerud and S. Lall, “A new approach for analysis and synthesis of time varying systems”, *IEEE Trans. Automat. Contr.*, vol. 44 (1999), pp. 1486–1497.

[5] A.Hassibi, S.P.Boyd and J.P.How, “Control of asynchronous dynamical systems with rate constraints on events”, Proc. *IEEE Conf. Decis. Contr.*, Phoenix, AZ, USA, (1999), pp. 1345–1351.

[6] Z. P. Jiang, A. R. Teel and L. Praly, “Small gain theorem for ISS systems and applications”, *Math. Contr. Sign. Syst.*, 7 (1994), pp. 95–120.

[7] C. M. Kellett and A. R.Teel, “Results on Converse Lyapunov Theorems for Difference Inclusions”, IEEE Conference on Decision and Control 2003, December 9-12, 2003, Maui, Hawaii, USA.

[8] V. Lakshmikantham and S. Leela, *Differential and integral inequalities: theory and applications*, vol. 1, Academic Press, New York, 1969.

[9] A. Loria E. Panteley, D. Popović, and A. R. Teel, “δ-persistency of excitation: a necessary and sufficient condition for uniform attractivity,” in Proc. 40th. IEEE Conf. Decision Contr., Las Vegas, CA, USA, (2002) Paper no. REG0623.

[10] E. Panteley, A. Loria and A. Teel, “Relaxed persistency of excitation for uniform asymptotic stability,” IEEE Trans. on Automat. Contr., vol. 46, no. 12, pp. 1874-1886, 2001.

[11] D. Nešić and A. R. Teel, “On $\mathcal{L}_p$ stability of networked control systems”, to appear in *IEEE Trans. Automat. Contr.*, 2004.

[12] D. Nešić and A. R. Teel, ”Matrosov theorem for parameterized families of discrete-time systems”, *Automatica*, vol. 40, No. 6 (2004), pp. 1025–1034.

[13] A.R. Teel, L. Moreau, D. Nešić. “A unified framework for input-to-state stability in systems with two time scales”, *IEEE Transactions on Automatic Control*, vol. 48 , No. 9 (2003), pp. 1526 - 1544.

[14] L. Praly and Y. Wang,“Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability”, *Math. Contr. Sign. Syst.*, 1996, 9:1-33.

[15] E. D. Sontag, “Smooth stabilization implies coprime factorization”, *IEEE Trans. Aut. Contr.*, Vol. 34, No. 4, April 1989, pp. 435–443.

[16] G. C. Walsh, O. Beldiman and L. G. Bushnell, “Asymptotic behaviour of nonlinear networked control systems”, *IEEE Trans. Automat. Contr.*, vol. 46, No. 7 (2001), 1093–1097.

[17] G. C. Walsh, H. Ye and L. G. Bushnell, “Stability analysis of networked control systems”, *IEEE Trans. Contr. Syst. Techn.*, vol. 10, No. 3 (2002), pp. 438–446.

[18] L. Zaccarian, A.R. Teel and D. Nešić, “On finite gain L_p stability of sampled-data nonlinear systems”, *Syst. Contr. Lett.*, vol. 49, No. 3 (2003), pp. 201–212.

[19] W. Zhang and M.S. Branicky, “Stability of networked control systems with time-varying transmission period”, Allerton Conf. Communication, Control, and Computing, Urbana, IL, Oct. 2001.

6 Appendix

Proof of Proposition 3: We will prove the result only for $\varepsilon \in (0,1]$ since the statement on Proposition 3 follows directly from this result. From the item 1, it follows that there exists a function $\widetilde{\varepsilon} \in \mathcal{K}_\infty$ such that $W(i,e_1) - W(i,e_2) \le \widetilde{\varepsilon}(|e_1 - e_2|)$, for all $e_1, e_2 \in \mathbb{R}^{n_e}$, $i \in \mathbb{N}$. Moreover, the item 2 implies that there exists a continuous, increasing positive function φ (not necessarily zero at zero) such that $|g(t,x,e,w)| \le \varphi(\max\{|x|,|e|,|w|\})$, for all $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $w \in \mathbb{R}^{n_w}$, $t \ge 0$. Define now $\tau_1^*(\varepsilon,c) := \frac{\widetilde{\varepsilon}^{-1}(\varepsilon)}{1+\varphi(\max\{c+1,\alpha_1^{-1}(c+1)\})}$, which is obviously a class $\mathcal{KL}$ function. Suppose for the purpose of showing contradiction that there exist $\hat{\varepsilon} \in (0,1]$, $\hat{c} > 0$ k, t_a, t_b, $\hat{x}(\cdot)$, $\hat{e}(\cdot)$ and $\hat{w}(\cdot)$ such that: $[t_a,t_b] \subseteq [t_{s_k}, t_{s_{k+1}}] \subseteq [t_{s_k}, t_{s_k} + \tau_1^*(\hat{\varepsilon},\hat{c})]$; $\max\{||\hat{x}||_\infty, ||\hat{w}||_\infty, W(k,\hat{e}(t_a))\} \le \hat{c}$; and $W(k,\hat{e}(t_b)) > W(k,\hat{e}(t_a)) + \hat{\varepsilon}$. Then, from the continuity of solutions of (3) and the uniform continuity of W it follows that there exists $\hat{t} \in [t_a,t_b]$ such that $\max\{||\hat{x}||_\infty, ||\hat{w}||_\infty, W(k,\hat{e}(t_a))\} \le \hat{c}$ imply

$$W(k,\hat{e}(t)) < W(k,\hat{e}(t_a)) + \hat{\varepsilon} \qquad \forall t \in [t_a,\hat{t}) \tag{15}$$

$$W(k,\hat{e}(\hat{t})) = W(k,\hat{e}(t_a)) + \hat{\varepsilon}\ . \tag{16}$$

But then we can write for $W(k,\hat{e}(t_a)) \le \hat{c}$ that $W(k,\hat{e}(t)) < W(k,\hat{e}(t_a)) + \hat{\varepsilon} \le \hat{c} + 1$, $\forall t \in [t_a,\hat{t})$, which implies that $|\hat{e}(t)| < \alpha_1^{-1}(\hat{c}+1), \forall t \in [t_a,\hat{t})$ and since also $\max\{||\hat{x}||_\infty, ||\hat{w}||_\infty\} < \hat{c}+1$, then $|g(t,\hat{x}(t),\hat{e}(t),\hat{w}(t))| < \varphi(\max\{\hat{c}+1, \alpha_1^{-1}(\hat{c}+1)\})$ for all $t \in [t_a,\hat{t})$. Since $\hat{t} - t_a \le \tau_1^*(\hat{\varepsilon},\hat{c})$, it follows that:

$$\begin{aligned}
W(k,\hat{e}(\hat{t})) &= W(k,\hat{e}(t_a)) + W(k,\hat{e}(\hat{t})) - W(k,\hat{e}(t_a)) \\
&\le W(k,\hat{e}(t_a)) + \widetilde{\varepsilon}\left(\left|\hat{e}(\hat{t}) - \hat{e}(t_a)\right|\right) \\
&= W(k,\hat{e}(t_a)) \\
&\quad + \widetilde{\varepsilon}\left(\left|\int_{t_a}^{\hat{t}} g(s,\hat{x}(s),\hat{e}(s),\hat{w}(s))ds\right|\right) \\
&< W(k,\hat{e}(t_a)) \\
&\quad + \widetilde{\varepsilon}(\varphi(\max\{\hat{c}+\hat{\varepsilon}, \alpha_1^{-1}(\hat{c}+\hat{\varepsilon})\})\tau_1^*(\hat{\varepsilon},\hat{c})) \\
&\le W(k,\hat{e}(t_a)) + \widetilde{\varepsilon}(\widetilde{\varepsilon}^{-1}(\hat{\varepsilon})) \\
&= W(k,\hat{e}(t_a)) + \hat{\varepsilon}\ ,
\end{aligned}$$

that is $W(k,\hat{e}(\hat{t})) < W(k,\hat{e}(t_a)) + \hat{\varepsilon}$, which contradicts (16). Since $\hat{\varepsilon} \in (0,1]$, $\hat{c}$, t_a, t_b, $\hat{x}$, $\hat{e}$, $\hat{w}$ and $\hat{t} \in [t_a,t_b]$ were arbitrary, it follows that (10) holds for any $\varepsilon \in (0,1]$. But then it is straightforward to show that (10) holds for all $\varepsilon > 0$. ■

Proof of Theorem 1 Let all conditions of Theorem 1 be satisfied. Let τ_1^* come from Proposition 3. We will, henceforth, use the notation $W_+(k) := W(k, e(t_{s_k}^+))$ and $W_-(k) := W(k, e(t_{s_k}^-))$. We prove Proposition for (ε, c) such that:

$$\varepsilon \le \min\left\{1, \widetilde{c}(c)\left(\frac{1}{\lambda} - 1\right)\right\} , \tag{17}$$

$$\widetilde{c}(s) := \max\{s, \alpha_2(s) + 1\} . \tag{18}$$

Once this is proved, then the result follows directly for arbitrary $\varepsilon > 0$ and $c > 0$.

Note first that since (17) implies $\varepsilon \le 1$, we can write using (6) that $\tau \le \tau^*(\varepsilon, c)$ and $\max\{||x||_\infty, ||w||_\infty, |e(t_\circ)|\} \le c$ imply

$$\begin{gathered}\tau \le \tau_1^*(\varepsilon, \widetilde{c}(c)) \\ \max\{||x||_\infty, ||w||_\infty, \alpha_2(|e(t_\circ)|) + \varepsilon\} \le \widetilde{c}(c) .\end{gathered} \tag{19}$$

To shorten notation, we use $\widetilde{c} := \widetilde{c}(c) = \max\{c, \alpha_2(c) + 1\}$. From item 2 with $(\varepsilon, \widetilde{c})$ we obtain from (10) with $t_a = t_{s_k}^+$ and $t_b = t_{s_{k+1}}^-$ that $\tau \le \tau_1^*(\varepsilon, \widetilde{c})$ and $\max\{||x||_\infty, ||w||_\infty, W_+(k)\} \le \widetilde{c}$ imply

$$W_-(k+1) \le W_+(k) + \varepsilon . \tag{20}$$

and write (7) as

$$W_+(k+1) \le \lambda W_-(k+1) . \tag{21}$$

The relations (20) and (21) can be combined to write that $\tau \le \tau_1^*(\varepsilon, \widetilde{c})$ and $\max\{||x||_\infty, ||w||_\infty, W_+(k)\} \le \widetilde{c}$ imply

$$W_+(k+1) \le \lambda(W_+(k) + \varepsilon) . \tag{22}$$

From (17) we have that $\varepsilon \le \widetilde{c}\left(\frac{1}{\lambda} - 1\right)$ and this implies using (22) and induction that if for some ℓ we have $W_+(\ell) \le \widetilde{c}$, then for all $k \ge \ell$ we have $W_+(k) \le \lambda(\widetilde{c} + \varepsilon) \le \widetilde{c}$. Using this we can write for each $k \ge \ell$ that $\tau \le \tau_1^*(\varepsilon, \widetilde{c})$ and $\max\{||x||_\infty, ||w||_\infty, W_+(\ell)\} \le \widetilde{c}$ imply $W_+(k) \le \lambda^{k-\ell} W_+(\ell) + \varepsilon \frac{\lambda}{1-\lambda}$. Next, taking into account the inter-sample behavior from (10) we can write that $t \in [t_{s_k}, t_{s_{k+1}})$, $t_\circ \in [t_{s_\ell}, t_{s_{\ell+1}})$, $t \ge t_\circ$, $\tau \le \tau_1^*(\varepsilon, \widetilde{c})$ and $\max\{||x||_\infty, ||w||_\infty, W(\ell, e(t_\circ)) + \varepsilon\} \le \widetilde{c}$ imply

$$\begin{aligned} W(k, e(t)) &\le \lambda^{k-\ell}\left(W(\ell, e(t_\circ)) + \varepsilon\right) + \varepsilon\frac{1}{1-\lambda} \\ &\le \lambda^{k-\ell} W(\ell, e(t_\circ)) + \varepsilon\frac{2-\lambda}{1-\lambda} . \end{aligned}$$

Next we observe that $t - t_\circ \le (k - \ell + 2)\tau$, i.e., $k - \ell \ge -2 + \frac{t - t_\circ}{\tau}$. Then, defining $\eta := -\ln(\lambda) > 0$, we get that $t \in [t_{s_k}, t_{s_{k+1}})$, $t_\circ \in [t_{s_\ell}, t_{s_{\ell+1}})$, $t \ge t_\circ$, $\tau \le \tau_1^*(\varepsilon, \widetilde{c})$ and $\max\{||x||_\infty, ||w||_\infty, W(\ell, e(t_\circ)) + \varepsilon\} \le \widetilde{c}$ imply

$$\begin{aligned} W(k, e(t)) &\le e^{2\eta} e^{-\frac{\eta(t - t_\circ)}{\tau}} W(\ell, e(t_\circ)) + \varepsilon\frac{2-\lambda}{1-\lambda} \\ &\le \max\left\{2e^{2\eta} e^{-\frac{\eta(t-t_\circ)}{\tau}} W(\ell, e(t_\circ)) , \varepsilon\frac{2(2-\lambda)}{1-\lambda}\right\} . \end{aligned}$$

Then we use (6) to write that $\tau \le \tau_1^*(\varepsilon, \widetilde{c})$ and $\max\{||x||_\infty, ||w||_\infty, \alpha_2(|e(t_\circ)|) + \varepsilon\} \le \widetilde{c}$ imply

$$\begin{aligned} |e(t)| &\le \max\left\{\alpha_1^{-1}\left(2e^{2\eta} e^{-\frac{\eta(t-t_\circ)}{\tau}} \alpha_2(|e(t_\circ)|)\right) ,\right. \\ &\quad \left.\alpha_1^{-1}\left(\varepsilon\tfrac{2(2-\lambda)}{1-\lambda}\right)\right\} \\ &= \max\left\{\beta_e\left(|e(t_\circ)|, \frac{t - t_\circ}{\tau}\right), \gamma_e(\varepsilon)\right\} , \end{aligned}$$

for all $t \ge t_\circ \ge 0$. The last inequality, together with (19), concludes the proof. ■

Proof of Lemma 1 We pick $\bar{x}(t_\circ) = x(t_\circ)$. Since the Lemma assumes $||x||_\infty \le c$, it follows that $|\bar{x}(t_\circ)| \le c$ and then, using item 3 of Theorem 2,

$$||\bar{x}||_\infty \le \widetilde{c}(c) := \max\{\beta(c, 0), \gamma(c)\} . \tag{23}$$

Note that $c \le \widetilde{c}(c)$ since $s \le \beta(s, 0)$. We define $z(t) := x(t) - \bar{x}(t)$. We have $z(t_\circ) = 0$ and, for almost all $t \in [t_\circ, t_\circ + T]$, $\dot{z} = f(t, x, w, e) - f(t, \bar{x}, w, 0)$. Using (14) $\max\{||x||_\infty, ||w||_\infty, ||e||_\infty\} \le c \le \widetilde{c}(c)$, together with (23) and Assumption 1, we have, for almost all $t \in [t_\circ, t_\circ + T]$, $|\dot{z}(t)| \le L(\widetilde{c}(c) + 1)|z(t)| + M(\widetilde{c}(c) + 1, |e(t)|)$. According to [1, Corollary IV.5] there exist two functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that $M(\widetilde{c}(c) + 1, |e(t)|) \le \gamma_1(\widetilde{c}(c) + 1) \cdot \gamma_2(|e(t)|)$. So we can write, for almost all $t \in [t_\circ, t_\circ + T]$, $\frac{d}{dt}|z(t)| \le |\dot{z}| \le L(\widetilde{c}(c) + 1)|z(t)| + \gamma_1(\widetilde{c}(c) + 1)\gamma_2(|e(t)|)$. Then we apply a standard comparison lemma, keeping in mind that $z(t_\circ) = 0$, to assert that, for all $t \in [t_\circ, t_\circ + T]$,

$$\begin{aligned} |z(t)| &\le \int_{t_\circ}^{t} e^{L(\widetilde{c}(c)+1)\cdot(t-r)} \gamma_1(\widetilde{c}(c) + 1)\gamma_2(|e(r)|)dr \\ &\le e^{L(\widetilde{c}(c)+1)\cdot(t-t_\circ)} \gamma_1(\widetilde{c}(c) + 1) \int_{t_\circ}^{t} \gamma_2(|e(r)|)dr \\ &\le e^{L(\widetilde{c}(c)+1)T} \gamma_1(\widetilde{c}(c) + 1) \int_{t_\circ}^{t_\circ + T} \gamma_2(|e(r)|)dr . \end{aligned} \tag{24}$$

The result of the Lemma will be established with the following claim:

Claim 1 *Under the conclusion of Theorem 1, there exists $\tau^* \in \mathcal{KLL}$ such that, for each triple of positive* *real numbers (ρ, T, c) and each $t_\circ \geq 0$ we have that $\tau \leq \tau^*(\rho, T, c)$ implies*

$$e^{L(\widetilde{c}(c)+1)T}\gamma_1(\widetilde{c}(c)+1)\int_{t_\circ}^{t_\circ+T}\gamma_2(|e(r)|)dr \leq \rho \ . \tag{25}$$

Proof. Using the result of Theorem 1, we have that, under the assumptions of Lemma 1, we have that $\tau \leq \tau_1^*(\varepsilon, \widetilde{c}(c))$ and $r \geq t_\circ$ imply

$$\gamma_2(|e(r)|) \leq \max\left\{\gamma_2\left(\beta_e\left(c, \frac{r-t_\circ}{\tau}\right)\right) \ , \ \gamma_2(\gamma_e(\varepsilon))\right\} . \tag{26}$$

We define $\widetilde{\rho}(\rho, T, c) := \frac{\rho}{\exp\left[L(\widetilde{c}(c)+1)\cdot T\right]\gamma_1(\widetilde{c}(c)+1)}$ and we note that $\widetilde{\rho} \in \mathcal{KLL}$. Next we define

$$\varepsilon(\rho, T, c) := \gamma_e^{-1} \circ \gamma_2^{-1}\left(\frac{\widetilde{\rho}(\rho, T, \widetilde{c}(c))}{2T}\right) \tag{27}$$

and we note that $\varepsilon \in \mathcal{KLL}$. We define

$$r^*(\rho, T, c) := \frac{\widetilde{\rho}(\rho, T, \widetilde{c}(c))}{2\gamma_2(c)} \tag{28}$$

and note that $r^* \in \mathcal{KLL}$. We let $\tau_2^* \in \mathcal{KLL}$ satisfy

$$\begin{aligned}\beta_e\left(c, \frac{r^*(\rho, T, c)}{\tau^*(\rho, T, c)}\right) &\leq \gamma_e(\varepsilon(\rho, T, \widetilde{c}(c))) \\ &= \gamma_2^{-1}\left(\frac{\widetilde{\rho}(\rho, T, \widetilde{c}(c))}{2T}\right) \ .\end{aligned} \tag{29}$$

Then we define

$$\tau^*(\rho, T, c) = \min\left\{\tau_2^*(\rho, T, c), \tau_1^*(\varepsilon(\rho, T, c), \widetilde{c}(c))\right\} \tag{30}$$

and we assume $\tau \leq \tau^*(\rho, T, c)$. Now we split the interval of integration in (25) into two pieces $r \in [t_\circ, t_\circ + r^*]$ and $r \in [t_\circ + r^*, t_\circ + T]$. For the first interval, using $||e||_\infty \leq c$, we get the bound

$$\begin{aligned}&e^{L(\widetilde{c}(c)+1)T}\gamma_1(\widetilde{c}(c)+1)\textstyle\int_{t_\circ}^{t_\circ+r^*(\rho,T,c)}\gamma_2(|e(r)|)dr \\ &\quad \leq e^{L(\widetilde{c}(c)+1)T}\gamma_1(\widetilde{c}(c)+1)r^*(\rho, T, c)\gamma_2(c) \\ &\quad \leq \tfrac{\rho}{2} \ .\end{aligned} \tag{31}$$

For the second interval, using that $|e(r)| \leq \gamma_e(\varepsilon(\rho, T, c))$ on this interval, we get

$$\begin{aligned}&e^{L(\widetilde{c}(c)+1)T}\gamma_1(\widetilde{c}(c)+1)\textstyle\int_{t_\circ+r^*(\rho,T,c)}^{t_\circ+T}\gamma_2(|e(r)|)dr \\ &\quad \leq e^{L(\widetilde{c}(c)+1)T}\gamma_1(\widetilde{c}(c)+1)T\gamma_2(\gamma_e(\varepsilon(\rho, T, c))) \\ &\quad \leq \tfrac{\rho}{2} \ .\end{aligned} \tag{32}$$

Adding (31) and (32) together establishes the result. ■

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