Summability characterizations of uniform exponential and asymptotic stability of sets for difference inclusions

A.R. Teel
Electrical and Computer Engineering Department,
The University of California, Santa Barbara,
CA 93106-9560, USA
teel@ece.ucsb.edu

D. Nešić*
Department of Electrical and Electronic Engineering,
The University of Melbourne,
Parkville, 3010, Victoria, Australia
d.nesic@ee.unimelb.edu.au

A. Loria and E. Panteley
LSS-Supelec, Plateau de Moulon, 91190, Gyf sur Yvette, France
loria@lss.supelec.fr, panteley@lss.supelec.fr

September 2, 2010

Abstract

We present several equivalent characterizations of uniform global exponential stability (UGES) and uniform global asymptotic stability (UGAS) of arbitrary closed (not necessarily compact) sets for nonlinear difference inclusions. In particular, we provide several characterizations of these stability properties via summability criteria that do not require the knowledge of a Lyapunov function. We apply our results to prove novel nested Matrosov theorems for UGES and UGAS of the origin for time-varying nonlinear difference inclusions.

1 Introduction

Qualitative stability theory of dynamical systems is an active research area since, at least, the seminal works of Poincaré [24] and Lyapunov [18]. The *qualitative* qualifier refers to the quest for methods which circumvent the generally impossible task of solving analytically nonlinear dynamical equations to draw conclusions on the behaviour of solutions.

Among the many definitions of stability (of solutions) of dynamical systems probably the most useful is Lyapunov stability and, more particularly, asymptotic stability; both introduced by Lyapunov in 1892 along with his *second* method, also known as 'direct method' which relies on the ability to find a function with certain monotonicity properties as well as its time derivative along the system's solutions. Since then, many refinements have been established including a range of definitions, pertaining to the size of the bassin of attraction (local *vs* global −*cf.* [1]), whether the attractor is a "point" or a set −*cf.* [29], *etc.* Definitions and theorems are suited for particular classes of systems: see for instance [10, 9] for the case of continuous-time continuous systems and [4] for systems with discontinuous right-hand sides. For recent

*This work was partially supported by the Australian Research Council under the Australian Professorial Fellow and Discovery Grant schemes.

examples see for instance [8] for non-autonomous continuous-time systems and [12] for discontinuous systems.

In this paper we study discrete-time systems with discontinuous right-hand sides, more precisely, systems described by difference inclusions

$$x^+ \in F(x) \ , \tag{1}$$

where $x \in \mathbb{R}^n$ and $F(\cdot)$ is in general a set-valued map (more specific conditions on it will be given later).

The study of such class of systems is important for a number of reasons: for example, they appear in the search for periodic solutions of continuous-time non-autonomous systems by defining a Poincaré map −*cf.* [5]. Also, analysis of discrete-time systems may appear as an intermediary step in the study of sampled-data systems −*cf.* [23] via approximately discretized models −*cf.* [7]. In that respect, it is generally accepted that discretized systems remain 'stable' under small time-step discretization. However, this entails specificities which must be studied in their own right such as the characterization of attractors, from a numerics viewpoint −*cf.* [14] and the introduction of appropriate definitions of stability for discrete-time systems −*cf.* [22].

Relatively early results on the study of difference inclusions extend linear systems theory −*cf.* [20]. See also [3] which establishes similar stability conditions, based on the so-called spectral radius. Yet, literature on Lyapunov-based methods remains the most developed −*cf.* [11, 6, 28]. Beyond its generality, maybe due to the possibility of establishing conditions for robustness −*cf.* [13] which is another strongly desirable property of dynamical systems from a control viewpoint.

Methods alternative to Lyapunov's, acoording to which the goal is to find an auxiliary function whose *derivative* along trajectories meets certain desired properties, are so-called *integrability* criteria. These are stated as conditions on functions of the trajectories that have to be (uniformly) integrable *i.e.*, the integral of a function of time over $\mathbb{R}_+$ with certain monotonicity properties, along the trajectories of the system, must be bounded from above by a quantity that does not depend on initial times. In the discrete-time context such conditions take the form of *summability* criteria and may also be considered as conditions on convergence of infinite series. Of broader interest are necessary and sufficient conditions for stability of sets.

Results that exploit the so-called integrability criteria for uniform global exponential stability (UGES) and uniform global asymptotic stability (UGAS) of arbitrary sets for continuous-time differential inclusions were recently reported in [27]. These conditions are useful in situations when the uniform stability has already been established (for instance, via a positive definite function whose derivative along the solutions of the system is negative semi-definite) and one only needs to check the uniform attractivity. An interesting application of such theorems is, also reported in [27], is a generalization of the so-called *Matrosov's theorem* [19]. Integrability and summability criteria establish a clear link with input-output stability −*cf.* [16] in the classical $\mathcal{L}_p$ sense. These results allow us to establish convergence rates for dynamical systems for which Lyapunov functions are very difficult to construct −*cf.* [15][1]. Closer to the realm of engineering science integrability conditions may result useful in applied control design; for instance in cases where Lyapunov-like methods fail, such is the case of stability analysis of tracking-controlled nonholonomic systems via time-varying controllers in [17].

Summability criteria for sampled-data nonlinear systems when the stability analysis is carried out via an approximate discrete-time model of the system have been obtained recently[2]. In particular, a sampled-data counterpart of the integrability criteria for differential inclusions from [27] was reported in [21] as well as a counterpart of extended Matrosov theorem for differential equations from [17] was presented in [22]. In this approach, one needs to establish stability of a family of approximate discrete-time models that are parameterized with the sampling period and the goal is to establish appropriate stability properties of this family that would guarantee that the family of exact discrete-time models will also be stable for sufficiently small sampling periods.

Results in [21, 22] can be modified and used in the special case when the exact discrete time model of the sampled-data system is known and one does not need to deal with families of discrete-time systems.

[1] In this respect it is convenient to stress that we are aware of a very recent unpublished (yet) result on the construction of Lyapunov functions for MRAC systems, by F. Mazenc.

[2] Since the exact discrete-time model of a nonlinear sampled-data system is often not available to the control designer.

However, straightforward modifications of results from [21, 22] to this special case would be unnecessarily restrictive and technical. Hence, we address in this paper the case when the exact discrete-time model of the system is known and we prove the results under weaker assumptions than what would be possible by doing the modifications of [21, 22]. In particular, we investigate various stability characterization of UGES and UGAS of arbitrary sets for nonlinear difference inclusions (1). We emphasize that very little will be assumed on the set-valued map $F(\cdot)$ and, in particular, we allow $F(\cdot)$ to be discontinuous. Hence, our results hold under very general assumptions and, moreover, the proofs are less technical than their continuous-time or sampled-data counterparts. The results in this paper are technically different from results in [27, 17, 21, 22] and, we believe, will prove very useful for systems of the form (1) as their sampled-data and continuous-time counterparts have.

The first part of this paper provides several summability criteria for checking UGES (Theorem 1) and UGAS (Theorem 2) of arbitrary closed sets for difference inclusions of the form (1). These results parallel recent results on integral characterizations of UGES and UGAS of arbitrary sets for continuous time nonlinear differential inclusions in [27] and constitute an outgrowth of the main results in [16] for difference equations. The second part of the paper provides further characterizations of UGES and UGAS for (1) using the notion of detectability (Theorem 1) and via Matrosov functions (Theorems 3 and 4). Matrosov functions are useful in situations when uniform stability of the system is already established and one only needs to check uniform attractivity.

Our results on stability of sets may be related to output stability $-$*cf.* [25] for the case when the output corresponds to the distance between the state and the set in question. We define such distance below.

2 Preliminaries

$\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the sets of real and natural (that includes zero) numbers. Given $c \in \mathbb{R}$ we denote as $\mathbb{R}_{\geq c}$ the set of all real numbers that are greater than or equal to c (similar notation is used for the set $\mathbb{N}$). Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, we denote the distance of an arbitrary $x \in \mathbb{R}^n$ from this set as:

$$|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z| \ .$$

Also, given $0 \leq \delta \leq \Delta$, we use the notation $\mathcal{H}_{\mathcal{A}}(\delta, \Delta) := \{x \in \mathbb{R}^n : \delta \leq |x|_{\mathcal{A}} \leq \Delta\}$.

Assumption 1 *Consider the system (1). We assume that for each x the set $F(x)$ is non-empty.* □

The solutions with initial condition $x_0 \in \mathbb{R}^n$ are denoted by $\phi(\cdot, x_0)$ hence, $\phi(0, x_0) = x_0$. When $F(x)$ is multi-valued, the solution generated by the initial condition x_0 is not uniquely defined. We denote the set of all possible solutions starting from x_0 as $\mathcal{S}(x_0)$ and for any function $(k, x_0) \mapsto \phi(k, x_0)$ we write $\phi \in \mathcal{S}(x_0)$ if we have that $\phi(k+1, x_0) \in F(\phi(k, x_0))$ for all $k \in \mathbb{N}$.

We also use the following standard definitions. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{PD}$ (also $\alpha \in \mathcal{PD}$) if it is continuous, zero at zero and positive for all other values of its argument. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{G}$ (also $\alpha \in \mathcal{G}$) if the function is continuous, zero at zero and nondecreasing. It is said to belong to class-$\mathcal{K}$ (also $\alpha \in \mathcal{K}$) if $\alpha \in \mathcal{G}$ and it is strictly increasing. It is said to belong to class-$\mathcal{K}_\infty$ (also $\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and it is unbounded. Note that class-$\mathcal{K}_\infty$ functions are globally invertible. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ (also $\beta \in \mathcal{KL}$) if the function is nondecreasing is its first argument, non-increasing in its second argument and $\lim_{s \to 0^+} \beta(s, k) = \lim_{k \to \infty} \beta(s, k) = 0$. The following lemma can be proved in a similar manner as the "Sontag's lemma" given in [26]:

Lemma 1 *Let $\beta \in \mathcal{KL}$ and $\lambda \in (0, 1)$. There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(\beta(s, k)) \leq \alpha_2(s) \lambda^k \qquad \forall s, k \ .$$

□

The following definition is not standard but we find it useful to state our main results.

Definition 1 *A pair of class-$\mathcal{G}$ functions (α_1, α_2) is said to be compatible for uniform global exponential stability (cUGES) if there exist $\underline{\lambda}, \overline{\lambda} \in (0,1)$ and $\overline{\Gamma} \geq 1$ such that*

$$\alpha_2(s) \leq \alpha_1(\overline{\Gamma} \cdot s); \qquad \underline{\lambda} \cdot \alpha_2(s) \leq \alpha_2(\overline{\lambda} \cdot s) \qquad \forall s \geq 0 \ .$$

□

For instance, functions $\alpha_i(s) = a_i s^p, i = 1,2$ are cUGES if $a_i > 0, i = 1,2$ and $p > 0$. Similarly, the functions $\alpha_1(s) = \alpha_2(s) = \arctan(s)$ are cUGES (for instance, we can take $\overline{\Gamma} = 1$, $\underline{\lambda} = \overline{\lambda} = 0.5$).

We can state the following lemma whose proof is given in the appendix:

Lemma 2 *If the pair of class-$\mathcal{G}$ functions (α_1, α_2) is cUGES, then for each $\mu \in [0,1)$ there exist $\Gamma \in \mathbb{R}_{\geq 1}$ and $\lambda \in [0,1)$ such that:*

$$\mu^k \alpha_2(s) \leq \alpha_1(s \cdot \Gamma \lambda^k) \qquad \forall s \in \mathbb{R}_{\geq 0}, \ k \in \mathbb{N} \ . \tag{2}$$

□

Remark 1 *It is straightforward to show that if the pair $(\alpha_1(s), \alpha_2(s))$ is cUGES, then for any $\Gamma \in \mathbb{R}_{\geq 1}$ we have that the pair $(\alpha_1(s), \alpha_2(\Gamma \cdot s))$ is cUGES.* □

3 Characterizations of stability of sets

In this section, we provide two main results that establish several equivalent characterizations of set UGES (Theorem 1) and set UGAS (Theorem 2). These results are later used to prove further characterizations of UGES/UGAS via the detectability conditions (Theorem 1) and Matrosov functions (Theorem 3 and 4). Proofs of all main results are presented in Section 4.

3.1 Uniform global exponential stability of sets

Let $\mathcal{A} \subset \mathbb{R}^n$ be a (given) closed (not necessarily compact) set. We introduce the following definitions for system (1).

Definition 2 *$\mathcal{A} \subset \mathbb{R}^n$ is uniformly globally exponentially stable (UGES) for the system (1) if there exist $\Gamma \in \mathbb{R}_{\geq 1}$ and $\lambda \in [0,1)$ such that for all $x_0 \in \mathbb{R}^n$ and all $\phi \in \mathcal{S}(x_0)$ we have*

$$|\phi(k, x_0)|_{\mathcal{A}} \leq \Gamma \lambda^k |x_0|_{\mathcal{A}} \qquad \forall k \in \mathbb{N} \ . \tag{3}$$

□

When $\mathcal{A} = \{0\}$ UGES boils down to the usual definition of uniform exponential stability of the origin –*cf.* [27].

Definition 3 *The closed set $\mathcal{A}$ is finite-step contractive (FSC) for the system (1) if there exist $k^* \in \mathbb{N}$ and $\lambda_\circ \in [0,1)$ such that for each $x_0 \in \mathbb{R}^n$ and each $\phi \in \mathcal{S}(x_0)$ there exists $k \in \{0, \ldots, k^*\}$ such that*

$$|\phi(k, x_0)|_{\mathcal{A}} \leq \lambda_\circ \cdot |x_0|_{\mathcal{A}} \ . \tag{4}$$

□

The following result establishes several equivalent characterizations of UGES for the system (1).

Theorem 1 *The statements enumerated below are equivalent:*

1. $\mathcal{A} \subset \mathbb{R}^n$ *is UGES for the system (1);*
2. $\mathcal{A} \subset \mathbb{R}^n$ *is FSC for the system (1) and there exists* $\Gamma_1 \in \mathbb{R}_{\geq 1}$ *such that*

$$|w|_{\mathcal{A}} \leq |x|_{\mathcal{A}} \cdot \Gamma_1 \qquad \forall x \in \mathbb{R}^n,\ w \in F(x)\,; \tag{5}$$

3. *there exist* $\alpha_1 \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{G}$ *such that the pair* (α_1, α_2) *is cUGES and for each* $x_0 \in \mathbb{R}^n$ *and each* $\phi \in \mathcal{S}(x_0)$ *we have*

$$\sum_{k=0}^{\infty} \alpha_1 \left(|\phi(k,x_0)|_{\mathcal{A}}\right) \leq \alpha_2 \left(|x_0|_{\mathcal{A}}\right); \tag{6}$$

4. *for each* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that* $(\alpha_1^{-1}, \overline{\Gamma} \cdot \alpha_2^{-1})$ *is cUGES for all* $\overline{\Gamma} \in \mathbb{R}_{\geq 1}$, *there exists* $\Gamma_2 \in \mathbb{R}_{\geq 1}$ *such that for each* $x_0 \in \mathbb{R}^n$ *and* $\phi \in \mathcal{S}(x_0)$ *we have*

$$\sum_{k=0}^{\infty} \alpha_1 \left(|\phi(k,x_0)|_{\mathcal{A}}\right) \leq \Gamma_2 \cdot \alpha_2 \left(|x_0|_{\mathcal{A}}\right) . \tag{7}$$

□

Remark 2 *A continuous-time counterpart of Theorem 1 is given in [27, Theorem 2]. A sampled-data counterpart of this results is given in [21]. We note that the notion of cUGES was not used in [27, 21].* □

3.2 Uniform global asymptotic stability of sets

For closed sets $\mathcal{A} \subset \mathbb{R}^n$ and solutions $\phi \in \mathcal{S}(x_0)$ of systems (1) we introduce the more general definitions of *asymptotic* stability.

Definition 4 *The closed set* $\mathcal{A} \subset \mathbb{R}^n$ *is uniformly stable if for each* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that for all* $x_0 \in \mathbb{R}^n$ *satisfying* $|x_0|_{\mathcal{A}} \leq \delta$, *we have* $|\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon$ *for all* $k \in \mathbb{N}$ *and all* $\phi \in \mathcal{S}(x_0)$. *The set* $\mathcal{A}$ *is uniformly* globally *stable (UGS) if moreover* δ *has the property that* $\delta \to \infty$ *as* $\epsilon \to \infty$. □

Remark 3 *Following [9], we can show that the set* $\mathcal{A}$ *is UGS if and only if there exists* $\rho \in \mathcal{K}_\infty$ *such that for all* $x_0 \in \mathbb{R}^n$ *and* $\phi \in \mathcal{S}(x_0)$ *we have*

$$|\phi(k,x_0)|_{\mathcal{A}} \leq \rho(|x_0|_{\mathcal{A}}) \qquad \forall k \in \mathbb{N}\ . \tag{8}$$

Moreover, if ρ *is a linear function we say that the set* $\mathcal{A}$ *is UGS (for the system (1)) with linear growth.* □

Definition 5 *The closed set* $\mathcal{A} \subset \mathbb{R}^n$ *is uniformly globally attractive (UGA) for the system (1) if for each pair of strictly positive reals* (r, ϵ) *there exists* $k^* \in \mathbb{N}$ *such that:*

$$|x_0|_{\mathcal{A}} \leq r,\ \phi \in \mathcal{S}(x_0), \quad k \geq k^* \implies |\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon\ . \tag{9}$$

□

Remark 4 *Similarly to UGS we have the following characterization of UGAS. The set* $\mathcal{A}$ *is UGAS if there exists* $\beta \in \mathcal{KL}$ *such that for all* $x_0 \in \mathbb{R}^n$ *and all* $\phi \in \mathcal{S}(x_0)$ *we have*

$$|\phi(k,x_0)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, k) \qquad \forall k \in \mathbb{N}\ . \tag{10}$$

When $\mathcal{A} = \{0\}$ *we recover Barbashin's definition of UGAS (of the null solution) –cf. [2].* □

The following result establishes several equivalent characterizations of UGAS of sets for the system (1).

Theorem 2 *The following statements are equivalent:*

1. *the closed set $\mathcal{A}$ is UGAS for the system (1);*
2. (a) *the closed set $\mathcal{A}$ is UGA for the system (1) and*
   (b) *there exists $\rho \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $w \in F(x)$ we have $|w|_{\mathcal{A}} \leq \rho(|x|_{\mathcal{A}})$;*
3. (a) *$\mathcal{A}$ is UGS for the system (1) and*
   (b) *there exists $\eta \in \mathcal{PD}$ and $\alpha_2 \in \mathcal{G}$ such that for all $x_0 \in \mathbb{R}^n$ and all $\phi \in \mathcal{S}(x_0)$ we have*

$$\sum_{k=0}^{\infty} \eta(|\phi(k, x_0)|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}}); \tag{11}$$

4. (a) *there exists $\rho \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $w \in F(x)$ we have $|w|_{\mathcal{A}} \leq \rho(|x|_{\mathcal{A}})$ and*
   (b) *there exist $\alpha_1 \in \mathcal{K}$, $\alpha_2 \in \mathcal{G}$ such that for each $x_0 \in \mathbb{R}^n$ and $\phi \in \mathcal{S}(x_0)$ we have*

$$\sum_{k=0}^{\infty} \alpha_1(|\phi(k, x_0)|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}}); \tag{12}$$

5. *there exist $\alpha_1 \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{G}$ such that for each $x_0 \in \mathbb{R}^n$ and $\phi \in \mathcal{S}(x_0)$ inequality (12) holds;*
6. (a) *$\mathcal{A}$ is UGS for the system (1) and*
   (b) *for each pair of strictly positive real numbers satisfying $\delta \leq \Delta$ there exists a continuous function $\omega_{\delta,\Delta} : \mathbb{R}^n \to \mathbb{R}$ and strictly positive real numbers ω_m and γ such that*
      i. *$\omega_{\delta,\Delta}(x) \geq \omega_m$ for all $x \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$ and*
      ii. *for all $x_0 \in \mathcal{H}_{\mathcal{A}}(\delta, \Delta)$, $\phi \in \mathcal{S}(x_0)$ and all $k \in \mathbb{N}$ we have*

$$\sum_{i=0}^{k} \omega_{\delta,\Delta}(\phi(i, x_0)) \leq \gamma. \tag{13}$$

□

Remark 5 *A continuous-time counterpart of Theorem 2 is given in [27, Theorem 1] and its sampled-data counterpart can be found in [21].* □

3.3 Detectability

In this subsection, we consider the system (1) with an output $y \in \mathbb{R}^p$ defined as:

$$y \in H(x) \ , \tag{14}$$

where $H(\cdot)$ is in general multi-valued. Given $x_0 \in \mathbb{R}^n$, we denote by $\mathcal{S}_H(x_0)$ all possible pairs of trajectories and outputs that satisfy equations (1), (14), that is we write $(\phi, y) \in \mathcal{S}_H(x_0)$ if $\phi \in \mathcal{S}(x_0)$ and $y(j, x_0) \in H(\phi(j, x_0))$ for all $j \in \mathbb{N}$.

Definition 6 *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{G}$. The closed set $\mathcal{A}$ is said to be $(\alpha_1, \alpha_2, \alpha_3)$-detectable for the system system (1), (14) if for each $x_0 \in \mathbb{R}^n$ and $(\phi, y) \in \mathcal{S}_H(x_0)$ we have*

$$\sum_{j=0}^{k} \alpha_1(|\phi(j, x_0)|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}}) + \sum_{j=0}^{k} \alpha_3(|y(j, x_0)|) \qquad \forall k \in \mathbb{N} \ . \tag{15}$$

□

The following statement follows as a corollary of previous theorems.

Corollary 1 *Suppose that there exist $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2, \alpha, \overline{\alpha} \in \mathcal{G}$ such that the following conditions hold:*

1. *the closed set $\mathcal{A}$ is $(\alpha_1, \alpha_2, \alpha)$-detectable for the system (1), (14);*
2. *for each $x_0 \in \mathbb{R}^n$ and $(\phi, y) \in \mathcal{S}_H(x_0)$ we have*

$$\sum_{k=0}^{\infty} \alpha(|y(k, x_0)|) \leq \overline{\alpha}(|x_0|_{\mathcal{A}}). \tag{16}$$

Then, the closed set $\mathcal{A}$ is UGAS for the system (1). If moreover the pair $(\alpha_1, \alpha_2 + \overline{\alpha})$ is cUGES, the set $\mathcal{A}$ is UGES for the system (1). □

Proof of Theorem 1: Combining (16) with $(\alpha_1, \alpha_2, \alpha)$-detectability, we have

$$\sum_{k=0}^{\infty} \alpha_1(|\phi(k, x_0)|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}}) + \overline{\alpha}(|x_0|_{\mathcal{A}}) \ . \tag{17}$$

UGAS of $\mathcal{A}$ follows directly from Theorem 2. If, moreover, the pair $(\alpha_1, \alpha_2 + \overline{\alpha})$ is cUGES, then UGES of $\mathcal{A}$ follows from Theorem 1. ■

Remark 6

- *It is possible to modify the definition of detectability so that (15) holds with $|y(j, x_0)|_{\mathcal{B}}$ instead of $|y(j, x_0)|$ where $\mathcal{B} \subset \mathbb{R}^p$ is a closed set. With this modification, we would need to modify the condition (16) in the same manner.*
- *The condition in item 2 of the corollary holds e.g., if there exists a positive definite function V and class $\mathcal{K}_\infty$ functions α_1 and α_2 such that $V(x) \leq \alpha_1(|x|_{\mathcal{A}})$ and $V(w) \leq V(x) - \alpha(|y|)$. Then, applying the sum from $k = 0$ to infinity on both sides of the latter inequality, we recover (16). Hence, Corollary 1 establishes UGAS and UGES under detectability and "Krasovskii-LaSalle-type" conditions. Correspondingly, in the next section we present results which generalize the latter in the spirit of Matrosov's theorem –cf. [19, 17].*

□

3.4 Matrosov Theorem for UGES

In this and the next subsection we apply our results to two Matrosov theorems, for UGES and UGAS of the origin, for time-varying systems

$$x^+ \in F(x, k) \ . \tag{18}$$

The above system can be rewritten as:

$$\begin{array}{rcl} x^+ & \in & F(x, p(k)) \\ k^+ & = & p(k) + 1 \ , \end{array} \tag{19}$$

where $p : \mathbb{R}_{\geq 0} \mapsto \mathbb{Z}_{\geq 0}$ is such that for all $k \in \mathbb{Z}_{\geq 0}$ we have that $p(s) = k, \forall s \in [k, k+1)$, and we can think of the system (19) as time invariant. In this case, we consider stability of the origin of the system (18) which is equivalent to considering stability of the set $\mathcal{A} := \{(x, k) : x = 0\}$ for the time invariant system (19). Hence, we can use all the definitions and results from the previous sections if we replace $|x|_{\mathcal{A}}$ by $|x|$.

Theorem 3 *The origin of the system (18) is UGES if the following conditions hold:*

1. *the system (18) is UGS with linear growth;*
2. *there exist* $\alpha_1 \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{G}$, $\mu \in \mathbb{R}_{\geq 1}$, *functions* $\varphi : \mathbb{R}^n \to \mathbb{R}^r$, $\psi : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^s$ *and* m *functions* $W_i : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ *and symmetric matrices* $Y_i \in \mathbb{R}^{(r+s)\times(r+s)}$, $i \in \{1, 2, \ldots, m\}$ *such that*
   (a) *the pair* $(\alpha_1, \overline{\Gamma} \cdot \alpha_2)$ *is cUGES for each* $\overline{\Gamma} \in \mathbb{R}_{\geq 1}$;
   (b) *for all* $x \in \mathbb{R}^n$ *and* $k \in \mathbb{N}$ *we have*
$$\alpha_1(|x|) \leq |\varphi(x)|^2 , \qquad |\psi(x,k)| \leq \mu \, |\varphi(x)| , \tag{20}$$
   (c) *for all* $i \in \{1, \ldots, m\}$, *all* $x \in \mathbb{R}^n$ *and all* $k \in \mathbb{N}$,
$$|W_i(x,k)| \leq \alpha_2(|x|) \tag{21}$$
   and for all $w \in F(x,k)$ *we have*
$$W_i(w, k+1) \leq W_i(x,k) + \begin{bmatrix} \varphi(x) \\ \psi(x,k) \end{bmatrix}^T Y_i \begin{bmatrix} \varphi(x) \\ \psi(x,k) \end{bmatrix} ; \tag{22}$$
   (d) *the following properties hold for the matrices* Y_i *and* $y = [\Phi^T \ \Psi^T]^T$ *for all* $\Phi \in \mathbb{R}^r$ *and* $\Psi \in \mathbb{R}^s$:
      i. *for each* $j \in \{1, \ldots, m\}$, $y^T Y_i y = 0$ *for all* $i \in \{1, \ldots, j-1\}$ *implies* $y^T Y_j y \leq 0$ *and*
      ii. $y^T Y_i y = 0$ *for all* $i \in \{1, \ldots, m\}$ *implies* $\Phi = 0$

□

Remark 7 *Theorem 3 provides conditions for UGES of difference inclusions (18). We are not aware whether a continuous-time version of this result has been published in the literature.* □

Next, we illustrate our Theorem 3 with an example.

Example 1 *We show that the origin of the following system*
$$x^+ = x - g(x,k) =: f(x,k) \tag{23}$$
is UGES under the following conditions:

1. *there exists* $\epsilon > 0$ *such that* $x^T g(x,k) \geq \frac{1}{2}(1+\epsilon) \, |g(x,k)|^2$ *for all* x, k;
2. *there exist* $c_1, c_2 > 0$ *and* $k^* \in \mathbb{N}$ *such that for all* $x \in \mathbb{R}^n$ *and* $k \in \mathbb{N}$ *we have*
$$c_1^2 \, |x|^2 \leq \sum_{j=k}^{k+k^*} |g(x,j)|^2 \leq c_2^2 \, |x|^2 ; \tag{24}$$
3. *there exists* $L > 0$ *such that for all* $k \in \mathbb{N}$ *and all* $x_1, x_2 \in \mathbb{R}^n$ *we have*
$$|g(x_1,k) - g(x_2,k)| \leq L \, |x_1 - x_2| . \tag{25}$$

Define $V_1(x) := |x|^2$. *Then, from item 1 above we have*
$$V_1(f(x,k)) = |x - g(x,k)|^2 \leq V_1(x) - \epsilon \, |g(x,k)|^2 \ ,$$
which establishes item 1 of Theorem 3. Let $\lambda \in (0,1)$ *and define* $W_1(x) := V_1(x)$,
$$W_2(x,k) := -\sum_{j=k}^{\infty} \lambda^{j-k} \, |g(x,j)|^2 \ .$$

Using item 2 above, we obtain $|g(x,j)|^2 \leq c_2^2 |x|^2$ *for all* $j \in \mathbb{N}$ *and all* $x \in \mathbb{R}^n$ *hence,*

$$|W_2(x,k)| \leq c_2^2 \frac{1}{1-\lambda} |x|^2 \ .$$

Then, using item 2 we obtain

$$W_2(x,k) \leq -\lambda^{k^*} c_1^2 |x|^2 \ .$$

Moreover, using item 3 we get

$$|g(x_1,k)|^2 - |g(x_2,k)|^2 \leq c_2(|x_1| + |x_2|)L |x_1 - x_2|$$

and, furthermore,

$$\begin{aligned} |W_2(x_1,k) - W_2(x_2,k)| &\leq \frac{1}{1-\lambda} \sup_{j \in \mathbb{N}_{\geq k}} \left| |g(x_1,j)|^2 - |g(x_2,j)|^2 \right| \\ &\leq \frac{1}{1-\lambda} \sup_{j \in \mathbb{N}_{\geq k}} (|g(x_1,j)| + |g(x_2,j)|) |g(x_1,j) - g(x_2,j)| \\ &\leq \frac{1}{1-\lambda} c_2(|x_1| + |x_2|) L |x_1 - x_2| \ . \end{aligned} \tag{26}$$

Next, using the definition of W_2 *and the fact that* $f(x,k) = x - g(x,k)$*, we obtain*

$$\begin{aligned} W_2(f(x,k), k+1) &= \lambda^{-1}(W_2(f(x,k),k) + |g(f(x,k),k)|^2) \\ &= W_2(f(x,k),k) + (\lambda^{-1} - 1) W_2(f(x,k),k) + \lambda^{-1} |g(f(x,k),k)|^2 \\ &\leq W_2(f(x,k),k) - (\lambda^{-1} - 1)\lambda^{k^*} c_1^2 |x|^2 + \lambda^{-1} |g(x,k)|^2 \\ &\quad + (\lambda^{-1} - 1) \frac{1}{1-\lambda} c_2(2 + c_2) |x| L |g(x,k)| + \lambda^{-1} c_2 (2 + c_2) |x| L |g(x,k)| \, . \end{aligned}$$

Let $\varphi(x) := |x|$, $\psi(x,k) := |g(x,k)|$, $\alpha_1(s) := s^2$, $\mu := c_2$, $\alpha_2(s) := \frac{c_2^2 \cdot s^2}{1-\lambda}$ *and*

$$Y_1 := \begin{bmatrix} 0 & 0 \\ 0 & -\sqrt{\epsilon} \end{bmatrix}; \qquad Y_2 := \begin{bmatrix} -(\lambda^{-1} - 1)\lambda^{k^*} c_1^2 & c_2(2 + c_2) L \lambda^{-1} \\ c_2(2 + c_2) L \lambda^{-1} & \lambda^{-1} \end{bmatrix} \ .$$

With these definitions, item 2 of Theorem 3 holds, which establishes UGES of the origin. □

3.5 Matrosov theorem for UGAS

The following statement is the discrete-time counterpart of the extended Matrosov theorem [17, Theorem 1] for continuous-time systems and of [22] for sampled-data systems.

Theorem 4 *The origin of system (1) is UGAS if:*

1. *there exist functions* $V : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ *and* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that*
   (a) $\alpha_1(|x|) \leq V(x,k) \leq \alpha_2(|x|)$ *for all* $x \in \mathbb{R}^n$, $k \in \mathbb{N}$*;*
   (b) $V(f, k+1) - V(x,k) \leq 0$ *for all* $x \in \mathbb{R}^n$, $k \in \mathbb{N}$*;*
2. *for each* $\Delta > 0$ *there exist*
   - *numbers* $m \in \mathbb{N}$, $\mu \in \mathbb{R}_{\geq 0}$,
   - *a function* $\psi : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^p$,
   - m *functions* $W_i : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$,
   - m *continuous functions* $Y_i : \mathbb{R}^{n+p} \to \mathbb{R}$

such that

(a) for all $i \in \{1, \ldots, m\}$*,* $|x| \leq \Delta$*,* $k \in \mathbb{N}$

$$\begin{array}{rcll} \max\{|W_i(x,k)|, |\psi(x,k)|\} & \leq & \mu & \\ W_i(f,k+1) - W_i(x,k) & \leq & Y_i(x,\psi(x,k)) & \forall f \in F(x,k)\ ; \end{array} \tag{27}$$

(b) the following properties hold for the functions Y_i*:*

i. for each $j \in \{1, \cdots, m\}$*,* $Y_i(x,z) = 0$ *for all* $i \in \{1, \ldots, j-1\}$ *and* $|x| \leq \Delta$*,* $|z| \leq \mu$ *imply* $Y_j(x,z) \leq 0$ *and*

ii. $Y_i(x,z) = 0$ *for all* $i \in \{1, \cdots, m\}$ *and* $|x| \leq \Delta$*,* $|z| \leq \mu$ *imply* $x = 0$*.*

□

4 Proofs

4.1 Proof of Theorem 1

($1 \implies 4$): Let item 1 and Definition 2 generate $\Gamma \in \mathbb{R}_{\geq 1}$ and $\lambda \in [0,1)$ such that (3) holds. Pick arbitrarily a cUGES pair $(\alpha_1^{-1}, \Gamma\,\alpha_2^{-1})$ and, for any $\mu \in [0,1)$, let Lemma 2 generate $\Gamma_* \in \mathbb{R}_{\geq 1}$ and $\lambda_* \in [0,1)$ such that

$$\mu^k \Gamma \alpha_2^{-1}(s) \leq \alpha_1^{-1}(s\Gamma_* \lambda_*^k)$$

The latter holds, in particular, for $\mu = \lambda$ hence

$$\alpha_1\left(\lambda^k \Gamma \alpha_2^{-1}(s)\right) \leq s\,\Gamma_* \lambda_*^k \qquad \forall s \in \mathbb{R}_{\geq 0},\ k \in \mathbb{N}. \tag{28}$$

Therefore, for $s = \alpha_2(|x_0|_{\mathcal{A}})$ and any $x_0 \in \mathbb{R}^n$ we have, using (3),

$$\alpha_1(|\phi(k,x_0)|_{\mathcal{A}}) \leq \alpha_1(\lambda^k \Gamma\,|x_0|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}})\Gamma_* \lambda_*^k.$$

Evaluating the sum from $k = 0$ to ∞ on both sides of the inequalities above, we obtain

$$\sum_{k=0}^{\infty} \alpha_1(|\phi(k,x_0)|_{\mathcal{A}}) \leq \Gamma_* \sum_{k=0}^{\infty} \lambda_*^k\ \alpha_2(|x_0|_{\mathcal{A}}), \tag{29}$$

so item 4 the theorem holds with $\Gamma_2 := \frac{\Gamma_*}{1-\lambda_*}$.

($4 \implies 3$): Let $\overline{\alpha}_1(s) = \overline{\alpha}_2(s) = s$. Then, $(\overline{\alpha}_1^{-1}, \Gamma \cdot \overline{\alpha}_2^{-1})$ is cUGES for each $\Gamma \in \mathbb{R}_{\geq 1}$. Hence, using item 4, there exists $\Gamma_2 \in \mathbb{R}_{\geq 1}$ such that

$$\sum_{k=1}^{\infty} \overline{\alpha}_1(|\phi(k,x_0)|_{\mathcal{A}}) \leq \Gamma_2 \cdot \overline{\alpha}_2(|x_0|_{\mathcal{A}})\ . \tag{30}$$

Define $\alpha_1(s) := \overline{\alpha}_1(s) = s$ and $\alpha_2(s) := \Gamma_2 \cdot \overline{\alpha}_2(s) = \Gamma_2 \cdot s$ and note that the pair (α_1, α_2) is cUGES. This and (30) immediately shows that item 3 of the theorem holds.

($3 \implies 2$): By assumption we have

$$\alpha_1(|\phi(1,x_0)|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}}), \quad \forall\ x_0 \in \mathbb{R}^n,\ \phi \in \mathcal{S}(x_0) \tag{31}$$

or, equivalently, for all $x \in \mathbb{R}^n$ and $w \in F(x)$ we have

$$|w|_{\mathcal{A}} \leq \alpha_1^{-1} \circ \alpha_2(|x|_{\mathcal{A}}) \tag{32}$$

Since (α_1, α_2) is cUGES by assumption, there exists $\Gamma_1 \in \mathbb{R}_{\geq 1}$ such that (5) holds.

We show next that the system (1) is FSC. By assumption the pair (α_1, α_2) is cUGES hence, by Lemma 2 there exist $\Gamma \in \mathbb{R}_{\geq 1}$ and $\lambda \in [0,1)$ such that for all $j \in \mathbb{N}$,

$$0.5^j \alpha_2(s) \leq \alpha_1(\Gamma \lambda^j s) \qquad \forall s \in \mathbb{R}_{\geq 0}\ . \tag{33}$$

Let $j^* \in \mathbb{N}$ satisfy $\Gamma \lambda^{j^*} \leq 0.5$ and $k^* \in \mathbb{N}$ satisfy $(k^*+1)^{-1} \leq 0.5^{j^*}$. It follows from (33) that

$$(k^*+1)^{-1}\alpha_2(s) \leq \alpha_1(0.5s) \qquad \forall s \in \mathbb{R}_{\geq 0} \tag{34}$$

or equivalently,

$$\alpha_2(s) \leq (k^*+1)\alpha_1(0.5s) \qquad \forall s \in \mathbb{R}_{\geq 0}\ . \tag{35}$$

Next, we show by *reductio ad absurdum* that FSC holds with this k^* and $\lambda_\circ = 0.5$. Assume it does not *i.e.*, suppose that there exists $x_0 \in \mathbb{R}^n$ and $\phi \in \mathcal{S}(x_0)$ such that $|\phi(k,x_0)|_{\mathcal{A}} > 0.5\,|x_0|_{\mathcal{A}}$ for all $k \in \{0,1,\ldots,k^*\}$. Then, since $\alpha_1 \in \mathcal{K}_\infty$, we have

$$(k^*+1)\alpha_1(0.5\,|x_0|_{\mathcal{A}}) < \sum_{k=0}^{k^*} \alpha_1(|\phi(k,x_0)|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}})\ , \tag{36}$$

which contradicts (35).

($2 \Longrightarrow 1$): Without loss of generality, assume that $\Gamma_1 \in \mathbb{R}_{\geq 1}$ and $\lambda_\circ \in (0,1)$. Then, from FSC we have that for any x_0 and $\phi \in \mathcal{S}(x_0)$ there exists a sequence of times $j_i, i \in \mathbb{Z}_{\geq 0}$, such that $j_{i+1} - j_i \leq k^*$ (and, consequently, $j_i \leq ik^*$) such that

$$|\phi(j_i,x_0)|_{\mathcal{A}} \leq \lambda_\circ^i |x_0|_{\mathcal{A}} \leq \left(\lambda_\circ^{\frac{1}{k^*}}\right)^{j_i} |x_0|_{\mathcal{A}} =: \lambda^{j_i}|x_0|_{\mathcal{A}}\ .$$

Moreover, from (5) we have that for any $k \in [j_i, j_{i+1}]$ the following holds:

$$\begin{aligned}
|\phi(k,x_0)|_{\mathcal{A}} &\leq \Gamma_1^{k-j_i}|\phi(j_i,x_0)|_{\mathcal{A}} \\
&\leq \Gamma_1^{k-j_i}\lambda^{j_i}|x_0|_{\mathcal{A}} \\
&\leq \Gamma_1^{k^*}\lambda^{-(k-j_i)}\lambda^k|x_0|_{\mathcal{A}} \\
&\leq \Gamma_1^{k^*}\lambda^{-k^*}\lambda^k|x_0|_{\mathcal{A}} \\
&= \frac{\Gamma_1^{k^*}}{\lambda_\circ}\lambda^k|x_0|_{\mathcal{A}}\,.
\end{aligned}$$

Hence, (3) holds with $\Gamma := \Gamma_1^{k^*}/\lambda_\circ$ and $\lambda = (\lambda_\circ)^{1/k^*}$. ■

4.2 Proof of Theorem 2

Without loss of generality we assume that functions $\rho \in \mathcal{K}_\infty$ satisfy $\rho(s) \geq s$, $\forall s \geq 0$. Throughout this proof ρ^k denotes the k-fold composition of the function ρ with itself:

$$\rho^k(s) := \underbrace{\rho \circ \rho \circ \cdots \circ \rho}_{k \text{ times}}(s), \qquad k \in \mathbb{Z}_{\geq 1}.$$

Clearly, if $\rho \in \mathcal{K}_\infty$ then $\rho^k \in \mathcal{K}_\infty$ for each $k \in \mathbb{Z}_{\geq 1}$.

($2 \Longrightarrow 1$):
Uniform stability: We first show that the origin is uniformly stable *i.e.*, for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_0 \in \mathbb{R}^n$ satisfying $|x_0|_{\mathcal{A}} \leq \delta$, we have $|\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon$ for all $k \in \mathbb{N}$. Pick $\epsilon > 0$ arbitrarily and let item 2(a) generate, via Definition 5, k^* such that (9) holds for the pair $(r,\epsilon) = (\epsilon,\epsilon)$. Let $\delta_0 > 0$ be such that $\rho^{k^*}(\delta_0) \leq \epsilon$ and define $\delta = \min\{\epsilon, \delta_0\}$. Using this and item 2(b) it is now straightforward to verify that $|x_0|_{\mathcal{A}} \leq \delta$ implies $|\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon$ for all $k \in \mathbb{N}$.

Uniform global boundedness: Next we establish uniform global boundedness *i.e.*, there exist $\mu \geq 0$ and $\gamma \in \mathcal{K}_\infty$ such that, for all initial conditions and all solutions we have $|\phi(k,x_0)|_{\mathcal{A}} \leq \mu + \gamma(|x_0|_{\mathcal{A}})$. We take $\mu = 1$. Let $\delta > 0$ come from uniform stability for $\epsilon = 1$. Then, for each $\Delta \geq \delta$, let $k^*(\Delta)$ come from uniform global attractivity for $(r,\epsilon) = (\Delta,\Delta)$. Without loss of generality, we can assume that $k^*(\cdot)$ is non decreasing on $[\delta,\infty)$. Then, it may be verified that for all $x_0 \in \mathbb{R}^n$ satisfying $|x_0|_{\mathcal{A}} \geq \delta$, we have

$$|\phi(k,x_0)|_{\mathcal{A}} \leq \rho^{k^*(|x_0|_{\mathcal{A}})}(|x_0|_{\mathcal{A}}) \qquad \forall k \in \mathbb{N} \ . \tag{37}$$

Finally, we let γ be any function in class-K_∞ satisfying $\rho^{k^*(s)}(s) \leq \gamma(s)$ for all $s \in [\delta,\infty)$. It now can be verified that for all initial conditions, solutions and $k \in \mathbb{N}$, we have $|\phi(k,x_0)|_{\mathcal{A}} \leq \mu + \gamma(|x_0|_{\mathcal{A}})$.

Uniform stability and uniform global boundedness imply uniform global stability:
This is seen as follows: We take the uniform stability relationship $\epsilon \mapsto \delta(\epsilon) > 0$ and find a class-$\mathcal{K}_\infty$ function η such that $\eta(\epsilon) \leq \delta(\epsilon)$ for all $\epsilon > 0$. Next we note that η can be inverted on its range, denoted $[0,\eta_\infty)$. If $\eta_\infty = \infty$ then we define $\rho_2 := \eta^{-1}$. Otherwise let $\eta^* \in (0,\eta_\infty)$ satisfy $\eta^{-1}(\eta^*) = \mu + \gamma(\eta^*)$ and define

$$\rho_2(s) := \begin{cases} \eta^{-1}(s) & s \in [0,\eta^*] \\ \mu + \gamma(s) & s \geq \eta^* \ . \end{cases} \tag{38}$$

It is straightforward to see that $\rho_2 \in \mathcal{K}_\infty$ and that the uniform global stability bound holds with ρ_2.

Uniform global stability and uniform global attractivity imply UGAS:
Regarding the mapping $(\epsilon,\Delta) \mapsto k^*(\epsilon,\Delta)$ that comes from uniform global attractivity, we can assume without loss of generality that
- for each $\Delta > 0$, $k^*(\cdot,\Delta)$ is non-increasing on $\mathbb{R}_{>0}$ and, with uniform global stability, $k^*(\epsilon,\Delta) = 0$ for ϵ sufficiently large *i.e.*, $\epsilon \geq \rho_2(\Delta)$;
- for each $\epsilon > 0$ $k^*(\epsilon,\cdot)$ is nondecreasing on $\mathbb{R}_{>0}$.

Let $\psi_\Delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function that is strictly decreasing and onto $\mathbb{R}_{>0}$ (hence invertible on $\mathbb{R}_{>0}$) and satisfies

$$\psi_\Delta(\epsilon) \geq k^*(\Delta,\epsilon) \qquad \forall \epsilon > 0 \ . \tag{39}$$

We claim that $|x_0|_{\mathcal{A}} \leq \Delta$ implies $|\phi(k,x_0)|_{\mathcal{A}} \leq \psi_\Delta^{-1}(k)$ for all $k \in \mathbb{Z}_{\geq 1}$. To see this, for each $k \in \mathbb{Z}_{\geq 1}$ let $\epsilon := \psi_\Delta^{-1}(k)$ and then note that, from (39), $\psi_\Delta(\epsilon) = k \geq k^*(\Delta,\epsilon)$. Therefore, from uniform global attractivity, $|\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon = \psi_\Delta^{-1}(k)$. Finally, for each $\Delta > 0$ define $\psi_\Delta^{-1}(0) := \infty$ and define

$$\beta(s,k) := \min\left\{\rho_2(s), \inf_{\Delta \in (s,\infty)} \psi_\Delta^{-1}(k)\right\} \ . \tag{40}$$

It is straightforward to verify that $\beta \in \mathcal{KL}$ and that $|\phi(k,x_0)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, k)$ for all $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

($3 \Longrightarrow 2$): We only need to establish uniform global attractivity. Let $\Delta > 0$ and $\epsilon > 0$ be given. Using the function $\rho \in \mathcal{K}_\infty$ from uniform global stability, let $\delta := \rho^{-1}(\epsilon)$. Then $|x_0|_{\mathcal{A}} \leq \delta$ implies $|\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon$ for all $k \in \mathbb{N}$. Define

$$\eta^* = \min_{s \in [\delta,\rho(\Delta)]} \eta(s) \ . \tag{41}$$

Then let k^* be the smallest nonnegative integer satisfying

$$k^* \geq \frac{\alpha_2(\Delta)}{\eta^*} - 1 \ . \tag{42}$$

We claim that for each x_0 satisfying $|x_0|_{\mathcal{A}} \leq \Delta$ and $\phi \in \mathcal{S}(x_0)$, there exists $\overline{k} \in \{0,1,\ldots,k^*\}$ such that $\left|\phi(\overline{k},x_0)\right|_{\mathcal{A}} \leq \delta$. If not then

$$\sum_{k=0}^{\infty} \eta(|\phi(k,x_0)|_{\mathcal{A}}) \geq \sum_{k=0}^{k^*} \eta(|\phi(k,x_0)|_{\mathcal{A}}) > (k^*+1)\eta^* \geq \alpha_2(\Delta) \geq \alpha_2(|x_0|_{\mathcal{A}}) \ . \tag{43}$$

Considerting $\phi(k,x_0)$ for $k \geq \overline{k}$ as a solution starting at $\phi(\overline{k},x_0)$, it follows that $|x_0|_{\mathcal{A}} \leq \Delta$ and $k \geq k^*$ imply $|\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon$.

($4 \implies 2$): This implication is very similar to the previous one. We just need to establish uniform global attractivity. Let $\Delta > 0$ and $\epsilon > 0$ be given. Let $\delta > 0$ be such that $\alpha_1(s) \leq \alpha_2(\delta)$ implies $s \leq \epsilon$. Such a δ exists since $\alpha_1 \in \mathcal{K}$ and $\alpha_2 \in \mathcal{G}$. Note that $|x_0|_\mathcal{A} \leq \delta$ implies $|\phi(k, x_0)|_\mathcal{A} \leq \epsilon$ for all $k \in \mathbb{N}$ since

$$\alpha_1(|\phi(k, x_0)|_\mathcal{A}) \leq \sum_{i=0}^{\infty} \alpha_1(|\phi(i, x_0)|_\mathcal{A}) \leq \alpha_2(|x_0|_\mathcal{A}) \leq \alpha_2(\delta) \ . \tag{44}$$

Next let k^* be the smallest nonnegative integer satisfying

$$k^* \geq \frac{\alpha_2(\Delta)}{\alpha_1(\delta)} - 1 \ . \tag{45}$$

We claim that for each x_0 satisfying $|x_0|_\mathcal{A} \leq \Delta$ and $\phi \in \mathcal{S}(x_0)$, there exists $\overline{k} \in \{0, \dots, k^*\}$ such that $|\phi(\overline{k}, x_0)|_\mathcal{A} \leq \delta$. If not then

$$\sum_{k=0}^{\infty} \alpha_1(|\phi(k, x_0)|_\mathcal{A}) \geq \sum_{k=0}^{k^*} \alpha_1(|\phi(k, x_0)|_\mathcal{A}) > (k^* + 1)\alpha_1(\delta) \geq \alpha_2(\Delta) \geq \alpha_2(|x_0|_\mathcal{A}) \ . \tag{46}$$

Considering $\phi(k, x_0)$ for $k \geq \overline{k}$ as a solution starting at $\phi(\overline{k}, x_0)$, it follows that $|x_0| \leq \Delta$ and $k \geq k^*$ imply $|\phi(k, x_0)|_\mathcal{A} \leq \epsilon$.

($5 \implies 4$): We only need to establish part (a) of item 4 since part (b) is obvious. By assumption $\alpha_1(|w|_\mathcal{A}) \leq \alpha_2(|x_0|_\mathcal{A})$ for all $x_0 \in \mathbb{R}^n$ and $w \in F(x_0)$. Thus $|w|_\mathcal{A} \leq \alpha_1^{-1} \circ \alpha_2(|x_0|_\mathcal{A}) =: \rho(|x_0|_\mathcal{A})$. This establishes the result.

($5 \implies 3$): We only need to establish part (a) of item 3 since part (b) is obvious. By assumption, for all $k \in \mathbb{N}$, all $x_0 \in \mathbb{R}^n$

$$\alpha_1(|\phi(k, x_0)|_\mathcal{A}) \leq \sum_{i=0}^{\infty} \alpha_1(|\phi(i, x_0)|_\mathcal{A}) \leq \alpha_2(|x_0|_\mathcal{A}) \tag{47}$$

or, equivalently, $|\phi(k, x_0)|_\mathcal{A} \leq \alpha_1^{-1} \circ \alpha_2(|x_0|_\mathcal{A})$ *i.e.*, the origin is uniformly globally stable.

($1 \implies 5$): According to Lemma 1, there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\beta(s, k)) \leq \alpha_2(s) \left(\frac{1}{2}\right)^{k+1} \qquad \forall (s, k) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \ . \tag{48}$$

Then

$$\sum_{k=0}^{\infty} \alpha_1(|\phi(i, x_0)|_\mathcal{A}) \leq \sum_{k=0}^{\infty} \alpha_2(|x_0|_\mathcal{A}) \left(\frac{1}{2}\right)^{k+1} = \alpha_2(|x_0|_\mathcal{A}) \ . \tag{49}$$

($3 \implies 6$): Since items 3(a) and 6(a) are identical, we only need to show that 3(b) implies 6(b). This is immediate with the following definitions $\omega_{\delta,\Delta}(x) := \eta(|x|_\mathcal{A})$, $\omega_m := \min_{s \in [\delta, \Delta]} \eta(s)$, $\gamma := \alpha_2(\Delta)$. Then, item 6(b)i holds by definition of ω_m and 6(b)ii holds because for all $x_0 \in \mathcal{H}_\mathcal{A}(\delta, \Delta)$, $\phi \in \mathcal{S}(x_0)$ and all $k \in \mathbb{N}$ we have

$$\sum_{i=0}^{k} \omega_{\delta,\Delta}(\phi(i, x_0)) \leq \sum_{i=0}^{\infty} \eta(|\phi(i, x_0)|_\mathcal{A}) \leq \alpha_2(|x_0|_\mathcal{A}) \leq \alpha_2(\Delta) = \gamma \ .$$

($6 \implies 1$): Since $\mathcal{A}$ is assumed to be UGS we only need to prove UGA. From item 6(a), let $\rho \in \mathcal{K}_\infty$ be such that for all $x_0 \in \mathbb{R}^n$, $\phi \in \mathcal{S}(x_0)$ and $k \in \mathbb{N}$ we have

$$|\phi(k, x)|_\mathcal{A} \leq \rho(|x_0|_\mathcal{A}) \ . \tag{50}$$

Let the strictly positive numbers (r, ϵ) be given. Define $\Delta := \rho(r)$ and $\delta := \min\{\Delta, \rho^{-1}(\epsilon)\}$. Let δ, Δ generate $\omega_{\delta,\Delta}(x)$, ω_m and γ. Define $k^* := \max\left\{j \in \mathbb{N} : j \leq \frac{2\gamma}{\omega_m}\right\}$. We claim that for all $x_0 \in \mathcal{H}_\mathcal{A}(0, r)$ and all $\phi \in \mathcal{S}(x_0)$ there exists $k' \in [0, k^*]$ such that $|\phi(k', x_0)|_\mathcal{A} \leq \rho^{-1}(\epsilon)$. This establishes the result since the time invariance of the system and (50) imply that for all $x_0 \in \mathcal{H}_{\mathcal{A}}(0,r)$ and $k \geq k^*$ we have $|\phi(k,x_0)|_{\mathcal{A}} \leq \epsilon$.

Assume that the claim is not true that is, there exists $x_0 \in \mathcal{H}_{\mathcal{A}}(0,r)$ and $\phi \in \mathcal{S}(x_0)$ such that $|\phi(k,x_0)|_{\mathcal{A}} > \rho^{-1}(\epsilon)$ for all $k \in [0,k^*]$. From (50) and the definition of δ it follows that

$$\phi(k,x_0) \in \mathcal{H}_{\mathcal{A}}(\delta,\Delta) \qquad \forall k \in [0,k^*] \ .$$

It follows that $\omega_{\delta,\Delta}(\phi(k,x_0)) \geq \omega_m$ for all $k \in [0,k^*]$. Hence,

$$\sum_{i=0}^{k^*} \omega_{\delta,\Delta}(\phi(i,x_0)) \geq (k^*+1)\omega_m \geq \frac{2\gamma}{\omega_m}\omega_m = 2\gamma \ ,$$

which contradicts 6(b). Hence, the set $\mathcal{A}$ is UGA for the system (1). ■

4.3 Proof of Theorem 3

Let $y = [\Phi^T \ \Psi^T]^T$. We shall use the following two claims.

Claim 1 *Suppose that all conditions of Theorem 3 hold. Then, for each $\mu > 0$ there exists $\epsilon > 0$ such that*

(A): $\quad \{ \ \Phi^T\Phi = 1 \quad \& \quad \Psi^T\Psi \leq \mu^2 \quad \& \quad y^T Y_i y = 0 \ \ \forall i \in \{1,\ldots,m-1\} \ \}$

implies

(B): $\quad \{ \ y^T Y_m y \leq -\epsilon \ \} \ .$

□

Proof of Claim 1: From item 2d(i) of Theorem 3 we have that $y^T Y_i y = 0$ for all $i \in \{1,\ldots,m-1\}$ implies $y^T Y_m y \leq 0$. Moreover, item 2d(ii) of Theorem 3 guarantees that $y^T Y_m y < 0$ since from (A) we have $\Phi^T\Phi = 1 \neq 0$. Note that the first two inequalities in (A) imply that y belongs to a compact set. Finally, we take ϵ to be the minimum of $-y^T Y_m y$ over this compact set, which proves (B). ■

Claim 2 *Suppose that all conditions of Theorem 3 hold. Let all conditions of Theorem 3 hold. Let $\ell \in \{2,\ldots,m\}$, $\mu > 0$, $\epsilon > 0$ and a continuous function $\widetilde{Y}_\ell : \mathbb{R}^{n+p} \to \mathbb{R}$ be given and satisfy the property that*

(1): (A) implies (B) where

(A): $\quad \{ \ \Phi^T\Phi = 1 \quad \& \quad \Psi^T\Psi \leq \mu^2 \quad \& \quad y^T Y_i y = 0 \ \ \forall i \in \{1,\ldots,\ell-1\} \ \}$

(B): $\quad \{ \ y^T \widetilde{Y}_\ell y \leq -\epsilon \ \}$

Then,

(2): there exists $K_{\ell-1} > 0$ such that

(A):

$$\{ \ \Phi^T\Phi = 1 \quad \& \quad \Psi^T\Psi \leq \mu^2 \quad \& \quad y^T Y_i y = 0 \ \ \forall i \in \{1,\ldots,\ell-2\} \ \}$$

implies that

(B): $\quad \{ \ y^T (K_{\ell-1} Y_{\ell-1} + \widetilde{Y}_\ell) y \leq -\frac{\epsilon}{2} \ \}.$

$\square$

Proof of Claim 2: Due to item 2d(i) of Theorem 3, we can write:

$$\left.\begin{array}{c} y^T Y_1 y = 0 \\ \vdots \\ y^T Y_{\ell-2} y = 0 \end{array}\right\} \implies y^T Y_{\ell-1} y \leq 0 \tag{51}$$

Hence, since the property 1B holds, the property 2B is automatically satisfied when $y^T Y_{\ell-1} y = 0$, in fact with the upper bound $-\epsilon$. By continuity, there exists $\eta \in \mathbb{R}_{>0}$ such that the bound in the property 2B holds for all $K_{\ell-1} \in \mathbb{R}_{\geq 0}$ and y satisfying $y^T Y_{\ell-1} y \in [-\eta, 0]$ in addition to the conditions in 2B. Finally, letting M denote the maximum of $y^T \widetilde{Y}_\ell y$ over the set $\Phi^T \Phi = 1$, $\Psi^T \Psi \leq \mu^2$, we can take

$$K_{\ell-1} := \frac{\frac{\epsilon}{2} + M}{\eta}. \tag{52}$$

$\blacksquare$

We now prove the theorem. We apply the first claim and then repeatedly apply the second claim to get values $K_i > 0$, $i = \{1, \ldots, m-1\}$ such that for $\Phi^T \Phi = 1$ and $\Psi^T \Psi \leq \mu^2$ we have

$$y^T \left(\sum_{i=1}^{m-1} K_i Y_i + Y_m \right) y \leq -\frac{\epsilon}{2^{m-1}} .$$

Next, let

$$V(x,k) := \sum_{i=1}^{m-1} K_i W_i(x,k) + W_m(x,k), \tag{53}$$

$\Phi = \varphi(x)$ and $\Psi = \psi(x,k)$. Then, from item 2(c) it follows that for all $|x| \neq 0$ (which implies $\varphi(x) \neq 0$), $k \in \mathbb{N}$ and $f \in F(x,k)$

$$\begin{aligned} V(f,k+1) - V(x,k) &\leq \begin{bmatrix} \varphi(x) \\ \psi(x,k) \end{bmatrix}^T \left(\sum_{i=1}^{m-1} K_i Y_i + Y_m \right) \begin{bmatrix} \varphi(x) \\ \psi(x,k) \end{bmatrix} \\ &= |\varphi(x)| \left(\begin{bmatrix} \frac{\varphi(x)}{|\varphi(x)|} \\ \frac{\psi(x,k)}{|\varphi(x)|} \end{bmatrix}^T \left(\sum_{i=1}^{m-1} K_i Y_i + Y_m \right) \begin{bmatrix} \frac{\varphi(x)}{|\varphi(x)|} \\ \frac{\psi(x,k)}{|\varphi(x)|} \end{bmatrix} \right) |\varphi(x)| . \end{aligned} \tag{54}$$

Using item 2(b) and the result of the claims, we get

$$V(f,k+1) - V(x,k) \leq -\frac{\epsilon}{2^{m-1}} |\varphi(x)|^2 \leq -\frac{\epsilon}{2^{m-1}} \alpha_1(|x|) . \tag{55}$$

This bound also holds when $|x| = 0$ since, by virtue of the system being stable with linear growth, we have $|f| = 0$ when $|x| = 0$ and $f \in F(x,k)$, and from (21) we have $W_i(x,k) = 0$ whenever $|x| = 0$.

Letting $\Gamma \in \mathbb{R}_{\geq 1}$ characterize the system being stable with linear growth (item 1) and defining $K := 1 + \sum_{i=1}^{m-1} K_i$, it now follows that, for all $x_0 \in \mathbb{R}^n$, $\phi \in \mathcal{S}(x_0, k_0)$, and $k \in \mathbb{N}$,

$$\begin{aligned} \textstyle\sum_{j=0}^{k} \alpha_1(|\phi(j,k_0,x_0)|) &\leq \tfrac{2^{m-1}}{\epsilon} \left(V(x_0,k_0) - V(\phi(k+1,k_0,x_0),k+1) \right) \\ &\leq \tfrac{2^{m-1}}{\epsilon} \left(1 + \textstyle\sum_{i=1}^{m-1} K_i \right) \left(\alpha_2(|x_0|) + \alpha_2(|\phi(k+1,k_0,x_0)|) \right) \\ &\leq \tfrac{2^{m-1}}{\epsilon} \left(1 + \textstyle\sum_{i=1}^{m-1} K_i \right) \left(\alpha_2(|x_0|) + \alpha_2(\Gamma \cdot |x_0|) \right) \\ &\leq \tfrac{2^m K}{\epsilon} \alpha_2 \left(\Gamma \cdot |x_0| \right) . \end{aligned}$$

Using item 2(a) we have that the pair $(\alpha_1, \frac{2^m K}{\epsilon} \cdot \alpha_2)$ is cUGES, and thus from Remark 1 the pair $\left(\alpha_1(s), \frac{2^m K}{\epsilon} \cdot \alpha_2(\Gamma \cdot s)\right)$ is also cUGES. The result now follows directly from item 3 of Theorem 1. $\blacksquare$

4.4 Proof of Theorem 4

Throughout this proof, given $\delta > 0$ we define

$$\mathcal{H} := \{(x,z) : \delta \leq |x| \leq \Delta \ , \ |z| \leq \mu\} \ . \tag{56}$$

We first establish two claims.

Claim 3 *Let all conditions of Theorem 4 hold. Given $\delta \in (0, \Delta]$, there exists $\epsilon > 0$ such that:*

(A): $\{ \ (x,z) \in \mathcal{H} \quad \& \quad Y_i(x,z) = 0 \ \ \forall i \in \{1, \dots, m-1\} \ \}$

implies

(B): $\{ \ Y_m(x,z) \leq -\epsilon \ \}$.

□

Proof of Claim 3: We prove the claim by contradiction. Suppose that for each integer n, there exist $(x_n, z_n) \in \mathcal{H}$ such that $Y_i(x_n, z_n) = 0$ for all $i \in \{1, \dots, m-1\}$, and $Y_m(x_n, z_n) > -\frac{1}{n}$. By compactness of $\mathcal{H}$, the continuity of $Y_m(\cdot, \cdot)$, and item 2(b)i, the sequence (x_n, z_n) has an accumulation point $(x^*, z^*) \in \mathcal{H}$ such that $Y_i(x^*, z^*) = 0$ for all $i \in \{1, \dots, m\}$. By item 2(b)ii, this implies that $x^* = 0$ which contradicts the fact that $(x^*, z^*) \in \mathcal{H}$. ■

Claim 4 *Let all conditions of Theorem 4 hold. Let $\ell \in \{2, \dots, m\}$, $\widetilde{\epsilon} > 0$ and a continuous function $\widetilde{Y}_\ell : \mathbb{R}^{n+p} \to \mathbb{R}$ be given and satisfy the property that*

(1): (A) implies (B) where

(A): $\{ \ (x,z) \in \mathcal{H} \quad \& \quad Y_i(x,z) = 0 \ \ \forall i \in \{1, \dots, \ell-1\} \ \}$

(B): $\{ \ \widetilde{Y}_\ell(x,z) \leq -\tilde{\epsilon} \ \}$

Then,

(2): there exists $K_{\ell-1} > 0$ such that

(A): $\{ \ (x,z) \in \mathcal{H} \quad \& \quad Y_i(x,z) = 0 \quad \forall i \in \{1, \dots, \ell-2\} \ \}$

implies that

(B): $\{ \ K_{\ell-1} Y_{\ell-1}(x,z) + \widetilde{Y}_\ell(x,z) \leq -\frac{\widetilde{\epsilon}}{2} \ \}.$

□

Proof of Claim 4: By item 2(b)i, Property 2A implies that $Y_{\ell-1}(z, \psi) \leq 0$. Therefore Property 2A implies

$$K_{\ell-1} Y_{\ell-1}(x,z) + \widetilde{Y}_\ell(x,z) \leq \widetilde{Y}_\ell(x,z) \quad \forall K_{\ell-1} \geq 0. \tag{57}$$

Now if $Y_{\ell-1}(x,z) = 0$ then, due to Property 1, Property 2B holds for all $K_{\ell-1} \geq 0$ whenever Property 2A holds. We claim further that there exists $\tau > 0$ such that Property 2B holds whenever Property 2A holds and $Y_{\ell-1}(x,z) > -\tau$. Suppose not *i.e.*, for each integer n there exists $(x_n, z_n) \in \mathcal{H}$ such that $Y_{\ell-1}(x_n, z_n) > -\frac{1}{n}$ and

$$\widetilde{Y}_\ell(x_n, z_n) > -\frac{\widetilde{\epsilon}}{2}. \tag{58}$$

Then, by compactness of $\mathcal{H}$, continuity of $Y_{\ell-1}$, and item 2(b)i, the sequence (x_n, z_n) has an accumulation point (x^*, z^*) such that $Y_{\ell-1}(x^*, z^*) = 0$. But then from Property 1 we have that $\widetilde{Y}_\ell(x^*, z^*) \leq -\tilde{\epsilon}$. By continuity of $\widetilde{Y}_\ell(\cdot,\cdot)$ this contradicts (58) when n is large and associated with a subsequence converging to the accumulation point. Then, from the continuity of $\widetilde{Y}_\ell$ and compactness of $\mathcal{H}$ that we can pick $K_{\ell-1} > 0$ large enough to satisfy

$$-\tau K_{\ell-1} + \max_{(x,z)\in\mathcal{H}} \widetilde{Y}_\ell(x,z) \;\leq\; -\frac{\widetilde{\epsilon}}{2}$$

then Property 2A implies Property 2B. ■

We now use these two claims to prove Theorem 4. In particular, we will show that the condition 6 in Theorem 2 holds.

From items 1(a) and 1(b) of the theorem, we conclude that the system is UGS, that is there exists $\rho \in \mathcal{K}_\infty$ such that for all $x(k_0) = x_0 \in \mathbb{R}^n, k_0 \in \mathbb{N}$ and $\phi \in \mathcal{S}(x_0, k_0)$ we have

$$|\phi(k, k_0, x_0)| \leq \rho(|x_0|) \qquad \forall k \geq k_0 \ .$$

Let $\delta \leq \Delta$ and introduce $\widetilde{\Delta} := \rho(\Delta)$. Let $\widetilde{\Delta}$ generate μ via item 2 of the theorem and we let $(\delta, \widetilde{\Delta}, \mu)$ generate the set $\mathcal{H}$. Using this set $\mathcal{H}$ we apply the two claims. According to Claim 3, Property 1 of Claim 4 holds when $\ell = m$, $\widetilde{\epsilon} = \epsilon$ and $\widetilde{Y}_\ell = Y_m$. An application of Claim 4 with these choices provides a value K_{m-1} such that Property 1 of Claim 4 holds when $\ell = m-1$, $\widetilde{\epsilon} = \epsilon/2$ and $\widetilde{Y}_\ell = K_{m-1}Y_{m-1} + Y_m$. Continuing with this iteration, it follows that for each $\delta > 0$ there exists $\varepsilon > 0$ and positive real numbers K_i, $i = 1, \ldots, m-1$ such that, for all $(x,z) \in \mathcal{H}$

$$Z(x,z) := \sum_{i=1}^{m-1} K_i Y_i(x,z) + Y_m(x,z) \leq -\frac{\epsilon}{2^{m-1}} \ . \tag{59}$$

Next define the function $W : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ as

$$W(x,k) := \sum_{i=1}^{m-1} K_i W_i(x,k) + W_m(x,k) \ . \tag{60}$$

We denote

$$\epsilon_1 := \max_{\delta \leq |x| \leq \widetilde{\Delta}} U(x) \ .$$

According to the conditions of the theorem and the discussion above, we have that, for all $(x,k) \in \mathbb{R}^n \times \mathbb{N}$ with $|x| \leq \widetilde{\Delta}$, $f \in F(x,k)$

$$|W(x,k)| \leq \mu \left(1 + \sum_{i=1}^{m-1} K_i\right) =: \eta \ , \tag{61}$$

$$W(f, k+1) - W(x,k) \leq Z(x, \psi(x,k)) \ , \tag{62}$$

and, using the bound on ψ in the first inequality of (27) together with (59) we obtain that for all $(x,k) \in \mathbb{R}^n \times \mathbb{N}$ with $\delta \leq |x| \leq \widetilde{\Delta}$,

$$Z(x, \psi(x,k)) \leq -\frac{\varepsilon}{2^{m-1}} =: -\epsilon_2 \ . \tag{63}$$

Next, we show that condition 6 of Theorem 2 holds with the following definitions:

$$\begin{aligned}
\omega_{\delta,\Delta}(x,k) &:= KU(x) - Z(x, \psi(x,k)) \\
\omega_m &:= \epsilon_2 \\
\gamma &:= K\alpha_2(\Delta) + 2\eta \ ,
\end{aligned}$$

where $K := \frac{\eta + \epsilon_2}{\epsilon_1}$.

Note that we have already showed that the condition 6(a) of Theorem 2 holds. Next, we show that the condition 6(b)i of Theorem 2 holds. Indeed, since $U(\cdot)$, $Z(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ are continuous functions, so is $\omega_{\delta,\Delta}(\cdot,\cdot)$. Moreover, since $\widetilde{\Delta} \geq \Delta$ and $KU(x) \geq 0$, we have for all x such that $\delta \leq |x| \leq \Delta$ and $k \in \mathbb{N}$ that the following holds:

$$\omega_{\delta,\Delta}(x,k) = KU(x) - Z(x,\psi(x,k)) \geq -Z(x,\psi(x,k)) \geq \epsilon_2 = \omega_m ,$$

as desired. Finally, we show that the condition 6(b)ii of Theorem 2 holds. We define

$$\widetilde{V}(x,k) := KV(x,k) + W(x,k) .$$

Then, for all $|x| \leq \widetilde{\Delta}$, $k \in \mathbb{N}$, $f \in F(x,k)$ we have

$$\widetilde{V}(f,k+1) - \widetilde{V}(x,k) \leq -\omega_{\delta,\Delta}(x,k) ,$$

which implies using item 1(a), UGS and (61) that for all $k_0 \in \mathbb{N}$, $x(k_0) = x_0$ with $\delta \leq |x_0| \leq \Delta$, $\phi \in \mathcal{S}(x_0,k_0)$ and $k \geq k_0$ we have

$$\begin{aligned}
\sum_{i=0}^{k} \omega_{\delta,\Delta}(\phi(i,k_0,x_0),i) &\leq \widetilde{V}(x_0,k_0) - \widetilde{V}(\phi(k,k_0,x_0),k) \\
&= KV(x_0,k_0) + W(x_0,k_0) - KV(\phi(k,k_0,x_0),k) - W(\phi(k,k_0,x_0),k) \\
&\leq K\alpha_2(|x_0|) + |W(x_0,k_0)| + |W(\phi(k,k_0,x_0),k)| \\
&\leq K\alpha_2(\Delta) + 2\eta \\
&= \gamma , \qquad (64)
\end{aligned}$$

which completes the proof. ■

5 Conclusions

We have provided several results that can be used to verify UGAS and UGES of arbitrary closed sets that do not require the knowledge of Lyapunov functions. Instead, we assume appropriate summability conditions on trajectories of the system. We used these results to prove two Matrosov type results that can be used to verify UGES and UGAS of the origin. The results and their proofs presented here parallel the continuous-time and sampled-data counterparts but they are more straightforward and derived under different assumptions.

References

[1] E. A. Barbashin. *Introduction to the theory of stability.* Wolters-Noordhoff, Groningen, The Netherlands, 1970. Translated by Transcripta Service, London, edited by T. Lukes.

[2] E. A. Barbashin and N. N. Krasovskiĭ. O cyschestvovanii functziĭ Lyapunova v cluchae asimptoticheskoĭ ustoĭchivostĭ v tzelom. *Prikl. Mat. i Mekh.*, 18:345–350, 1954. English title: “On the existence of Lyapunov fuctions in the case of asymptotic stability in the large”. English apropriate translation: On the existence of Lyapunov fuctions in the case of asymptotic stability in the whole.

[3] P. Diamond and V. I. Opoitsev. Stability of linear difference and differential inclusions. *Automation and Remote Control*, 62(5):695–703, 2001.

[4] A. F. Filippov. *Differential equations with discontinuous right-hand sides.* Kluwer academic publisher, 1988.

[5] P. Giesel. Necessary conditions for the basin of attraction of a periodic orbit of non-smooth periodic systems. *Discrete and Continuous Dynamical Systems*, 18(2& 3):355–373, 2007.

[6] Rafal Goebel, Tingshu Hu, and Andrew R. Teel. Dual matrix inequalities in stability and performance analysis of linear differential/difference inclusions. In *Current trends in nonlinear systems and control*, Systems Control Found. Appl., pages 103–122. Birkhäuser Boston, Boston, MA, 2006.

[7] G. Grammel. Towards fully discretized differential inclusions. *Set-valued Analysis*, 11:1–8, 2003.

[8] L. Grüne, P. R. Kloeden, S. Siegmund, and F. R. Wirth. Lyapunov's second method for nonautonomous differential equations. *Discrete and Continuous Dynamical Systems*, 18(2& 3):375–403, 2007.

[9] W. Hahn. *Theory and Application of Liapunov's Direct Method*. Prentice Hall, New York, 1963.

[10] W. Hahn. *Stability of motion*. Springer-Verlag, New York, 1967.

[11] S. Hui and S. Zak. On the Lyapunov stability of discrete-time processes modeled by difference inclusions. *Syst. & Contr. Letters*, 10(3):207–299, 1988.

[12] C. Kellett. *Advances in converse and control Lyapunov functions*. PhD thesis, University of California, Santa Barbara, CA, USA, June 2002.

[13] C. M. Kellett and A. R. Teel. Sufficient conditions for robustness of $\mathcal{KL}$-stability for difference inclusions. *Math. of Cont. Sign. and Syst.*, 19:183–205, 2007.

[14] P. E. Kloeden and B. Schmalfuss. Lyapunov functions and attractors under variable time-step discretization. *Discrete and Continuous Dynamical Systems*, 2(2):163–172, 1996.

[15] A. Loría. Explicit convergence rates for MRAC-type systems. *Automatica*, 40(8):1465–1468, 2004.

[16] A. Loría F. Lamnabhi-Lagarrigue, and D. Nesic. Summation-type conditions for uniform asymptotic convergence in discrete-time systems: applications in identification. In *Proc. 44th. IEEE Conf. Decision Contr.*, pages 6591–6595, 2005.

[17] A. Loría E. Panteley, D. Popovic, and A. Teel. A nested Matrosov theorem and persistency of excitation for uniform convergence in stable non-autonomous systems. *IEEE Trans. on Automat. Contr.*, 50(2):183–198, 2005.

[18] A. M. Lyapunov. *The general problem of stability of motion*. Academic Press, New York, 1996. Commented reprint of the English translation by A. T. Fuller, first appeared in *Int. J. Contr.* , Vol. 55, 1992, made from the facsimile reprint (published in Ann. math. Studies **17**, Princeton 1947) of the French translation of the original which was published in Russian, in Kharkov 1892.

[19] V. M. Matrosov. On the stability of motion. *J. Appl. Math. Mech.*, 26:1337–1353, 1962.

[20] A. P. Molchanov and E. S. Pyatnitskĭ. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Syst. & Contr. Letters*, 13:59–64, 1989.

[21] D. Nesic, A. Loría E. Panteley, and A. R. Teel. On stability of sets for sampled-data nonlinear inclusions via their approximate discrete-time models and summability criteria. *SIAM J. on Contr. and Opt.*, 2009 (to appear).

[22] D. Nešić and A. Teel. Matrosov theorem for parameterized families of discrete-time systems. *Automatica*, 40(6):1025–1034, 2004.

[23] D. Nešić and A. R. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Trans. on Automat. Contr.*, 49(7):1103–1122, 2004.

[24] H. Poincaré. Sur le problème des trois corps et les équations de la dynamique. *Acta Mathematica*, 13:5–266, 1890.

[25] E. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Syst. & Contr. Letters*, 30:177–183, 1997.

[26] E. D. Sontag. Comments on integral variants of ISS. *Syst. & Contr. Letters*, 34:93–100, 1998.

[27] A. Teel, E. Panteley, and A. Loría. Integral characterizations of uniform asymptotic and exponential stability with applications. *Math. of Cont. Sign. and Syst.*, 15:177–201, 2002.

[28] S. K. Zavriyev and A. G. Perevozchikov. The direct Lyapunov method in investigating the attraction of trajectories of finite-difference inclusions. *USSR Computational Mathematics and Mathematical Physics*, 30(1):15–22, 1990.

[29] V. I. Zubov. *Mathematical Theory of the motion stability.* The St. Petersburg state university, St. Petersburg, Russia, 1997.

6 Appendix

Proof of Lemma 2: Let $\underline{\lambda}, \overline{\lambda} \in (0,1)$ and $\overline{\Gamma} \in \mathbb{R}_{\geq 1}$ come from the definition of cUGES. Let $p \in \mathbb{N}$ satisfy $\mu^{p+1} \leq \underline{\lambda}$. Such a p exists since $\underline{\lambda} \in (0,1)$ and $\mu \in [0,1)$. We claim that (2) holds with

$$\Gamma := \frac{\overline{\Gamma}}{\overline{\lambda}^{\frac{p}{p+1}}}; \qquad \lambda := \overline{\lambda}^{\frac{1}{p+1}} \ .$$

Indeed, for $k \in \mathbb{N}$, let $j \in \mathbb{N}$ satisfy $k \in \{j(p+1), (j+1)(p+1)-1\}$ and note that

$$\begin{array}{rcl} \mu^k \alpha_2(s) & \leq & \mu^{j(p+1)} \alpha_2(s) \\ & \leq & \underline{\lambda}^j \alpha_2(s) \\ & \leq & \underline{\lambda}^j \alpha_1(\overline{\Gamma} \cdot s) \\ & \leq & \alpha_1(\overline{\lambda}^j \overline{\Gamma} \cdot s\ ) \\ & \leq & \alpha_1\left(s \cdot \overline{\Gamma}\ \overline{\lambda}^{\frac{k-p}{p+1}} \right) \\ & = & \alpha_1\left(s \cdot \dfrac{\overline{\Gamma}}{\overline{\lambda}^{\frac{p}{p+1}}} \left(\overline{\lambda}^{\frac{1}{p+1}} \right)^k \right) \ . \end{array}$$