Advanced topics in control systems theory II

Lecture notes from FAP 2005

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1

Sampled-Data Control of Nonlinear Systems

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Summary. This chapter provides some of the main ideas resulting from recent developments in sampled-data control of nonlinear systems. We have tried to bring the basic parts of the new developments within the comfortable grasp of graduate students. Instead of presenting the more general results that are available in the literature, we opted to present their less general versions that are easier to understand and whose proofs are easier to follow. We note that some of the proofs we present have not appeared in the literature in this simplified form. Hence, we believe that this chapter will serve as an important reference for students and researchers that are willing to learn about this area of research.

1.1 Introduction

Technological advances in digital electronics that occurred in the second half of the 20^{th} century have led to a rapid development in computer technology and this has made a great impact on a range of engineering areas, including control engineering. Nowadays, most control systems exploit a digital computer as their crucial part and computer controlled systems are prevalent in engineering practice. Hence, the theory for analysis and design of computer controlled systems is a crucial part of the control engineer's toolbox.

A general configuration of a computer controlled feedback system is illustrated in Figure 1.1. A continuous-time plant (process) is interfaced with the computer via analog-to-digital (A/D) and digital-to-analog (D/A) converters that are often referred to as *sampler* and *hold* devices respectively. The A/D converter produces the samples $y(t_k)$ of the continuous plant output $y(t)$ at sampling times t_k and sends them to a control algorithm within the computer. The control algorithm processes the measured sequence $y(t_k)$ and produces a sequence of control inputs $u(t_k)$. This control sequence is converted in the D/A converter into a piecewise continuous control signal $u(t)$ that is applied

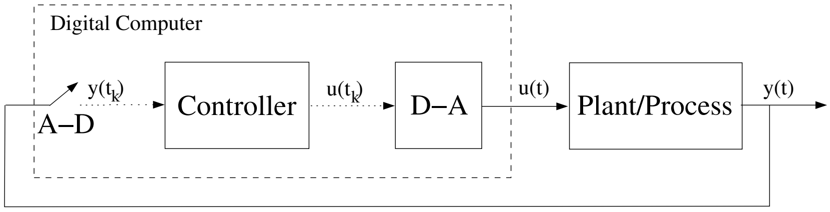

Fig. 1.1. General computer controlled system configuration.

to the plant. This is typically done by holding the value of the control signal constant during the sampling intervals (*zero-order-hold*). An internal clock synchronizes the operation of the system. The sampling instants t_k are typically equidistant, i.e. $t_k = kT$, $k = 0, 1, 2, \ldots$, where $T > 0$ is the *sampling period*.

The computer controlled system in Figure 1.1 is often referred to as a *sampled-data control system* to emphasize the sampling process as its crucial feature. Note that due to the hybrid nature of sampled-data systems, that involve continuous-time (plant) dynamics and discrete-time (controller) dynamics, their analysis and design are harder than those of continuous-time systems[3]. Indeed, this has led to several distinct approaches to controller design for sampled-data systems.

1. *Emulation:* Design a continuous-time controller for the continuous-time plant model and then discretize the controller for digital implementation. This approach involves an approximation (discretization) of the controller that is valid only for small sampling periods T and, typically, the system loses stability for large sampling periods. Advanced emulation techniques also use controller redesign for digital implementation and they are better behaved for larger sampling periods.
2. *Discrete-time design:* Design a controller in discrete-time using the discrete-time plant model. This method exploits an approximation (discretization) of the plant model that ignores the inter-sample behaviour. While this method does not require fast sampling to maintain stability, performance of the sampled-data system is not automatically guaranteed since the inter-sample behaviour may be unacceptable.
3. *Sampled-data design:* Using an exact sampled-data model of the plant[4], design a controller that achieves both stability and required performance for the sampled-data system. This method uses no approximations of the

[3] This is the case, for instance, when the plant is a continuous-time system and the controller is realized via analog electronics using operational amplifiers.
[4] See for instance, [13] where the lifting technique is used to obtain models for sampled-data systems that model the inter-sample behaviour.

plant model or controller and, hence, it maintains stability and performance for arbitrarily large sampling periods T.

Emulation is regarded as the simplest method, while sampled-data design requires the most advanced techniques. On the other hand, satisfactory system performance can be achieved using the sampled-data design, whereas emulation is typically inferior to the other two methods in terms of stability and/or achievable performance.

Analysis and design of *linear* sampled-data control systems date back to the 1950's, that marked the beginning of the digital revolution. The early works concentrated on input-output approaches involving z-transform and they were parallel to the corresponding continuous-time developments. In the 1960's and 1970's, state space approaches involving state difference equations have become popular and optimal regulation and Kalman filtering for discrete-time systems were developed during that time. This material has become a standard part of many undergraduate curricula. The 1980's and 1990's have seen several new developments for linear systems that have led to H_∞ theory for discrete-time systems, advanced emulation techniques based on optimization, the use of δ-transform and H_∞ sampled-data controller design based on lifting techniques (for more details on all of these developments see [6, 13, 25, 32]).

Linear sampled-data control theory is now a mature area with a range of undergraduate textbooks that cover different analysis and design approaches. On the other hand, *nonlinear* sampled-data control theory is quite underdeveloped compared to its linear counterpart. While it is often possible to use a linear sampled-data control theory for solving nonlinear control problems via the linearization technique, there are many important situations where nonlinearities cannot be neglected. For instance, wide ranges of operating conditions typically prevent control designers from ignoring important nonlinearities, such as saturation, that are commonly present in the system. Moreover, hysteresis, dead-zone and dry friction are but a few examples of common nonlinearities that often can not be ignored in practice (see [52] for details). Indeed, there is a wide area of applications where nonlinear phenomena cannot be avoided. These applications range from vertical take-off and landing (VTOL) aircraft systems, ship or submarine vehicle control, position control for robotic systems in a precision manufacturing process, autonomous vehicle systems, biochemical reactors, power plants and many others. Finally, many control algorithms, such as adaptive and sliding mode controllers, are inherently nonlinear. Therefore, nonlinear sampled-data control systems form an important class of systems that arises in applications. Emulation for nonlinear sampled-data systems has been studied in some detail and general results that provide a justification for this approach are available (see [29] and references cited therein). Due to a variety of tools for nonlinear continuous-time controller design (see for instance [23, 24, 53, 52]) and its inherent simplicity, the emulation method is quite attractive to practitioners. Unfortunately, emulated controllers are prone to instability in nonlinear systems. As a result, one typically needs to use smaller sampling periods in emulation design for nonlinear systems. In particular, the required sampling may sometimes exceed the hardware limitations and in such cases one may need to use methods other than emulation.

On the other hand, due to the complexity of the underlying nonlinear sampled-data model, results on sampled-data design for nonlinear systems that would parallel the linear results presented in [13] are scarce (we are not aware of any) and it appears that they will be hard to develop in the future. Hence, it appears that discrete-time design techniques for nonlinear sampled-data systems provide a nice tradeoff between the possible conservatism of emulation design and the difficulty of developing direct sampled-data design.

The literature on discrete-time design methods for nonlinear sampled-data systems can be classified into two large groups:

1. *Exact discrete-time design methods.* The majority of results in this direction, for example [2, 16, 21, 33, 34, 56, 60], assume that the *exact discrete-time plant model is known* and it is available to the designer. Hence, these papers start directly from discrete-time models of the form:
$$x(k+1) = F(x(k), u(k)) \ ,$$
where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and the control input of the system and $F(\cdot,\cdot)$ is a known vector function. This assumption, however, is rarely justified for nonlinear sampled-data systems (such as the one illustrated by Figure 1.1) as will be discussed in Section 1.2 and, hence, results that belong to this group have very limited applicability.
2. *Approximate discrete-time design methods.* Some earlier research, for instance [15, 17, 19, 31], recognize the fact that the exact discrete-time model for nonlinear systems is typically unavailable to the controller designer and they instead base their controller design on an *approximate discrete-time plant model.* While this approach is closer to reality and it is most natural to use in practice, due to the limited theoretical results, the majority of the published works in this area are *ad hoc* and they do not carefully investigate the interplay between the controller design and the plant model approximation. In particular, we show in Section 1.4 that there may exist controllers that stabilize a seemingly good approximate discrete-time plant model but destabilize the sampled-data system for arbitrarily small sampling periods. Hence, great care is needed when pursuing this approach.

The main purpose of this chapter is to provide a rigorous framework for sampled-data nonlinear controller design via approximate discrete-time plant models. Our framework is fully consistent with what most engineers would do in practice but our analysis provides a framework and guidelines for such design to be successful. Moreover, this framework can be used to justify the emulation method for general nonlinear systems (see Section 1.7.1). Several controller design techniques are presented for classes of nonlinear systems that are fully consistent with our framework. Our approach benefits from selected topics in numerical analysis literature [51, 59]. In particular, we adapted the notion of *consistency*, commonly used in numerical analysis, to develop our controller design framework.

We emphasize that this chapter is not intended to serve as a literature survey and the material presented summarizes just a subset of recent results in nonlinear sampled-data control that reflect the authors research interests. Moreover, we emphasize that our results are often presented in a simpler form than that in the original references in order to achieve clarity and simplicity of exposition. We have tried to achieve this without sacrificing the rigor of our arguments. More complete and details results in this area and the closely related works are listed in the references.

1.2 Mathematical Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing, and of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Note that linear functions $\varphi(s) = Ks$ for some $K > 0$ are of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class $\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. The function β is of class exp-$\mathcal{KL}$ if there exist $K, \lambda > 0$ such that $\beta(s,t) = Ks\exp(-\lambda t)$. Class $\mathcal{K}$ and $\mathcal{KL}$ functions are useful to characterize stability properties of nonlinear systems [23]. For instance, suppose that there exists $\beta \in \mathcal{KL}$ such that the solutions $\phi(t, x_\circ)$ of the continuous-time system $\dot{x} = f(x)$ satisfy

$$|\phi(t, x_\circ)| \leq \beta(|x_\circ|, t) \qquad \forall t \geq 0,\ x(0) = x_\circ \in \mathbb{R}^n \ .$$

Then, the origin of the system is *globally asymptotically stable (GAS)*. Moreover, if $\beta \in$ exp-$\mathcal{KL}$, then the origin of the system is *globally exponentially stable (GES)*.

A function $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is of order $O(T^p), p > 0$, if there exist $\varphi \in \mathcal{K}_\infty$ and $T^* > 0$ such that for all $T \in (0, T^*)$ and all $x \in \mathbb{R}^n$ we have $|f(x,T)| \leq \varphi(|x|)T^p$. We will use the Mean Value Theorem several times in the sequel and we state it below for the sake of completeness.

If x and y are two distinct points in $\mathbb{R}^n$, then the (open) line segment $L(x, y)$ joining two distinct points x and y in $\mathbb{R}^n$ is

$$L(x,y) = \{z | z = \theta x + (1-\theta)y,\ 0 < \theta < 1\} \ .$$

Theorem 1.1 (Mean Value Theorem). *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at each point x of an open set $S \subset \mathbb{R}^n$. Let x and y be two points of S such that the line segment $L(x, y) \subset S$. Then there exists a point z of $L(x, y)$ such that*

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x) .$$

■

1.3 Zero-Order-Hold Equivalent Models

In this section we present results on discretization of sampled-data systems assuming the use of zero-order-hold devices. These results provide a basis for the controller design framework via approximate discrete-time models presented in the next section. Consider the sampled-data system in Figure 1.1 where we assume that the plant dynamics are linear, i.e.

$$\dot{x} = Ax + Bu , \tag{1.1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are the state and control vectors respectively. The plant is assumed to be between a sampler (A/D converter) and zero-order-hold (D/A converter). The control signal is assumed to be piecewise constant, i.e.

$$u(t) = u(kT) =: u(k), \qquad \forall t \in [kT, (k + 1)T), \ k \in \mathbb{N} \tag{1.2}$$

where $T > 0$ is the sampling period. Moreover, we assume that the state measurements $x(k)$, where[5]

$$x(k) := x(kT) . \tag{1.3}$$

are available at sampling instants.

A classical approach to controller design for the system (1.1) is to first discretize the model and then design a controller for the discretized model. Using the variations of constant formula for the linear system (1.1) we can compute the solution x at time $t \geq kT$ that starts from the initial state $x(k)$ at time kT, while keeping the control constant $u(t) \equiv u(k)$,

$$x(t) = e^{A(t-kT)}x(k) + \int_{kT}^{t} e^{A(t-s)}Bu(k)ds .$$

Evaluating the above equation for $t = (k + 1)T$, we have

[5] One can also assume that only outputs $y(k) = Cx(k) + Du(k)$ are measured but in this section we want to keep the presentation as simple as possible and do not consider this case.

$$x(k+1) = \Phi_T x(k) + \Gamma_T u(k) \ , \tag{1.4}$$

where

$$\Phi_T := e^{AT}; \qquad \Gamma_T := \int_0^T e^{As} B ds \ .$$

The discretized model (1.4) describes the sampled-data system (1.1), (1.2), (1.3) *exactly* at sampling instants kT and, in particular, it describes how the state $x(k+1)$ of the system at the time instant $(k+1)T$ depends on the state $x(k)$ at the previous sampling instant kT and control $u(k)$ on the sampling interval $[kT, (k+1)T)$.

Note that the model (1.4) is a linear difference equation that is parameterized by the sampling period T. The sampling period T is assumed to be a design parameter which can be arbitrarily assigned. In practice, there is a range of allowable sampling periods T that depends on the hardware limitations (e.g. the DAQ 2000 I/O card can achieve any sampling periods from 0.01 seconds to 30 minutes). Note that the discrete-time model ignores the inter-sample behaviour and any controller that is designed using this model may lead to poor inter-sample behaviour.

Consider now the nonlinear continuous-time control system

$$\dot{x} = f(x, u) \ , \qquad x(0) = x_\circ \ . \tag{1.5}$$

The function f is assumed to be such that, for each initial condition and each constant control, there exists a unique solution defined on some (perhaps bounded) interval of the form $[0, \tau)$. We can compute the solution x at time $t \geq kT$ that starts from the initial state $x(k)$ while keeping the control constant $u(t) \equiv u(k)$ as

$$x(t) = x(k) + \int_{kT}^{t} f(x(s), u(k)) ds \ .$$

Suppose that the solutions are well defined and evaluate the above equations for $t = (k+1)T$,

$$x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(x(s), u(k)) ds =: F_T^e(x(k), u(k)) \ . \tag{1.6}$$

The equation (1.6) represents the exact discrete-time model of the nonlinear sampled-data system (1.5), (1.2), (1.3) and it is the nonlinear counterpart of (1.4). We emphasize that F_T^e is not known in most cases since computing F_T^e explicitly will require an analytic solution of a nonlinear initial value problem. On the other hand, one can easily write down a range of approximate models. For example, the forward Euler approximate model of the sampled-data system (1.5), (1.2), (1.3),

$$x(k+1) = x(k) + T f(x(k), u(k)) =: F_T^{Euler}(x(k), u(k)) \ , \tag{1.7}$$

is often used in the sequel. A range of other approximate models (e.g. using Runge-Kutta integration methods) can be found in standard books on numerical analysis [59].

In the sequel, we consider the difference equations corresponding to the exact and approximate discrete-time models of the sampled data system (1.5), (1.2), (1.3) that are denoted respectively as

$$x(k+1) = F_T^e(x(k), u(k)) \tag{1.8}$$
$$x(k+1) = F_T^a(x(k), u(k)) \tag{1.9}$$

and which are parameterized by the sampling period T. We will think of F_T^e and F_T^a as being defined globally for all small T even though the initial value problem (1.5) may exhibit finite escape times. In general, one needs to use small sampling periods T since the approximate plant model is a good approximation of the exact model mainly only for small T.

It turns out that most sampled-data literature [2, 16, 21, 33, 34, 56, 60] uses the following assumption.

Assumption 1 *The exact discrete-time model (1.8) for the sampled-data system (1.5), (1.2), (1.3) is known and it is available to the designer. In other words, the controller design can be carried out using the exact discrete-time model (1.8).*

Indeed, this assumption is the starting point in the exact discrete-time design method discussed in the Introduction. On the other hand, Assumption 1 is not justified in most cases. The exact discrete-time model can not be analytically computed since it requires solving a nonlinear initial value problem explicitly. Hence, our results are useful in cases when the following more realistic assumption holds.

Assumption 2 *The exact discrete-time model (1.8) for the sampled-data system (1.5), (1.2), (1.3) is not known exactly and it is not available to the designer. Therefore, the controller design needs to be carried out using an approximate discrete-time model (1.9).*

We note that Assumption 2 is more natural to use for most nonlinear systems. Moreover, even in the linear case we use approximate models that come from numerically computing the matrices Φ_T and Γ_T in (1.4).

1.4 Motivating Counter-Examples

The approximate model (1.9) is parameterized by T and, in general, we need to be able to obtain a family of controllers which is parameterized by T and which is defined for all small T. There are two reasons for this: (i) F_T^a is a good approximation for F_T^e only for small T and, hence, the designed controller will have to achieve stability of F_T^a for all small T; (ii) finding a controller that does not depend on T and that stabilizes the approximate family F_T^a for all small T is a harder problem than when the controller is allowed to depend on T. Hence, we will concentrate in the sequel on controllers of the form[6]

$$u(k) = u_T(x(k)) \ . \tag{1.10}$$

The goal of this section is to show that there may exist a family of controllers of the form (1.10) that stabilizes the family of approximate models (1.9) for all small T whereas it *destabilizes* the family of exact models (1.8) for all small T. We identify several indicators of lack of stability robustness that typically lead to these undesirable behaviours. In the next section, we will introduce conditions that rule out each of these non-robustness indicators and this will lead to a framework for sampled-data controller design via approximate discrete-time models.

Examples in this section can be interpreted in the following manner. Assume that we want to pursue an *ad hoc* approach to controller design that many practitioners and researchers have considered. Consider an approximate plant model (1.9), such as (forward) Euler model, that is a good approximation for (1.8) when the two models are regarded as "open-loop". Suppose, moreover, that we want to first reduce T sufficiently to guarantee that F_T^a is a good approximation of F_T^e and then we design a controller (1.10) that stabilizes F_T^a, hoping that it will stabilize F_T^e because T is already small enough. Examples presented in this section show that in general this approach is flawed and no matter how small sampling period T we choose, we can always find a controller (1.10) that stabilizes the approximate model (1.9) but it destabilizes the exact system (1.8). The following examples (taken from [43]) illustrate that a careful investigation is needed if controller design is to be carried out on approximate models.

Example 1.1. **(Control with excessive force)** Consider the sampled-data control of the triple integrator

$$\dot{x}_1 = x_2; \qquad \dot{x}_2 = x_3; \qquad \dot{x}_3 = u \ . \tag{1.11}$$

Although the exact discrete-time model of this system can be computed, we base our control algorithm on the family of the Euler approximate discrete-time models in order to illustrate possible pitfalls in control design based on approximate discrete-time models. The family of Euler approximate discrete-time models for this system is given by (1.7). A minimum time dead beat controller for the Euler discrete-time model is given by

[6] For simplicity we consider only static state feedback controllers, while results on dynamic controllers can be found in the cited references.

$$u = u_T(x) = \left(-\frac{x_1}{T^3} - \frac{3x_2}{T^2} - \frac{3x_3}{T} \right) . \tag{1.12}$$

The closed-loop system (1.7), (1.12) has all eigenvalues equal to zero for all $T > 0$ and hence this discrete-time Euler based closed-loop system is asymptotically stable for all $T > 0$. On the other hand, the closed-loop system consisting of the exact discrete-time model of the triple integrator and controller (1.12) has an eigenvalue at ≈ -2.644 for all $T > 0$. Hence, the closed-loop sampled-data control system is unstable for all $T > 0$.

Note that in Example 1.1 we have the following properties.

1. **Nonuniform bound on overshoot**. The solutions of the family of approximate models with the given controller satisfy for all $T > 0$ a stability estimate of the type
$$|\phi_T(k, x_\circ)| \le b_T e^{-kT} |x_\circ| , \quad k \in \mathbb{N}$$
and $b_T \to \infty$ as $T \to 0$. Hence, the overshoot in the stability estimate for the family of approximate models is not uniformly bounded in T.
2. **Nonuniform bound on control**. The control is not uniformly bounded on compact sets with respect to the parameter T and in particular we have for all $x \neq 0$ that $|u_T(x)| \to \infty$ as $T \to 0$.

Example 1.2. **(Control with excessive finesse)** Consider the system

$$\dot{x} = x + u . \tag{1.13}$$

Again, the exact discrete-time model of the system can be computed, but we consider a control design based on the "partial Euler" model

$$x(k+1) = (1+T)x(k) + (e^T - 1)u(k) . \tag{1.14}$$

The control

$$u = u_T(x) = -\frac{T(1 + \frac{1}{2}T)x}{e^T - 1} \tag{1.15}$$

stabilizes the family of approximate models (for $T \in (0, 2)$) by placing the eigenvalue of the closed-loop at $1 - \frac{1}{2}T^2$. On the other hand, the eigenvalue of the exact discrete-time closed-loop is located at $e^T - T - \frac{1}{2}T^2 > 1$, $\forall T > 0$.

Note that in Example 1.2 we have the following properties.

- **Nonuniform attractive rate**. For all $T > 0$, the family of approximate discrete-time models satisfies
$$|\phi_T(k, x_\circ)| \le b e^{-kT^2} |x_\circ| , \quad k \in \mathbb{N} ,$$

where $b > 0$ is independent of T. Therefore the overshoot is uniformly bounded in T. However, if we think of $kT = t$ as "continuous-time", then as $T \to 0$, the rate of convergence of solutions is such that for any $t > 0$ we have $e^{-tT} \to 1$. In other words, the rate of convergence in continuous-time is not uniform in the parameter T.

Conditions in our framework for controller design in the next section will rule out all of the above non-robustness indicators.

1.5 Preliminary Results on Stability and Stabilization

This section contains two main results. In Proposition 1.1 we show under natural and general conditions that stability of the exact discrete-time model implies stability of the sampled-data system. Proposition 1.2 provides Lyapunov conditions to analyze the stability of the exact discrete-time model.

These results are important in proving that stability of approximate model will guarantee, under appropriate conditions, stability of the sampled-data system. Indeed, we show in the next section that stability of the approximate model implies, under certain checkable conditions, the stability of the exact model and, consequently, we can conclude stability of the sampled-data system using the results proved in this section.

Note that if Assumption 1 was satisfied, the results of this section could be used to conclude stability of the sampled-data systems directly from stability of its exact discrete-time model. However, since we use Assumption 2, more work will be needed to investigate when stability of the approximate model implies stability of the exact.

Suppose for simplicity that a parameterized family of control laws (1.10) was designed for the system so that the closed-loop sampled-data system becomes

$$\dot{x}(t) = f(x(t), u_T(x(kT))) \qquad t \in [kT, (k+1)T) \ . \tag{1.16}$$

Hence, with the control (1.10) the closed-loop exact model of this sampled-data system is

$$x(k+1) = F_T^e(x(k), u_T(x(k))) = \mathcal{F}_T^e(x(k)) \ . \tag{1.17}$$

Proposition 1.1 given below states that if the sampled-data system (1.16) has bounded inter-sample behaviour (condition 2), then GAS of the exact discrete-time model (condition 1) implies UGAS of the sampled-data system[7]. The proof of this proposition is presented in [46].

[7] Note that the exact discrete-time model (1.17) is time invariant whereas the sampled-data system (1.16) is periodically time varying because of sampling. Hence, we talk about "uniform" GAS for the sampled-data system where uniformity is with respect to the initial time instant $t_\circ$.

Proposition 1.1. *Consider a sampled-data system (1.16) and suppose that the sampling period $T > 0$ is such that the following two conditions hold.*

1. *There exists $\widetilde{\beta} \in \mathcal{KL}$ such that the trajectories of the exact discrete-time closed-loop system (1.17) satisfy*

$$|x(k)| \leq \widetilde{\beta}(|x_\circ|, kT) \qquad \forall k \in \mathbb{N}, \quad x(0) = x_\circ \in \mathbb{R}^n \ . \tag{1.18}$$

2. *There exists $\kappa \in \mathcal{K}_\infty$ such that the solutions of the sampled-data system (1.16) satisfy*

$$|x(t)| \leq \kappa(|x_\circ|) \qquad \forall t \in [t_\circ, t_\circ + T], \ t_\circ \geq 0, \ x(t_\circ) = x_\circ \in \mathbb{R}^n \ . \tag{1.19}$$

Then there exists $\beta \in \mathcal{KL}$ such that the trajectories of the sampled-data system satisfy[8]

$$|x(t)| \leq \beta(|x_\circ|, t - t_\circ) \qquad \forall t \geq t_\circ \geq 0, \quad x(t_\circ) = x_\circ \in \mathbb{R}^n \ . \tag{1.20}$$

Moreover, if $\widetilde{\beta} \in$ exp-$\mathcal{KL}$ and $\kappa \in \mathcal{K}_\infty$ is linear, we can take $\beta \in$ exp-$\mathcal{KL}$. ■

Remark 1.1. If the function f is globally Lipschitz then condition 2 of Proposition 1.1 always holds. It is important to note that condition 2 holds for any locally Lipschitz discrete-time model F_T^e in an appropriate relaxed (semiglobal practical) sense if the sampling period T is sufficiently reduced. We decided not to state these more general conditions to simplify the presentation. The more general semiglobal practical stability results (that are also more natural in this context) can be found in [46].

Condition 2 of Proposition 1.1 holds under natural and general conditions and it only remains to see how one can satisfy condition 1. The following result that will help verifying condition 1 in Proposition 1.1 is presented with a proof.

Proposition 1.2. *Suppose there exists a family of Lyapunov functions $V_T(x)$ parameterized by T and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the following conditions hold for all $x \in \mathbb{R}^n$.*

$$\begin{aligned} \alpha_1(|x|) \leq V_T(x) &\leq \alpha_2(|x|) \ , \\ \frac{V_T(\mathcal{F}_T^e(x)) - V_T(x)}{T} &\leq -\alpha_3(|x|) \ . \end{aligned} \tag{1.21}$$

[8] It was shown in [12] that the state of the sampled-data system (1.16) at any time instant $t_\circ \in [kT, (k+1)T)$ consists of $x(t_0)$ and $u_T(x(k))$. Hence, strictly speaking the stability bound (1.20) is not equivalent to uniform global asymptotic stability of the sampled-data system. However, if $|u_T(x)| \leq \varphi(|x|)$ for some $\varphi \in \mathcal{K}_\infty$, our conditions imply uniform global asymptotic stability of the sampled-data system. To conclude $\beta \in$ exp-$\mathcal{KL}$, we also need that the function φ is linear.

Then there exists $\widetilde{\beta} \in \mathcal{KL}$ such that condition 1 of Proposition 1.1 holds. That is, the solutions of the exact discrete-time model (1.17) satisfy (1.18). Moreover, if there exist $a_i > 0$ and $p > 0$ such that $\alpha_i(s) = a_i s^p$ for $i = 1, 2, 3$, then condition 1 of Proposition 1.1 holds with $\widetilde{\beta} \in$ exp-$\mathcal{KL}$. ■

Proof: Note that (1.21) implies

$$\frac{V_T(\mathcal{F}_T^e(x)) - V_T(x)}{T} \le -\alpha_3 \circ \alpha_2^{-1}(V_T(x)) =: -\alpha(V_T(x)) \ .$$

Denote $V_T(kT) := V_T(x(kT))$. We introduce a variable $t \in \mathbb{R}$ and define $y(t) := V_T(kT) + (t - kT)\frac{V_T((k+1)T) - V_T(kT)}{T}, t \in [kT, (k+1)T], k \ge 0$. Note that $0 \le y(kT) = V_T(kT), k \ge 0$ and $y(t)$ is a continuous function of the "time" t. Moreover, it is absolutely continuous in t (in fact, piecewise linear) and we can write for almost all t,

$$\begin{aligned} \frac{d}{dt}y(t) &= \frac{V_T((k+1)T) - V_T(kT)}{T} \\ &\le -\alpha(V_T(kT)) \ , \quad \text{for } t \in [kT, (k+1)T), \quad k \ge 0 \ , \\ &\le -\alpha(y(t)) \ , \qquad \text{for } t \ge 0 \ . \end{aligned} \tag{1.22}$$

Let $v(t) = \beta(v_0, t)$ be the (unique) solution of $\dot{v} = -\alpha(v)$, $v(t_0) = v_0$. It is shown in Lemma 6.1 in [58] that $\beta \in \mathcal{KL}$. By standard comparison theorems (see for instance [30, Theorem 1.10.2]) we have for $y_0 = v_0$ that

$$y(t) \le v(t) = \beta(y_0, t - t_0), \ \forall t \ge t_0 \ ,$$

which implies using $V_T(kT) = y(kT)$ with $t = kT, t_0 = k_0 = 0, y_0 = V_T(0)$ that

$$|x(k)| \le \alpha_1^{-1}(V_T(kT)) \le \alpha_1^{-1}(\beta(V_T(0), kT)) \le \alpha_1^{-1}(\beta(\alpha_2(|x_0|), kT)), \ k \ge 0 \ ,$$

which proves (1.18) with $\widetilde{\beta}(s,t) := \alpha_1^{-1}(\beta(\alpha_2(s), t))$. Proving that $\widetilde{\beta} \in$ exp-$\mathcal{KL}$ under stronger conditions is easy following the same steps. ■

Remark 1.2. The above results hold for arbitrarily large T. In other words, they are not fast sampling results. However, to satisfy some of these conditions we will need to reduce T in general. For example, to satisfy condition 2 of Proposition 1.1 on a compact subset of $\mathbb{R}^n$ in case f is locally Lipschitz in x, we need to reduce T sufficiently. Similarly, the results of the next section will require fast sampling to show that under certain conditions stability of an approximate model implies stability of the exact model.

1.6 Framework for Controller Design

In this section we show how one can conclude, under certain checkable conditions, that a controller that stabilizes the approximate model F_T^a is guaranteed to also stabilizes the exact model F_T^e. Then, we can conclude that the sampled-data model is also stabilized using the results from the previous section. We will start from the simplest case of exponential stability design which we will prove in detail. While the proof of this result is quite easy to follow, the used conditions are quite strong for general nonlinear systems. In Subsection 1.6.2 we present without proof a more general result on semiglobal practical stability that uses more natural and less restrictive conditions.

1.6.1 Global Exponential Stabilization

Suppose that Assumption 2 holds and we want to achieve global exponential stability (GES) of F_T^e by stabilizing F_T^a. To do this, we assume for convenience that the function $f(\cdot,\cdot)$ in the continuous-time plant model is globally Lipschitz (this can be relaxed).

We need to find conditions that guarantee global exponential stability of the exact discrete-time closed loop system (1.17) via the following discrete-time approximate closed-loop system

$$x(k+1) = F_T^a(x(k), u_T(x(k))) \ , \tag{1.23}$$

where the family of controllers (1.10) that is parameterized by T is designed using the family of approximate discrete-time models (1.9). In the sequel, we refer to the exact (1.17) and approximate (1.23) closed loop systems respectively as (F_T^e, u_T) and (F_T^a, u_T). Using Proposition 1.2, it is reasonable to aim to design the family of controllers (1.10) so that the following holds for some Lyapunov function family (these conditions are also strong and can be relaxed).

$$\begin{gathered} a_1 |x|^c \leq V_T(x) \leq a_2 |x|^c \\ \frac{V_T(F_T^a(x, u_T(x))) - V_T(x)}{T} \leq -a_3 |x|^c \ , \end{gathered} \tag{1.24}$$

for some $c > 0$, all x and all $T \in (0, T^*)$ where $T^* > 0$ is fixed. Hence, Proposition 1.2 guarantees that (F_T^a, u_T) is GES for all small $T \in (0, T^*)$. The reason for requiring this condition to hold for all small T is going to become clear soon.

We want to see when the above conditions imply that all conditions of Proposition 1.2 hold for the closed-loop exact discrete-time model if we perhaps further reduce T. In order to see when this can be achieved, add and subtract $\frac{1}{T} V_T(F_T^e(x, u_T(x)))$ to (1.24) yielding

$$\frac{V_T(F_T^e(x, u_T(x))) - V_T(x)}{T} \leq -a_3 |x| + \frac{V_T(F_T^e(x, u_T(x))) - V_T(F_T^a(x, u_T(x)))}{T} \ . \tag{1.25}$$

Suppose also that for all $x, y \in \mathbb{R}^n$ and all $T \in (0, T^*)$ the following two conditions hold.

$$|V_T(x) - V_T(y)| \leq L\,|x - y| \tag{1.26}$$

and

$$|F_T^e(x, u_T(x)) - F_T^a(x, u_T(x))| \leq T\rho(T)\,|x| \tag{1.27}$$

where $\rho \in \mathcal{K}_\infty$. Then, from (1.25), (1.26) and (1.27) we obtain

$$\begin{aligned}\frac{V_T(F_T^e(x, u_T(x))) - V_T(x)}{T} &\leq -a_3\,|x| + \frac{L\,|F_T^e(x, u_T(x)) - F_T^a(x, u_T(x))|}{T} \\ &\leq -a_3\,|x| + \frac{LT\rho(T)\,|x|}{T} \\ &= -a_3\,|x| + L\rho(T)\,|x| \ .\end{aligned} \tag{1.28}$$

It is now obvious that for all $T \in (0, T_1^*)$ with $T_1^* := \min\{T^*, \rho^{-1}(a_3/2L)\}$ we have that

$$\frac{V_T(F_T^e(x, u_T(x))) - V_T(x)}{T} \leq -\frac{1}{2}a_3\,|x| \ , \tag{1.29}$$

and, hence, we can conclude from Proposition 1.2 that the closed-loop exact model (F_T^e, u_T) is GES. Before discussing this result in detail, we state our findings in the following proposition.

Proposition 1.3. *Suppose there exists $T^* > 0$ such that for all $T \in (0, T^*)$ the following holds.*

1. *The closed-loop approximate model (F_T^a, u_T) satisfies (1.24). Moreover, condition (1.26) holds uniformly in $T \in (0, T^*)$.*
2. *Condition (1.27) holds.*

Then for all $T \in (0, T_1^*)$, with $T_1^* := \min\{T^*, \rho^{-1}(a_3/2L)\}$, we have that the closed-loop exact model (F_T^e, u_T) satisfies (1.21) with $\alpha_i(s) = a_i s$ for $i = 1, 2$ and $\alpha_3(s) = \frac{a_3}{2}s$. ■

The condition (1.27) quantifies the mismatch between the exact and approximate closed-loop models and similar conditions are named *consistency* in the numerical analysis literature [59]. Note that (1.27) is not easy to use since we need to first design u_T to check it. Hence, it would be better if a condition involving the (open-loop) exact (1.8) and approximate (1.9) models is used. This condition is now stated.

Proposition 1.4. *Suppose there exist $\rho_1 \in \mathcal{K}_\infty$, $K > 0$ and $T^* > 0$ such that for all $T \in (0, T^*)$ and all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ the following conditions hold.*

$$|F_T^e(x, u) - F_T^a(x, u)| \leq T\rho_1(T)[|x| + |u|] \tag{1.30}$$

and

$$|u| := |u_T(x)| \leq K\,|x| \ . \tag{1.31}$$

Then condition (1.27) holds for all $T \in (0, T^*)$, with $\rho(s) := \rho_1(s) \cdot [1 + K]$. ■

We emphasize that the conditions (1.30) and (1.31) are easier to use than (1.27).

Combining the statements of Propositions 1.3 and 1.4, these results can be paraphrased as follows. The exact model (F_T^e, u_T) is exponentially stable if the following conditions hold.

1. Lyapunov exponential stability of (F_T^a, u_T) with a globally Lipschitz Lyapunov function (i.e. (1.24) and (1.26)).
2. Consistency between the approximate F_T^a and exact F_T^e models of the open-loop systems (i.e. (1.30)).
3. Uniform boundedness of control law u_T with respect to small T (i.e. (1.31)).

We emphasize that all of the above conditions can be checked without knowing the explicit expression of F_T^e. Indeed, it is obvious that the first and the third conditions only use the knowledge of F_T^a and u_T. The second condition is defined using F_T^e but we note that we do not need to know F_T^e in order to verify that the bound (1.30) holds. Indeed, we can state the following result.

Proposition 1.5. *Suppose that the system (1.5) is globally Lipschitz and* $f(0,0) = 0$. *Suppose, moreover, that* F_T^a *is consistent with* F_T^{Euler} *defined in (1.7). That is, there exists* $T^* > 0$ *and* $\rho_1 \in \mathcal{K}_\infty$ *such that for all* $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$ *we have*

$$\left|F_T^a(x,u) - F_T^{Euler}(x,u)\right| \le T\rho_1(T)[|x| + |u|] \ .$$

Then, F_T^e *is consistent with* F_T^a *in the sense of (1.30).* ■

Proof: First, we show that F_T^{Euler} is consistent with F_T^e under the given conditions on f. Indeed, since f is globally Lipschitz and zero at zero, we have $|f(x,u)| \le L(|x| + |u|)$ and the solution $\phi(t,x,u)$ of the system (1.5) starting from x with the constant control $u(t) \equiv u$ exists for all time, is unique and satisfies

$$|\phi(t,x,u)| \le \exp(Lt)\,|x| + (\exp(Lt) - 1)\,|u| \qquad \forall t \ge 0, \forall x, u \ .$$

Denote $\phi(t,x,u)$ shortly as $\phi(t)$. Then, using the above bound on $\phi(t)$ and the Lipschitzity of f we can write

$$
\begin{aligned}
&|F_T^e(x,u) - F_T^{Euler}(x,u)| \\
&\qquad = \left| x + \int_0^T f(\phi(s),u)ds - x - Tf(x,u) \right| \\
&\qquad = \left| \int_0^T [f(\phi(s),u) - f(x,u)]ds \right| \\
&\qquad \leq \int_0^T |f(\phi(s),u) - f(x,u)|\, ds \\
&\qquad \leq \int_0^T L\,|\phi(s) - x|\, ds \\
&\qquad = \int_0^T L \left| \int_0^s f(\phi(\tau),u)d\tau \right| ds \\
&\qquad \leq \int_0^T \int_0^s L\,|f(\phi(\tau),u)|\, d\tau ds \\
&\qquad \leq \int_0^T \int_0^s L[L\,|\phi(\tau)| + L\,|u|]d\tau ds \\
&\qquad \leq \int_0^T \int_0^T L[L\exp(LT)\,|x| + L(\exp(LT)-1)\,|u| + L\,|u|]d\tau ds \\
&\qquad = \frac{1}{2}T^2L^2\exp(LT)(|x| + |u|)\ ,
\end{aligned}
\tag{1.32}
$$

which completes the proof of consistency between F_T^e and F_T^{Euler}. Finally, by adding and subtracting F_T^{Euler} and using the triangular inequality, we obtain

$$
\begin{aligned}
|F_T^e(x,u) - F_T^a(x,u)| &= \left| F_T^e(x,u) - F_T^{Euler}(x,u) + F_T^{Euler}(x,u) - F_T^a(x,u) \right| \\
&\leq \left| F_T^e(x,u) - F_T^{Euler}(x,u) \right| + \left| F_T^{Euler}(x,u) - F_T^a(x,u) \right|
\end{aligned}
$$

and the conclusion immediately follows since F_T^e is consistent with F_T^{Euler} and by assumption F_T^a is consistent with F_T^{Euler}. ∎

Remark 1.3. The conditions in Propositions 1.3 and 1.4 provide a prescriptive framework for controller design via approximate models. Indeed, the first step in this approach is to pick F_T^a that is consistent with F_T^e in the sense of (1.30). Then, one would like to design a family of controllers of the form (1.10) that are bounded in the sense of (1.31) for the family of approximate models that satisfies the Lyapunov conditions (1.24) and (1.26). All of these conditions are checkable without knowing the explicit expression of F_T^e. Note that we do not say how one can design such controllers and that is why we refer to this framework as "prescriptive" rather than "constructive". However, we will show in Section 1.7 that one can obtain a variety of constructive procedures within this framework for certain classes of nonlinear systems, such as separable Hamiltonian systems and systems in strict feedback form.

1.6.2 Semiglobal Practical Stability

The purpose of this subsection is to present several definitions of stability and consistency that are more general than the ones in the previous subsection and use them to provide a more general framework for controller design via approximate models.

Semiglobal practical asymptotic stability property naturally arises when we relax the conditions of global Lipschitzity on f and GES for (F_T^a, u_T) that we used in the previous subsection. For simple illustration, we consider a parameterized family of discrete-time nonlinear systems

$$x(k+1) = F_T(x(k), u_T(x(k))) \ . \tag{1.33}$$

Semiglobal practical asymptotic stability and semiglobal practical asymptotic stability Lyapunov function for the system (1.33) are defined as follows.

Definition 1.1 (Semiglobal practical asymptotic (SPA) stability). *The family of systems (1.33) is SPA stable if there exists $\beta \in \mathcal{KL}$ such that for any strictly positive real numbers (Δ, δ) there exists $T^* > 0$ such that for all $T \in (0, T^*)$, all initial states $x(0) = x_\circ$ with $|x_\circ| \le \Delta$, the solutions of the system satisfy*

$$|x(k)| \le \beta(|x_\circ|, kT) + \delta, \quad \forall k \in \mathbb{N} \ . \tag{1.34}$$

■

Definition 1.2 (SPAS Lyapunov function). *A continuously differentiable function $V_T : \mathbb{R}^n \to \mathbb{R}$ is called SPAS Lyapunov function for the system F_T if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$ such that for any strictly positive real numbers (Δ_x, ν), there exist $L, T^* > 0$ such that for all $T \in (0, T^*)$ and for all $x, y \le \Delta_x$ and $T \in (0, T^*)$ the following holds.*

$$\underline{\alpha}(|x|) \le V_T(x) \le \overline{\alpha}(|x|) \ , \tag{1.35}$$

$$V_T(F_T(x, u_T(x))) - V_T(x) \le -T\alpha(|x|) + T\nu \tag{1.36}$$

$$|V_T(x) - V_T(y)| \le L\,|x - y| \tag{1.37}$$

In this case, we say that the pair (V_T, u_T) is Lyapunov SPA stabilizing for the system F_T. ■

We now state a more general notion of consistency.

Definition 1.3 (One-step consistency). *The family F_T^a is said to be one-step consistent with F_T^e if there exist functions $\rho, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$ such that given any strictly positive real numbers (Δ_x, Δ_u) there exists $T^* > 0$ such that, for all $T \in (0, T^*)$, $|x| \le \Delta_x$, $|u| \le \Delta_u$ we have*

$$|F_T^e(x, u) - F_T^a(x, u)| \le T\rho(T)[\varphi_1(|x|) + \varphi_2(|u|)] \ . \tag{1.38}$$

■

Definition 1.4. *The family of controllers u_T is bounded, uniformly in small T, if there exist $\kappa \in \mathcal{K}_\infty$ and for any $\Delta > 0$ there exists $T^* > 0$ such that for all $|x| \leq \Delta$ and $T \in (0, T^*)$ we have*

$$|u_T(x)| \leq \kappa(|x|) \ .$$

■

Using the above definitions, we can now state the following result.

Theorem 1.2. *Suppose the following conditions hold.*

1. *F_T^a is one-step consistent with F_T^e.*
2. *u_T is bounded, uniformly in small T.*
3. *There exists a SPAS Lyapunov function for the system (F_T^a, u_T).*

Then the system (F_T^e, u_T) is SPA stable and, hence, the sampled-data system (1.16) is SPA stable. ■

The statement of the above theorem is fully consistent with the result presented in the previous subsection but here we use much weaker (and hence more general) conditions that yield weaker conclusions. Hence, Theorem 1.2 is much more widely applicable than the results of the previous subsection.

Remark 1.4. Theorem 1.2 can be strengthened in different ways to either obtain global stability (as opposed to semiglobal) or to achieve local exponential stability. This can be done by combining stronger conditions in Proposition 1.3 with conditions in Theorem 1.2.

Remark 1.5. While conditions of Theorem 1.2 are sufficient (not necessary in general), they are tight in the sense that if we try to relax any of them, then we can find a counterexample where the exact closed-loop is not stabilized for small T. Example 1 and Example 2 in Section 1.4 can be used to illustrate this.

Indeed, Lyapunov SPA stability of the approximate closed-loop implies via Proposition 1.2 that[9] the approximate closed-loop system is SPA stable in the sense of Definition 1.1. This rules out two of the non-robustness indicators shown in Examples 1 and 2: non-uniform overshoot and non-uniform convergence rate. Moreover, the second condition in Theorem 1.2 requires uniform boundedness of the control law in small T. Hence, conditions of Theorem 1.2 rule out all indicators of non-robustness that we observed in Section 1.4. Another example that shows the need for the use of continuous Lyapunov function is presented in [43].

[9] Actually, we need a slightly more general statement than Proposition 1.2 that can be found in [43, 45].

Remark 1.6. Various extensions and variations of Theorem 1.2 have been published in the literature. First, alternative proofs that do not require the knowledge of a Lyapunov function and use SPA stability of closed-loop approximate model can be found in [45] for time invariant systems and in [40] for time-varying systems. These results use a slightly different notion of consistency than the one given in Definition 1.3. A framework for achieving input-to-state stability (ISS) and integral input-to-state stability (iISS) for systems with exogenous inputs via approximate discrete-time models can be found in [38] and [35], respectively. Moreover, similar results for sampled-data differential inclusions are presented in [43].

1.7 Controller Design Within the Framework

In this section, we present several simple design tools via approximate discrete-time models that rely on the framework presented in Section 1.6. We emphasize that any techniques for continuous time controller design can be revisited within our framework and new control laws will be obtained as a result (e.g. see the backstepping design in Subsection 1.7.4).

In Subsection 1.7.1 we show that emulation of continuous time controllers can be regarded as a special case of controller design that fits within our framework. In this case, we design a continuous-time controller $u^{ct}(x)$ for the continuous-time plant and then implement

$$u(t) = u_T^{dt}(x(k)) \qquad t \in [kT, (k+1)T) \ , \tag{1.39}$$

where[10]

$$u_T^{dt}(x) = u^{ct}(x) \ , \tag{1.40}$$

i.e. the discrete-time controller is identical to the continuous time controller. Note that we can still think of emulation as a design via an approximate model (the continuous-time plant model).

Subsections 1.7.2 and 1.7.3 show that our framework can be used for continuous-time controller *redesign* for sampled-data implementation. In this case, we first design a continuous-time controller $u^{ct}(x)$ for the continuous-time plant model (ignoring sampling) and then in the second step we parameterize the controller in the following manner:

$$u_T^{dt}(x) = u^{ct}(x) + \sum_{i=1}^{M} T^i u_i \ , \tag{1.41}$$

where $M \geq 1$ is a fixed integer and then we use an approximate model, such as the Euler model, to design $u_i = u_i(x)$. This *redesign* of continuous-time controllers can be directed to achieve different objectives and we will present two

[10] We introduce u_T^{dt} to be able to compare emulation with other design techniques.

cases of this controller redesign technique. In Subsection 1.7.2 the Lyapunov function for the continuous-time closed-loop is used as a *control Lyapunov function* for the approximate discrete-time model, assuming the redesigned controller follows the form (1.41). After substituting the term u^{ct} that is known from (1.40), the extra terms u_i's are regarded as new controls. Once the u_i's have been computed, the controller (1.41) is implemented. In Subsection 1.7.3, controller redesign is done starting from a passivity based design for a class of Hamiltonian systems namely the interconnection and damping assignment − passivity based control (IDA-PBC) design method. The modified energy function of the system is used as control Lyapunov function and design is carried out in a similar way as in Subsection 1.7.2.

Backstepping based on the Euler approximate model is presented in Subsection 1.7.4. In this case, we do not design a continuous controller as a first step in design/redesign but rather we use the Euler approximate model directly to design $u_T^{dt}(x)$ using our framework and then implement it using (1.39). It is interesting to observe that although we *do not assume* that the controller has the form (1.41), we show in our example that the obtained controller has the form (1.41) where $u^{ct}(x)$ is a controller that could be obtained using a continuous-time backstepping design (but we do not need to design it first). Moreover, we show in our example that in simulations $u_T^{dt}(x)$ performs better than the emulated $u^{ct}(x)$.

1.7.1 Emulation

Suppose that a static state feedback controller $u = u^{ct}(x)$ has been designed for the continuous-time system (1.5) ignoring sampling, so that there exists a smooth Lyapunov function V satisfying the following conditions:

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{1.42}$$

$$\frac{\partial V}{\partial x} f(x, u^{ct}(x)) \le -\alpha_3(|x|) \ , \tag{1.43}$$

with $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$. These conditions guarantee GAS of the continuous-time closed-loop system (f, u^{ct}). Suppose also that $u^{ct}(x)$ is bounded on compact sets of the state space. Then, suppose that the controller is "emulated" using (1.40). Suppose, moreover that the sampled-data system

$$\dot{x}(t) = f(x(t), u^{ct}(x(k))) \qquad t \in [kT, (k+1)T) \ , \tag{1.44}$$

has solutions that are well defined[11] for all initial conditions $x(0) = x_\circ \in \mathbb{R}^n$ and all $t \in [0, T]$. Denote $F_T^{Euler} := x(k) + Tf(x(k), u^{ct}(x(k)))$.

[11] Typically, for locally Lipschitz f the solutions would be defined only in a semiglobal sense, i.e. for any bounded set of initial conditions there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and all initial conditions from the set we have that the solutions are well defined for $t \in [0, T]$.

We will show next that the sampled-data system (1.44) is stable in an appropriate sense under appropriate conditions. In particular, we can state the following result.

Theorem 1.3. *Suppose that we have found a (locally bounded) controller $u^{ct}(x)$ and a smooth $V(x)$ that satisfy (1.42) and (1.43). Then, (V, u^{ct}) is a Lyapunov SPA stabilizing pair for F_T^{Euler}. Hence, (V, u^{ct}) is a Lyapunov SPA stabilizing pair for F_T^e, consequently, the sampled-data system (1.44) is SPA stable.* ∎

Proof: We first prove that (V, u^{ct}) is a Lyapunov SPA stabilizing pair for the Euler model of the system (1.7). Adding and subtracting $\frac{V(F_T^{Euler})-V(x)}{T}$ to (1.43) and using the Mean Value Theorem twice, we obtain

$$\begin{aligned}
&\frac{V(F_T^{Euler})-V(x)}{T} \\
&\quad \le -\alpha_3(|x|) + \frac{V(F_T^{Euler})-V(x)}{T} - \frac{\partial V}{\partial x}(x) f(x, u^{ct}(x)) \\
&\quad = -\alpha_3(|x|) + \left[\frac{\partial V}{\partial x}(x + \theta_1 T f(x, u^{ct}(x))) - \frac{\partial V}{\partial x}(x)\right] f(x, u^{ct}(x)) \\
&\quad \le -\alpha_3(|x|) + \left|\frac{\partial V}{\partial x}(x + \theta_1 T f(x, u^{ct}(x))) - \frac{\partial V}{\partial x}(x)\right| \cdot \left|f(x, u^{ct}(x))\right| \\
&\quad \le -\alpha_3(|x|) + \theta_1 T \left|\frac{\partial^2 V}{\partial x^2}(x + \theta_2 T f(x, u^{ct}(x)))\right| \cdot \left|f(x, u^{ct}(x))\right|^2 \\
&\quad \le -\alpha_3(|x|) + T\kappa(|x|) \ ,
\end{aligned}$$

where $\theta_1, \theta_2 \in (0,1)$, $\kappa \in \mathcal{K}_\infty$, we assumed that T is bounded and the first and the second derivatives of V are continuous (V is smooth). Hence, (V, u^{ct}) is a Lyapunov SPA stabilizing pair for F_T^{Euler}. Note that F_T^{Euler} is one-step consistent with F_T^e and $u^{ct}(x)$ is assumed to be bounded on compact sets and, hence, bounded uniformly in small T (since $u^{ct}(x)$ is independent of T). Since V has continuous first derivative, it is locally Lipschitz and we can conclude in a similar manner like in the proof of Proposition 1.3 that

$$\frac{V(F_T^e)-V(x)}{T} \le -\alpha_3(|x|) + T\kappa_1(|x|) \ , \tag{1.45}$$

for some $\kappa_1 \in \mathcal{K}_\infty$. Hence, (V, u^{ct}) is a Lyapunov SPA stabilizing pair for the exact model F_T^e. Finally, we conclude that the sampled-data system (1.44) is SPA stable from Theorem 1.2. ∎

Remark 1.7. The analysis given above can be carried out with more generality and one can prove that emulation leads to preservation of arbitrary dissipation inequalities in an appropriate sense (see [29] for more details).

The following example will be used to illustrate all our controller design and redesign methods. The reason for considering this simple system in strict feedback form is that we can use backstepping to systematically design a control law and a Lyapunov function that are needed to apply our framework.

Example 1.3. Consider the continuous-time plant

$$\begin{aligned} \dot{\eta} &= \eta^2 + \xi \\ \dot{\xi} &= u \ . \end{aligned} \tag{1.46}$$

We design a continuous-time backstepping controller [24]. Note that the first subsystem can be stabilized with the "control" $\phi(\eta) = -\eta^2 - \eta$ with the Lyapunov function $W(\eta) = \frac{1}{2}\eta^2$. Using this information and applying [24, Lemma 2.8 with c=1], we obtain

$$u^{ct}(\eta, \xi) = -2\eta - \eta^2 - \xi - (2\eta + 1)(\xi + \eta^2) \ , \tag{1.47}$$

which globally asymptotically stabilizes the continuous-time system (1.46) and moreover

$$V(\eta, \xi) = \frac{1}{2}\eta^2 + \frac{1}{2}(\xi + \eta + \eta^2)^2 \tag{1.48}$$

is a Lyapunov function for the continuous-time closed-loop system. Hence, we conclude from Theorem 1.3 that the sampled-data system (1.44) is SPA stable. Simulations for the sampled-data system with the emulated controller are presented in Subsection 1.7.4 and a comparison to other controllers obtained in the sequel is presented.

1.7.2 Continuous-Time Controller Redesign

In this subsection we illustrate the Lyapunov based redesign and we refer to [36] for more details. We assume that a continuous-time controller

$$u = u^{ct}(x) \tag{1.49}$$

has been designed and a Lyapunov function V satisfying (1.42), (1.43) was found for the closed-loop continuous-time system. Suppose that we want to implement a controller of the form (1.41) and we want to further design u_i so that the controller is "better" in some sense than u^{ct}. For simplicity, let us assume that

$$u_T^{dt}(x) := u^{ct}(x) + T u_1(x) \ ,$$

and $u_1(x)$ is a new control input that we want to design (i.e. we redesign $u^{ct}(x)$). We do that by using the continuous-time Lyapunov function V as a control Lyapunov function for an approximate discrete-time model F_T^a that is one step consistent with the exact model F_T^e. That is, we consider

$$\frac{V(F_T^a(x, u^{ct}(x) + Tu_1(x))) - V(x)}{T} \ ,$$

where F_T^a is one step consistent with F_T^e, and u^{ct} and V were obtained from an arbitrary continuous-time design. There are different possible objectives that one may try to achieve by designing u_1 and we discuss here one obvious choice. Let us first note that we can easily compute

$$\frac{V(F_T^a(x, u^{ct}(x))) - V(x)}{T} \ .$$

One way to design u_1 is to require that

$$\frac{V(F_T^a(x, u^{ct}(x) + Tu_1(x))) - V(x)}{T} < \frac{V(F_T^a(x, u^{ct}(x))) - V(x)}{T} \ . \qquad (1.50)$$

In other words, we can design u_1 to achieve more decrease for the Lyapunov function along solutions of the closed-loop approximate model with the redesigned controller (see [36]). However, not all Lyapunov functions that satisfy (1.42) and (1.43) are appropriate for doing the redesign with the aim of achieving the objective (1.50). Indeed, increasing the rate of convergence in this way may lead to increasing the overshoots for some Lyapunov functions, which is highly undesirable (see [36, Example 4.1]). To avoid creating unacceptable overshoots in this manner, we need to assume that V is "well behaved", that is the overshoot estimates that can be obtained using V for the closed-loop system are acceptable (see [36, Assumption 2.2]). We acknowledge that finding an appropriate V that satisfies this assumption is difficult in general. With this assumption, the above described redesign will yield acceptable overshoots while it will typically improve the rate of convergence of the approximate and sampled data closed-loop systems.

Finally, note that if $u_1 = u_1(x)$ is designed to satisfy (1.50) and it is bounded on compact sets, then we can conclude from our Theorem 1.2 that the sampled-data system with the redesigned controller is SPA stable. We revisit Example 1.3 to illustrate this approach.

Example 1.4. Consider the system in Example 1.3 and assume that we have already designed the controller (1.47) and found the Lyapunov function (1.48). Assume that the plant (1.46) is between a sampler and a zero-order-hold and let us use for redesign, its Euler approximate model

$$\begin{aligned} \eta(k+1) &= \eta(k) + T(\eta^2(k) + \xi(k)) \\ \xi(k+1) &= \xi(k) + Tu(k) \ . \end{aligned} \qquad (1.51)$$

Denote $x := (\eta \ \xi)^T$. Suppose for simplicity that $u^{dt}(x) = u^{ct}(x) + Tu_1(x)$ and it is then not hard to compute

$$\frac{V(F_T^{Euler}(x, u^{ct}(x) + Tu_1)) - V(x)}{T} = -\eta^2 - (\xi + \eta + \eta^2)^2 + Tp_1(u_1, x) + O(T^2)$$

where

$$p_1(u_1,x)=\frac{1}{2}(\eta^2+\xi)^2+(\xi+\eta+\eta^2)(u_1+(\eta^2+\xi)^2)+\frac{1}{2}(2\eta+\eta^2+\xi)^2\ , \quad (1.52)$$

and $O(T^2)$ contains higher order terms in T. Since T will have to be chosen small, we neglect $O(T^2)$ and we chose u_1 so that the term $p_1(u_1,x)$ is made more negative (note that there are some terms in p_1 that can not be made negative using u_1). One obvious choice is

$$u_1(x)=-(\eta^2+\xi)^2-(\xi+\eta+\eta^2)\ , \quad (1.53)$$

which cancels one term and then provides extra damping to yield

$$p_1(u_1(x),x)=\frac{1}{2}(\eta^2+\xi)^2-(\xi+\eta+\eta^2)^2+\frac{1}{2}(2\eta+\eta^2+\xi)^2\ .$$

We will simulate this controller in the next subsection and make some comparisons with other designs.

1.7.3 Discrete-Time Interconnection and Damping Assignment – Passivity Based Control (IDA-PBC)

In this subsection, the second tool for continuous-time controller redesign is discussed. While in Subsection 1.7.2 we consider general nonlinear system, now we consider a class of nonlinear system namely Hamiltonian systems. The technique used for the controller design is a type of passivity based control design known as IDA-PBC.

IDA-PBC design is a powerful tool for solving the stabilization problem for Hamiltonian systems [47, 48, 50]. Although IDA-PBC design is applicable to a broader class of systems (see [1, 49, 50]), it applies naturally to Hamiltonian systems due to the special structure of this class of systems.

Consider continuous-time Hamiltonian systems whose dynamics can be written as

$$\begin{bmatrix}\dot q\\ \dot p\end{bmatrix}=\begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}\begin{bmatrix}\nabla_q H\\ \nabla_p H\end{bmatrix}+\begin{bmatrix}0\\ G(q)\end{bmatrix}u\ , \quad (1.54)$$

where $p\in\mathbb{R}^n$ and $q\in\mathbb{R}^n$ are the states, and $u\in\mathbb{R}^m$, $m\le n$, is the control action. The matrix $G(q)\in\mathbb{R}^{n\times m}$ is determined by the way control u enters the system. The function $H(q,p)$ is called the Hamiltonian function of the system, and is defined as the sum of the kinetic energy $K(q,p)$ and the potential energy $P(q)$[12], i.e.

[12] Note that in many references the potential energy is commonly denoted with V. However, we use the notation P instead, to avoid confusion with the notations V and V_T that we have used to denote Lyapunov functions.

$$H(q,p) = K(q,p) + P(q) = \frac{1}{2}p^\top M^{-1}(q)p + P(q) \ , \tag{1.55}$$

where $M(\cdot)$ is the symmetric inertia matrix.

We consider a simple case when system (1.54) is a *separable Hamiltonian system*. For this class of systems, the inertia matrix M is constant, and hence the kinetic energy and the potential energy of the system are decoupled, i.e.

$$H(q,p) = K(p) + P(q) = \frac{1}{2}p^\top M^{-1}p + P(q) \ . \tag{1.56}$$

We also consider only fully actuated systems, i.e when $G(q)$ is full rank ($m = n$). In this setting, $\nabla_q H(q,p) = \nabla_q P(q)$ and $\nabla_p H(q,p) = M^{-1}p$. The idea of IDA-PBC design is to construct a controller for system (1.54) so that the stabilization is achieved assigning a desired energy function

$$H_d(q,p) = K_d(p) + P_d(q) = \frac{1}{2}p^\top M_d^{-1}p + P_d(q) \ , \tag{1.57}$$

that has an isolated minimum at the desired equilibrium point $(q^e, 0)$ of the closed-loop system. IDA-PBC design consists of two steps. First, design the energy shaping controller u_{es} to shape the total energy of the system to obtain the target dynamics; second, design the damping injection controller u_{di} to achieve asymptotic stability. Hence, an IDA-PBC controller is of the form

$$u = u_{es}(q,p) + u_{di}(q,p) \ . \tag{1.58}$$

The energy shaping controller u_{es} is obtained by solving the equation

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u_{es} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \ . \tag{1.59}$$

The first row of (1.59) is directly satisfied, and the second row can be written as

$$G u_{es} = \nabla_q H - M_d M^{-1} \nabla_q H_d \ . \tag{1.60}$$

Since we consider G full rank (and hence invertible), u_{es} is obtained as

$$u_{es} = G^{-1}(\nabla_q H - M_d M^{-1} \nabla_q H_d) \ . \tag{1.61}$$

Moreover, the damping injection controller u_{di} is constructed as

$$u_{di} = -k_v G^\top \nabla_p H_d = -k_v G^\top M_d^{-1} p, \quad k_v > 0 \ . \tag{1.62}$$

For more details and more general results about IDA-PBC design for continuous-time systems, we refer to [48, 49, 50].

In this subsection, we present a discrete-time IDA-PBC controller redesign to obtain a discrete-time IDA-PBC controller from a controller that is first obtained via continuous-time design. This redesign is based on the Euler approximate model of system (1.54), namely

$$\begin{aligned} q(k+1) &= q(k) + T\nabla_p H(q(k),p(k)) \\ p(k+1) &= p(k) - T\Big(\nabla_q H(q(k),p(k)) - Gu(k)\Big) \ . \end{aligned} \tag{1.63}$$

Suppose all conditions of the continuous-time design hold, and we have assigned the desired energy function (1.57) for the system. As in Subsection 1.7.2, we assume the Lyapunov function to be well behaved [36] and we are now ready to state the following theorem.

Theorem 1.4. *Consider the Euler model (1.63) of the separable Hamiltonian system (1.54) with Hamiltonian (1.56) and matrix G invertible. Suppose the inertia matrix M is diagonal and the desired potential energy V_d is positive definite. Then the discrete-time controller $u^T = u^T_{es} + u^T_{di}$ where*

$$u^T_{es} = G^{-1}\Big(\nabla_q H(q,p) - M_d M^{-1}\overline{\nabla}_q H_d(q,p)\Big) \tag{1.64}$$

$$u^T_{di} = -k_v G^\top \nabla_p H_d(q,p) = -k_v G^\top M_d^{-1} p \ , \tag{1.65}$$

with $k_v > 0$ and $\overline{\nabla}_q H_d(q,p) = \nabla_q H_d(q,p) + T\kappa L_v M^{-1}p$, where $\kappa > 0$ and $L_v = \nabla_{qq} P_d(q) \geq 0$ is a SPA stabilizing controller for the Euler model (1.63). Moreover, there exists a function

$$V(p,q) = H_d(p,q) + \epsilon p^\top q \ , \tag{1.66}$$

with $\epsilon > 0$ sufficiently small which is a SPAS Lyapunov function for the system (1.63), (1.64), (1.65). ∎

Remark 1.8. It is known that Euler approximation is not Hamiltonian conserving. To avoid confusion about the motivation of using this method in our construction we emphasize that IDA-PBC design does not involve Hamiltonian conservation as in the numerical analysis context and we need to distinguish these two different issues. Constructing u_{es} is not aimed to conserve the Hamiltonian of the system, but to transform the system to another Hamiltonian system by using feedback and shaping the energy of the system (defining the desired Hamiltonian). Therefore, the use of Euler approximation in this context is justified.

From the construction of the controller (1.64), it is obvious that the discrete-time controller is a modification of the controller obtained by emulation of the continuous-time IDA-PBC controller, with the extra term

$$\begin{aligned} Tu_1 &= -G^{-1}M_d M^{-1}\Big(\overline{\nabla}_q H_d(q,p) - \nabla_q H_d(q,p)\Big) \\ &= -TG^{-1}M_d M^{-1}\kappa L_v M^{-1} p \ . \end{aligned} \tag{1.67}$$

Moreover, assuming that $\epsilon > 0$ is of order T, the contribution of the extra term (1.67) to the Lyapunov difference is

$$\Delta V = -T^2 p^\top \overline{M} p + O(\epsilon T^2) + O(T^3) = -T^2 p^\top \overline{M} p + O(T^3) \ , \tag{1.68}$$

with $\overline{M} := \kappa M^{-1} L_v M^{-1}$ positive semidefinite. Therefore, it is guaranteed that for $\epsilon > 0$ and $T > 0$ sufficiently small, the Lyapunov difference with the discrete-time redesigned controller is more negative than it is with the emulation of the continuous-time controller.

Remark 1.9. It is obvious that this IDA-PBC redesign construction follows the approximate based design framework presented in Section 1.6. The setting we presented in this subsection is a simple illustration when a strict Lyapunov function for Hamiltonian system can be constructed in a systematic way. In a more general situation, especially for the case of underactuated control [27], finding a strict Lyapunov function is still an open problem.

Example 1.5. Consider the nonlinear pendulum shown in Figure 1.2, which is a separable Hamiltonian system with dynamic model given as

$$\dot{q} = p, \quad \dot{p} = -\sin(q) + u \ . \tag{1.69}$$

The Hamiltonian of this system is

$$H = K(p) + P(q) = \frac{1}{2}p^2 - \cos(q) \ , \tag{1.70}$$

and the equilibrium point to be stabilized is the origin. By choosing $M_d = M = I$ and

$$P_d = -\cos(q) + \frac{k_1}{2}q^2 + 1 \ , \quad k_1 \geq 1 \ ,$$

the desired energy function of the system is

$$H_d = K_d(p) + P_d(q) = \frac{1}{2}p^2 - \cos(q) + \frac{k_1}{2}q^2 + 1 \ . \tag{1.71}$$

Applying (1.61) and (1.62), the continuous-time energy shaping and the damping injection controller for system (1.69) are obtained as

$$u_{es}(t) = \nabla_q H - M_d M^{-1} \nabla_q H_d = -k_1 q \ , \tag{1.72}$$

$$u_{di}(t) = -k_v G^\top \nabla_p H_d = -k_v p, \quad k_v > 0 \ . \tag{1.73}$$

Choose the Lyapunov function as

$$V(q,p) = H_d(p,q) + \epsilon q p \tag{1.74}$$

with $\epsilon > 0$ sufficiently small. The Lyapunov derivative is obtained as

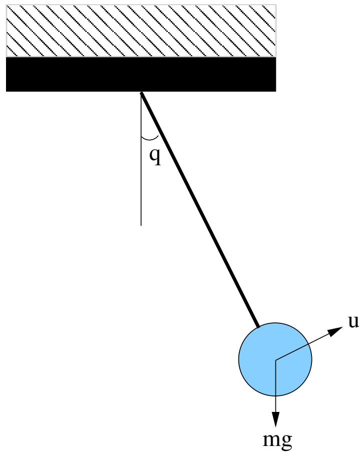

Fig. 1.2. Nonlinear pendulum

$$\begin{aligned}\dot{V}(q,p) &= \dot{H}_d(p,q) + \epsilon(\dot{q}p + q\dot{p}) \\ &= -k_v p^2 + \epsilon(p^2 - q\sin(q) - k_1 q^2 - k_v qp) \\ &\le -(k_v - \epsilon(1+\frac{1}{2}k_v))p^2 - \epsilon q \sin(q) - \epsilon(k_1 - \frac{1}{2}k_v)q^2 \ .\end{aligned} \tag{1.75}$$

By choosing k_v and k_1 appropriately, it can be shown that V is a strict AS Lyapunov function for the system (1.69), (1.72), (1.73). Moreover, using Theorem (1.3) we can conclude that the emulation controller $u(k) := u_{es}(k) + u_{di}(k)$ obtained by sample and hold of the continuous-time controller $u(t)$ is a SPA stable controller for the plant (1.69).

Now we redesign the controller (1.72) using Theorem 1.4. Applying (1.64) yields

$$\begin{aligned}u^T_{es}(k) &= \nabla_q H - M_d M^{-1}(\nabla_q H_d + T\kappa L_v M^{-1} p) \\ &= -k_1 q - T\kappa(\cos(q) + k_1)p \ ,\end{aligned} \tag{1.76}$$

and (1.65) gives $u^T_{es}(k) = -k_v p$. Applying the discrete-time controller

$$u^T(k) := u^T_{es}(k) + u^T_{di}(k) \tag{1.77}$$

and using the same Lyapunov function (1.74) as in the continuous-time case, we obtain the Lyapunov difference

$$\begin{aligned}\Delta V &:= V(q(k+1), p(k+1)) - V(q(k), p(k)) \\ &\le -T\left((k_v - \epsilon(1+\frac{1}{2}k_v))p^2 + \epsilon q\sin(q) + \epsilon(k_1 - \frac{1}{2}k_v)q^2\right) \\ &\quad - T^2\kappa(k_1 + \cos(q))p^2 + O(T^2) \ .\end{aligned} \tag{1.78}$$

By choosing k_v, k_1 and κ appropriately, we can show that for sufficiently small $T > 0$ and $\epsilon > 0$, V is a strict SPA Lyapunov function for the Euler model with discrete-time controller.

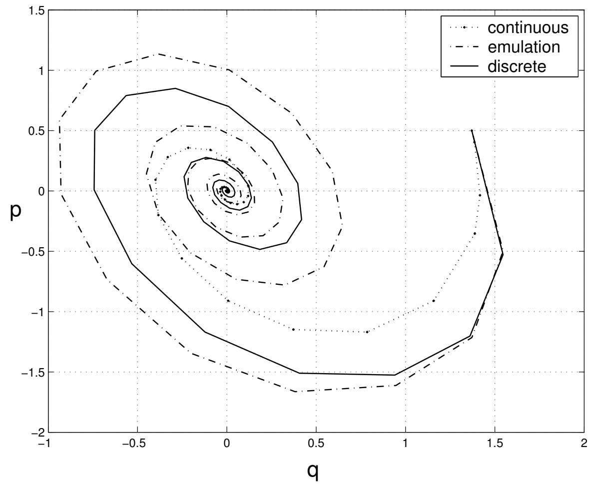

Fig. 1.3. Response of the nonliner pendulum

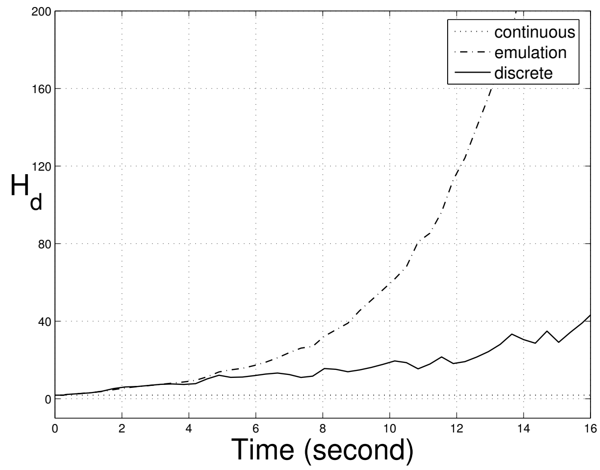

Fig. 1.4. The desired energy function H_d with $k_v = 0$.

Taking the trajectory of the continuous-time system as reference, Figure 1.3 shows that applying (1.77) keeps the trajectory of the closed-loop system closer to the reference than using the emulation controller. In the simulation we have used the initial state $(q_\circ, p_\circ) = (\pi/2 - 0.2, 0.5)$, $k_1 = 1$, $k_v = 1$ and $T = 0.35$. Figure 1.4 displays the desired Hamiltonian function when applying only the energy shaping controller to the plant. In continuous-time IDA-PBC, $u_{es}(t)$ conserves the Hamiltonian in closed-loop and hence the closed-loop system is *critically* stable. Applying the emulation controller $u_{es}(k)$ immediately destroys closed-loop stability. On the other hand, the discrete-time controller $u_{es}^T(k)$ tries to recover Hamiltonian conservation, making the closed-loop system *less unstable* than with $u_{es}(k)$.

Applying each controller to the Euler model of (1.69) and then computing the difference of the Lyapunov differences, we obtain that

$$\Delta V^{u_{es}^T} - \Delta V^{u_{es}} = -T^2\kappa(k_1+\cos(q))p^2 - \epsilon T^2\kappa(k_1+\cos(q))qp + O(T^3) \ . \tag{1.79}$$

Suppose that ϵ is of order T, then we can write

$$\Delta V^{u_{es}^T} - \Delta V^{u_{es}} = -T^2\kappa(k_1+\cos(q))p^2 + O(T^3) \ , \tag{1.80}$$

which shows that for $\epsilon > 0$ and $T > 0$ sufficiently small, $\Delta V^{u_{es}^T}$ is more negative than $\Delta V^{u_{es}}$ in a practical sense. This explains why the discrete-time controller performs better than the emulation controller.

1.7.4 Backstepping via The Euler Model

Backstepping is a systematic controller design technique for a special class of nonlinear systems in feedback form [24]. The goal is to exploit the special structure of the system to systematically construct a control law u_T for the Euler approximate model of the system and a Lyapunov function V_T that satisfy all conditions of Theorem 1.2 in Section 1.6. Results of this subsection are based on [41].

We consider discrete-time backstepping design based on the Euler model of the system since the Euler model preserves the strict feedback structure of the continuous-time system that is needed in backstepping. Consider the continuous-time system

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\xi \\ \dot{\xi} &= u \ . \end{aligned} \tag{1.81}$$

The Euler approximate model of (1.81) has the following form.

$$\eta(k+1) = \eta(k) + T\left[f(\eta(k)) + g(\eta(k))\xi(k)\right] \tag{1.82}$$

$$\xi(k+1) = \xi(k) + Tu(k) \ . \tag{1.83}$$

The main result of this subsection is stated next.

Theorem 1.5. *Consider the Euler approximate model (1.82), (1.83). Suppose that there exists $\hat{T} \geq 0$ and a pair (α_T, W_T) that is defined for all $T \in (0, \hat{T})$ and that is a SPA stabilizing pair for the subsystem (1.82), with $\xi \in \mathbb{R}$ regarded as a control. Moreover, suppose that the pair (α_T, W_T) has the following properties.*

1. *α_T and W_T are continuously differentiable for any $T \in (0, \hat{T})$.*
2. *there exists $\tilde{\varphi} \in \mathcal{K}_\infty$ such that*
$$|\alpha_T(\eta)| \leq \tilde{\varphi}(|\eta|) \ . \tag{1.84}$$
3. *for any $\tilde{\Delta} > 0$ there exist a pair of strictly positive numbers $(\tilde{T}, \tilde{M})$ such that for all $T \in (0, \tilde{T})$ and $|\eta| \leq \tilde{\Delta}$ we have*
$$\max\left\{\left|\frac{\partial W_T}{\partial \eta}\right|, \left|\frac{\partial \alpha_T}{\partial \eta}\right|\right\} \leq \tilde{M} \ . \tag{1.85}$$

Then, there exists a SPA stabilizing pair (u_T, V_T) for the Euler model (1.82), (1.83). In particular, we can take

$$u_T = -c(\xi - \alpha_T(\eta)) - \frac{\widetilde{\Delta W}_T}{T} + \frac{\Delta\alpha_T}{T} \tag{1.86}$$

where $c > 0$ is arbitrary, and

$$\Delta\alpha_T := \alpha_T(\eta + T(f + g\xi)) - \alpha_T(\eta) \tag{1.87}$$

$$\widetilde{\Delta W}_T := \begin{cases} \frac{\overline{\Delta W}_T}{(\xi - \alpha_T(\eta))}, & \xi \neq \alpha_T(\eta) \\ T\frac{\partial W_T}{\partial \eta}(\eta + T(f + g\xi))g, & \xi = \alpha_T(\eta) \end{cases} \tag{1.88}$$

$$\overline{\Delta W}_T := W_T(\eta + T(f + g\xi)) - W_T(\eta + T(f + g\alpha_T)) \tag{1.89}$$

and the Lyapunov function is

$$V_T(\eta, \xi) = W_T(\eta) + \frac{1}{2}(\xi - \alpha_T(\eta))^2 \ .$$

■

Remark 1.10. The control law (1.86) is in general different from continuous-time backstepping controllers as the next example will illustrate. Interestingly, we show in the next example that our control law can be written in the form

$$u_T^{Euler}(x) = u^{ct}(x) + Tu_1^{Euler}(x) \ ,$$

where $u^{ct}(x)$ is a backstepping controller obtained from continuous-time backstepping. We show for the example that u_T^{Euler} yields better performance (better transients and larger domain of attraction) than the emulated backstepping controller $u^{ct}(x)$. While we observed this trend in simulations for any control law designed within our framework, we were unable to prove that this is true in general.

Remark 1.11. Not every backstepping controller that stabilizes the Euler model will stabilize the exact model. Indeed, the dead beat controller in our first motivating example in Section 1.4 can be obtained using backstepping and we saw that it was destabilizing the sampled-data system for all sampling periods T. This further illustrates the importance of using our framework for controller design via approximate discrete-time models.

Example 1.6. [41] We revisit the system in Example 1.3 but now we want to use Theorem 1.5 based on the Euler model (1.51) of the system (1.46).

Again, the control law $\phi(\eta) = -\eta^2 - \eta$ globally asymptotically stabilizes the η-subsystem of (1.51) with the Lyapunov function $W(\eta) = \frac{1}{2}\eta^2$. Using the construction in Theorem 1.5, we obtain the controller

$$u_T^{Euler}(\eta,\xi) = u^{ct}(\eta,\xi) + T u_1^{Euler} \ , \tag{1.90}$$

where $u^{ct}(x)$ is the same as in Examples 1.3 and 1.4 and the following u_1^{Euler} is obtained as

$$u_1^{Euler} = -\frac{1}{2}(\xi - \eta + \eta^2) - (\xi + \eta^2)^2 \ . \tag{1.91}$$

From Theorem 1.5 we see that u_T^{Euler} SPA stabilizes the Euler model (1.51). This can be proven with the Lyapunov function $V(\eta,\xi) = \frac{1}{2}\eta^2 + \frac{1}{2}(\xi+\eta+\eta^2)^2$. Hence, using Theorem 1.2 we conclude that the same controller SPA stabilizes the exact model and consequently the sampled-data system.

Next we compare the controller (1.90) with the controllers that were designed in Examples 1.3 and 1.4. First, note that the terms u_1 in (1.53) and u_1^{Euler} in (1.91) are different. Moreover, all controllers become the same and equal to $u^{ct}(x)$ for $T = 0$. Hence, it makes sense to compare the controllers for small T.

Figure 1.5 shows the time response of the system (1.81) when applying respectively the emulation controller, the redesigned controller and the Euler based discrete-time controller. The response using continuous-time controller is used as reference. In the simulation, we set $x_\circ = (2 \;\; 2)^\top$ and $T = 0.5$. It is shown that the emulation controller destabilizes the system, whereas the redesign and the Euler based controllers maintain the response of the system relatively close to the continuous-time response.

In Figure 1.6 we show the simulation result when we increase the initial state to $x_\circ = (400 \;\; 400)^\top$. We do not plot the response of the system with emulation controller since it is obviously unstable. Interestingly, with the redesign controller and the Euler based controller, the stability of the closed-loop

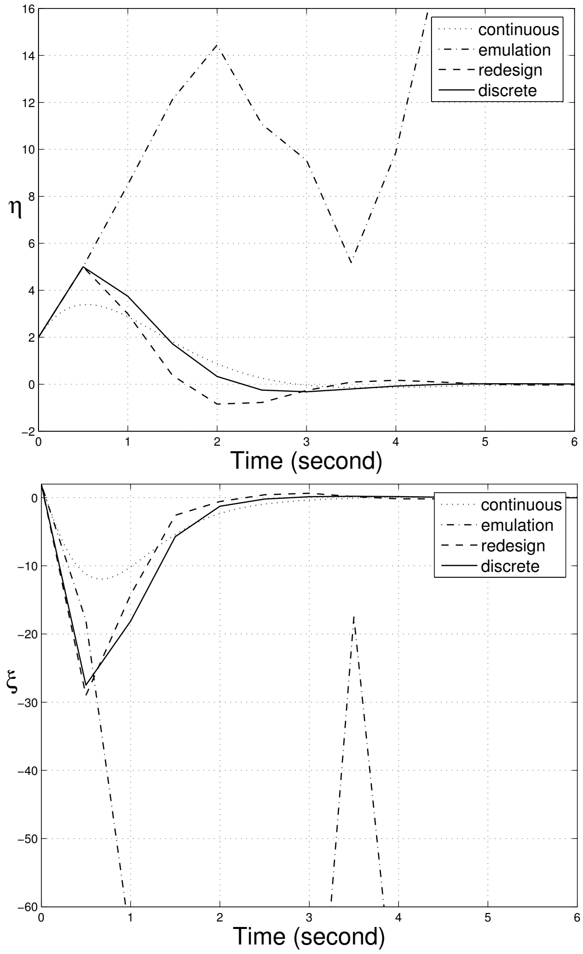

Fig. 1.5. Closed-loop responses of the system (1.81) for small initial states.

system is preserved although the initial state in this simulation is 200 time larger than the one used in Figure 1.5. In fact, for these two controllers, the stability is still maintained for larger initial states in any direction in the state space.

1.8 Design Examples

In this section two examples are presented to illustrate the various design tools we have discussed in Section 1.7. It will also be shown that the design fits

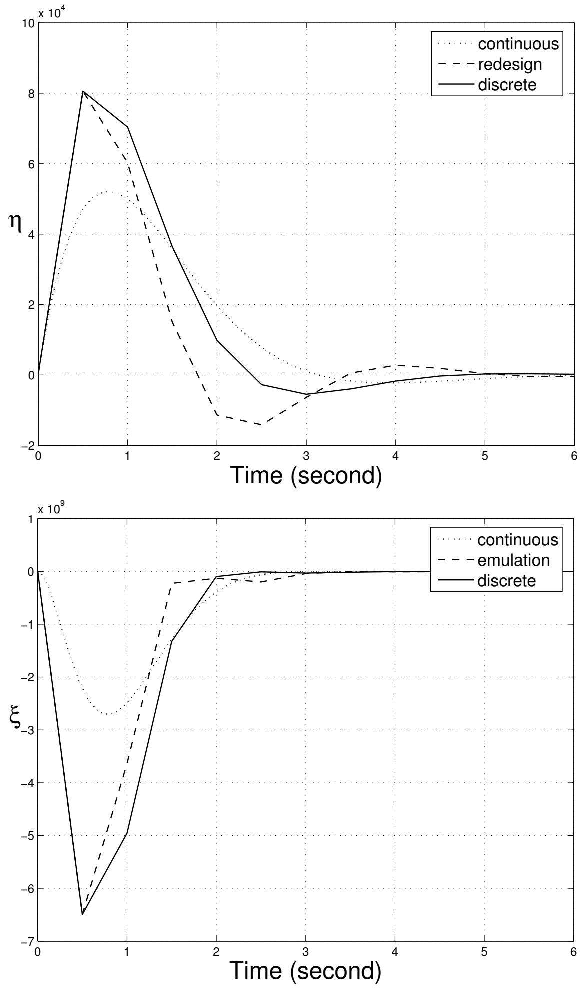

Fig. 1.6. Closed-loop response of the system (1.81) for large initial states.

with the framework proposed in Section 1.6. In the first example a jet engine system is considered, and the emulation and the Euler based backstepping design are applied to solve the stabilization problem of the jet engine. In the second example, a stabilization design for an inverted pendulum is studied. A backstepping design and an IDA-PBC design are applied to the system. Simulation results are presented to show the performance of each controller designed.

1.8.1 Jet Engine System

A simplified Moore-Greitzer model of a jet engine with the assumption of no stall is given by

$$\begin{aligned} \dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ \dot{x}_2 &= -u \ , \end{aligned} \tag{1.92}$$

where x_1 and x_2 are respectively related to the mass flow and the pressure rise through the engine after an appropriate change of coordinates (see [24] for more details). We will apply both the continuous-time and the Euler based backstepping design discussed in Subsection 1.7.4 to this system, and compare the performance of the controller obtained by the Euler based backstepping design with the one obtained by emulation of the continuous-time controller.

Choose $\phi(x_1) = -\frac{3}{2}x_1^2 + x_1$ and $W(x_1) = \frac{1}{2}x_1^2$. Applying [24, Lemma 2.8] and choosing $c = 1$, the continuous-time controller is obtained as

$$u^{ct}(x_1, x_2) = -x_1 + c(x_2 + \frac{3}{2}x_1^2 + \frac{1}{2}x_1^3) + (3x_1 - 1)(-x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3) \ . \tag{1.93}$$

Moreover, using the Euler approximate model of (1.92) and applying Theorem 1.5, we obtain the discrete-time Euler-based controller

$$u_T^{Euler}(x_1, x_2) = u^{ct}(x_1, x_2) + Tu_1(x_1, x_2) \ , \tag{1.94}$$

where

$$u_1(x_1, x_2) = \frac{1}{2}(x_2 + x_1 + \frac{3}{2}x_1^2 + x_1^3) \ .$$

We implement the controller (1.94) and the discrete-time emulation of (1.93) to control the continuous-time plant (1.92), comparing the performance. The simulation results with parameters $c = 1$, $x_\circ = (2,\ 3)^\top$ and $T = 0.2$ are illustrated in Figure 1.7.

It is shown that the Euler-based controller outperforms the emulation controller and for the chosen simulation parameters, it keeps the response of the closed-loop system close to the response of the continuous-time closed-loop system.

1.8.2 Inverted Pendulum

Consider a nonlinear dynamic model of an inverted pendulum as illustrated in Figure 1.8, namely

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \sin(q) + u \ . \end{aligned} \tag{1.95}$$

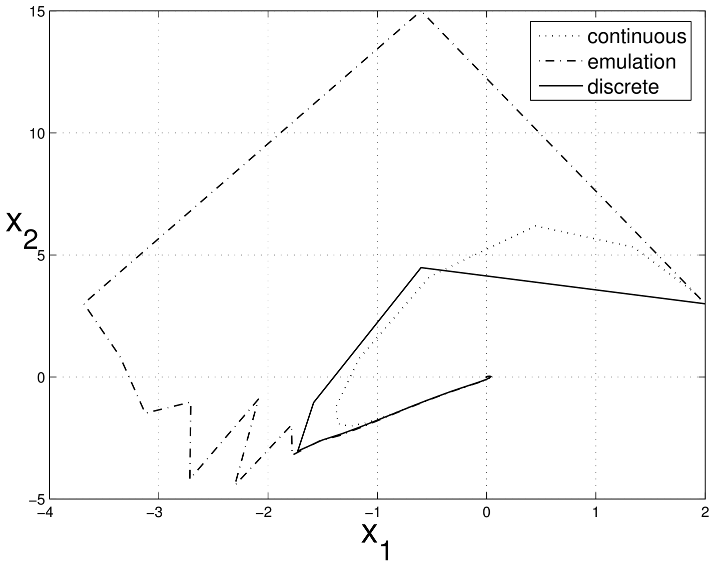

Fig. 1.7. Phase plot of the jet engine system.

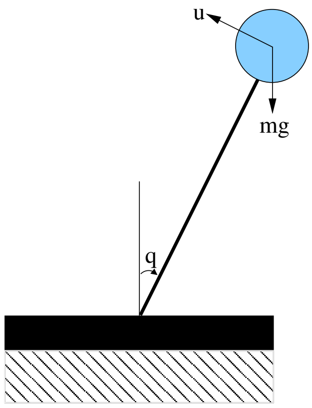

Fig. 1.8. Inverted pendulum

This dynamic model is in strict feedback form. This system also belongs to the class of separable Hamiltonian systems with the Hamiltonian function

$$H = \frac{1}{2}p^2 + \cos(q) \ . \tag{1.96}$$

Therefore, we can apply both the backstepping design and the IDA-PBC redesign to construct a stabilizing controller for this system. Note that $q = 0$

is the equilibrium point of this system and is an unstable equilibrium. The control design in this case is aiming at stabilizing this equilibrium point.

Backstepping Design

Choose $\phi(q) = -q$ and $W(q) = \frac{1}{2}q^2$. The continuous-time controller is obtained as

$$u^{ct}(q,p) = -\sin(q) - (1+c)(p+q) \ . \tag{1.97}$$

Applying Theorem 1.5, we design a discrete-time controller for the Euler approximate model of the system (1.95)

$$\begin{aligned} q(k+1) &= q(k) + Tp(k) \\ p(k+1) &= p(k) + T(\sin(q) + u) \ . \end{aligned} \tag{1.98}$$

We obtain the Euler based controller

$$u_T^{Euler}(q,p) = u^{ct}(q,p) + Tu_1(q,p) \ , \tag{1.99}$$

with

$$u_1(q,p) = -\frac{1}{2}(p-q) \ .$$

Implementing the controller (1.99) and the discrete-time emulation of (1.97) to the continuous-time plant (1.95), we compare the performance of the two controllers, using the continuous-time controller performance as reference. The simulation results with parameters $c = 1$, $(q_\circ,\ p_\circ) = (\frac{\pi}{2} - 0.2,\ \frac{1}{2})$ and $T = 0.5$ are displayed in Figure 1.9.

IDA-PBC Redesign

We apply the results discussed in Subsection 1.7.3 to design a stabilizing controller for the inverted pendulum (1.95). From the Hamiltonian (1.96) we have that $M = I$ and $P = \cos(q)$. To bring the energy to the minimum level at the equilibrium point, we assign a new energy function H_d for the pendulum, by keeping $M_d = M = I$ and choosing the new potential energy $P_d = -\cos(q) + \frac{1}{2}k_1q^2 + 1$. Hence,

$$H_d = \frac{1}{2}p^2 - \cos(q) + \frac{1}{2}k_1q^2 + 1 \ . \tag{1.100}$$

Using the continuous-time IDA-PBC design, we obtain the controller

$$u_{ct}(q,p) = u_{es}(q,p) + u_{di}(q,p) \ , \tag{1.101}$$

with

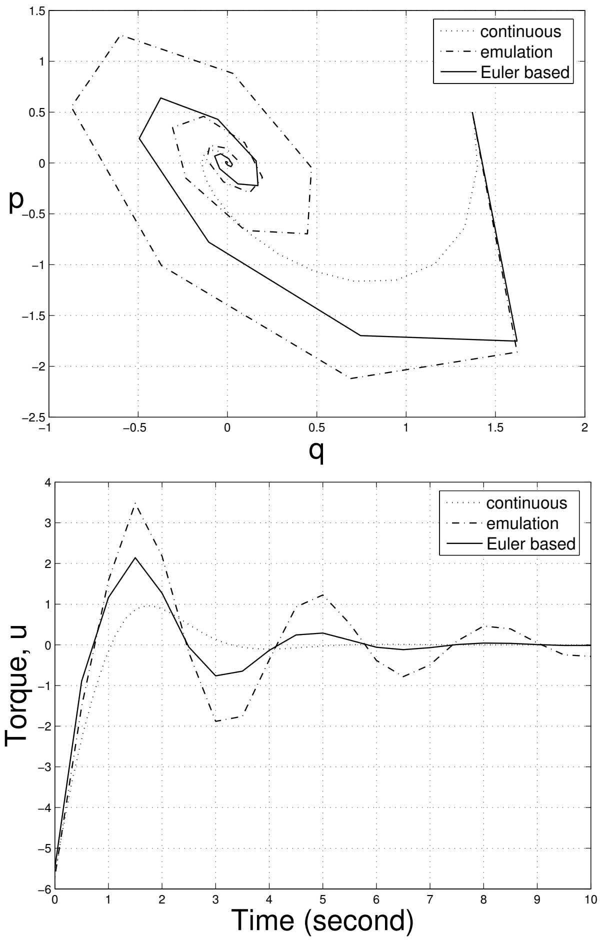

Fig. 1.9. Responses of the inverted pendulum with backstepping controllers.

$$u_{es}(q,p) = -2\sin(q) - k_1 q \tag{1.102}$$

$$u_{di}(q,p) = -k_v p \ . \tag{1.103}$$

With this controller, we obtain

$$\dot{H}_d = -k_v p^2 \ . \tag{1.104}$$

Utilizing La Salle Invariance Principle we can show that the closed-loop approximate model is asymptotically stable. Moreover, using Theorem 1.3, we can conclude that the discrete-time controller obtained by emulation of (1.101) is a SPA stabilizing controller for the inverted pendulum (1.95).

Moreover, using Theorem 1.4 we will redesign the emulation controller, to improve the performance of the system. Consider the Euler model (1.98) and applying (1.64) and (1.65), the redesigned controller is obtained as

$$\begin{aligned} u_{dt}^T(q,p) &= u_{ct}(q,p) + Tu_1(q,p) \ , \\ u_1(q,p) &= -G^{-1}M_dM^{-1}\kappa L_V(q)M^{-1}p = -\kappa(\cos(q)+k_1)p \ . \end{aligned} \tag{1.105}$$

While in the continuous-time design we can use the desired Hamiltonian as a Lyapunov function and utilize La Salle Invariance Principle to conclude stability of the continuous-time system, the same approach cannot be applied in this controller redesign. In order to apply the framework provided in Theorem 1.2 a strict Lyapunov function for the closed-loop approximate model is required, whereas the desired Hamiltonian does not satisfy this. For that we need to construct a strict Lyapunov function applying (1.66), and we choose such function to be

$$V(q,p) = H_d(q,p) + \epsilon qp \ . \tag{1.106}$$

Applying the controller (1.105) to stabilize the Euler model (1.98), the Lyapunov difference is obtained as

$$\begin{aligned} \Delta V &:= V(q(k+1),p(k+1)) - V(q(k),p(k)) \\ &\le -T\left((k_v - \epsilon(1+\frac{1}{2}k_v))p^2 + \epsilon q\sin(q) + \epsilon(k_1 - \frac{1}{2}k_v)q^2\right) \\ &\quad - T^2\kappa(k_1+\cos(q))p^2 + O(T^2) \ . \end{aligned} \tag{1.107}$$

By choosing k_v, k_1 and κ appropriately, we can show that for sufficiently small $T > 0$ and $\epsilon > 0$, V is a strict SPAS Lyapunov function for the Euler model with the discrete-time controller. Moreover, using Theorem 1.6 we can conclude SPA stability of the exact model and the sampled-data system (1.95), (1.105). The sampled-data simulation results with parameters $k_1 = 1$, $k_v = 2$, $\kappa = 1$, $(q_\circ,\ p_\circ) = (\frac{\pi}{2} - 0.2,\ \frac{1}{2})$ and $T = 0.5$ are illustrated in Figure 1.10.

1.9 Overview of Related Literature

The results we have presented in the earlier sections are only the basic of research that has been done in the topic covered by this chapter. Indeed there is a lot more research done in parallel directions, both for the direct discrete-time design and the emulation (re)design. A similar and more general design framework than what has been provided in Section 1.6 is presented in [45]. This framework uses trajectory based analysis and instead of using one step consistency, a multistep consistency property is utilized. More general design frameworks are presented in [43] where nonlinear systems represented as differential inclusion are considered, and in [39] where nonlinear systems with exogenous inputs are studied.

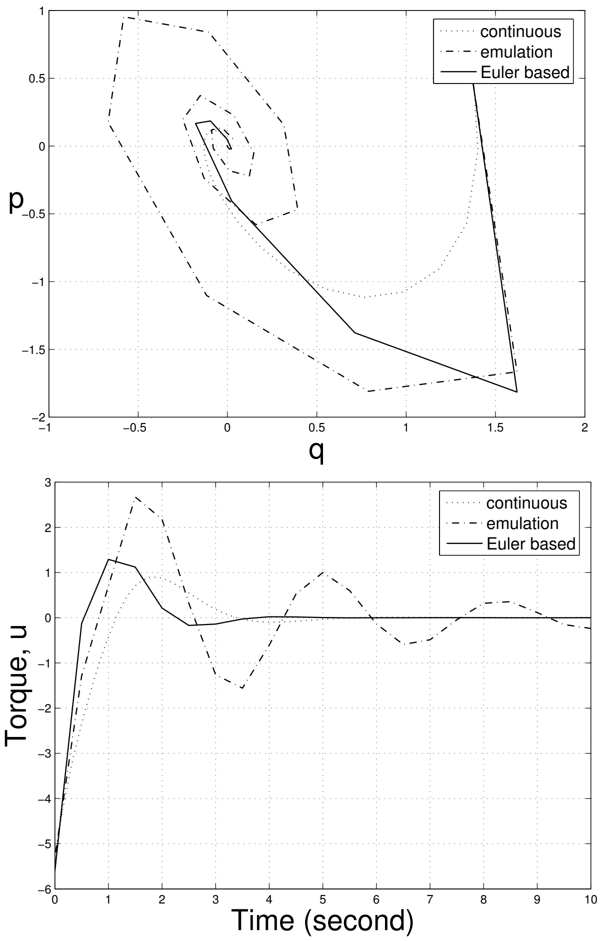

Fig. 1.10. Response of the inverted pendulum with IDA-PBC controller.

Recently, researcher has started to build design tools within the various frameworks mentioned above. Design exploring model predictive control or receding horizon techniques are presented in [20, 37].

Although the frameworks consider only time invariant systems, the extension to time-varying systems is direct. Results presented in [40] on asymptotic stabilization for time-varying cascaded systems and in [26] on input-to-state stabilization of systems in power form using time varying control are examples of this extension.

A problem that one may face in applying the framework is that it requires the knowledge of a strict Lyapunov function for the system. While for linear systems a strict Lyapunov function is available for free, in the sense that a quadratic Lyapunov function can always be used, it is not the case for nonlinear systems in general. Moreover, when the controller is designed based on an approximate model, powerful tools to conclude stability, either SP-AS or SP-ISS, for continuous-time systems, such as La Salle Invariance Principle and Matrosov Theorem are not directly applicable for sampled-data system when stability is attained in a semiglobal practical sense (see discussion in Subsection 1.8.2). Hence results from [42] that provide a partial construction of Lyapunov functions, that in some sense generalizes the construction used in Theorem 1.4 are very useful to replace La Salle Invariant Principle. In [28] a Lyapunov function construction for interconnected systems is proposed utilizing a nonlinear small gain theorem. In [44] a result similar to Matrosov theorem is developed.

There are more research and studies related to the topic presented in this chapter that follow a different framework. Approaches using feedback linearization are discussed for instance in [3, 18] and references therein. A geometric framework for feedback linearization is utilized in [8, 11]. Singular perturbation is used as the main tool to solve sampled-data control problems in [7, 9]. Adaptive control approach based on Euler model is used in [31] and robust stabilization using discontinuous control is studied in [22] (see also references therein).

While we only consider static state feedback in this chapter, assuming the availability of all states is sometimes not realistic. The issue of observability, as well as controllability, of discrete-time systems is studied in [55, 57]. Results on discrete-time controller design and stabilization using output feedback are presented for instance in [5, 10, 14, 54]. A framework for designing a discrete-time observer based on the approximate model of the plant is presented in [4]. When implementing the observer to build a controller for the plant, this result can also be considered as a framework for designing a dynamic feedback. This framework can be seen as an extension of the controller design framework presented in Section 1.6.

Due to the increasing interest of research on nonlinear sampled-data control systems, the list of related literature will always grow longer. What we have cited in this section is in any way not a complete list of reference but just a glimpse of available results on various directions that aims to help readers to see the variety and fertility of research in this topic.

1.10 Open Problems

There is a wide range of open research problems that one could address.

- Constructive designs for classes of nonlinear systems and their approximate models need to be further developed within our framework. Any continuous-time design technique can be revisited within our framework. If the Euler model is used for design, then the structure of the approximate model is the same as the structure of the continuous-time system and in this case the discrete-time design is easier. However, if higher order approximate models are used for controller design then the structure of the approximate discrete-time model may be very different from the structure of the continuous-time model and design becomes harder. In this case, it seems more natural to use model predictive control that does not exploit the structure of the model to design the controller.

- The quantitative relationship between the choice of approximate model used in design and the performance of the obtained controller is unclear. There is an obvious tradeoff between the complexity of the controller design and the accuracy of the approximations. Typically, the design is easiest for the Euler model but we expect that better performance could be obtained if a better approximation was used for controller design. Quantifying this possible improvement in performance appears to be an important issue.
- Obtaining non-conservative estimates of T^* in our theorems would be quite useful for practicing engineers since choosing an appropriate T is an important step in our approach. While we do compute T^* in our proofs, our estimates are very conservative and, hence, not useful in practice. We are not aware of any papers that attempt to address this problem.
- In the presented results, so far we use full state feedback, assuming that all states are available for measurement. In reality, this is not always the case due to the physical meaning of the states or the available sensors and measurement devices may be too expensive. To overcome this situation, observer design and developing results based on output feedback are potential solutions.
- Case studies and practical implementations of our algorithms are needed to motivate new theoretical issues in this area and to assess the developed theory in practice.

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