Changing supply rates for input-output to state stable discrete-time nonlinear systems with applications

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Abstract

We present results on changing supply rates for input output to state stable (IOSS) discrete-time nonlinear systems. Our results can be used to combine two Lyapunov functions, none of which can be used to verify that the system has a certain property, into a new composite Lyapunov function from which the property of interest can be concluded. The results are stated for parameterized families of discrete-time systems that naturally arise when an approximate discrete-time model is used to design a controller for a sampled-data system. We present several applications of our results: (i) a Lasalle criterion for input to state stability (ISS) of discrete-time systems; (ii) constructing ISS lyapunov functions for time-varying discrete-time cascaded systems; (iii) testing ISS of discrete-time systems using positive semidefinite Lyapunov functions; (iv) observer based input to state stabilization of discrete-time systems. Our results are exploited in a case study of a two link manipulator and some simulation results that illustrate advantages of our approach are presented.
Keywords: Discrete-time; Input-to-State Stability; Lyapunov method; Nonlinear.

1 Introduction

The Lyapunov method is one of the most important and useful methods in stability analysis and design of nonlinear control systems (see for example [17, 27]). Lyapunov functions, which are the main tool in this method, can be used to characterize various properties of control systems, such as stability, detectability and passivity. Unfortunately, there is no general systematic way of finding Lyapunov functions. Hence, developing methods for constructing Lyapunov functions are of utmost importance.

A very useful method for a partial construction of Lyapunov functions was introduced in [29] where it was shown how it is possible to combine two Lyapunov functions, none of which can be used to conclude a property of interest, into a new composite Lyapunov function from which the desired property follows. Results in [29] apply to the analysis of input to state stability (ISS) property of continuous-time cascade-connected systems. In [1] a similar proof technique was used to combine a Lyapunov function whose derivative is negative semidefinite and another Lyapunov function that characterizes a detectability property, which is called input-output to state stability (IOSS) (see [31]), into a new Lyapunov function from which ISS of a continuous-time system follows. A discrete-time counterpart of results in [29] was presented in [25]. These results and proof techniques were used in discrete-time backstepping [24], stability of continuous-time cascades [29, 3], stability of discrete-time cascades [25], continuous-time stabilization of robot manipulators [1] and L_p stability of time-varying nonlinear sampled-data systems [36]. A related Lyapunov based method for interconnected ISS continuous-time systems satisfying a small-gain condition can be found in [14].

The purpose of this paper is to present a general and unifying framework for partial constructions of Lyapunov functions for families of discrete-time systems parameterized by a positive parameter (sampling

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period). We consider semiglobal practical stability properties of these systems that arise naturally when approximate discrete-time models are used to design controllers for sampled-data nonlinear systems. Motivation for our approach is presented in the next section and more information can be found in [23, 22, 26, 24].

Our main technical result is contained in Lemma 4.1 where we prove a general result on changing supply rates for a generalized notion of input output to state stability (IOSS) for discrete-time nonlinear systems. We refer to this property as "IOSS with measuring functions" (see Definition 3.1) and show that many important properties considered in the literature are special cases of this general property. The general result of Lemma 4.1 is central to the paper since it allows us to prove several new results and generalize several existing results in the literature in a unified framework. Lemma 4.1 is a discrete-time version (as well as generalization) of the continuous-time result in [1] and a generalization of the discrete-time result in [25].

Using Lemma 4.1 we present two partial constructions of a Lyapunov function from two other Lyapunov functions in Theorems 4.1 and 4.2. The construction in Theorem 4.1 was used in [1] for continuous-time systems, whereas the construction in Theorem 4.2 was used in [29] and [25] for continuous-time and discrete-time systems respectively. However, because of the generality of the "IOSS with measuring functions property" that we use, we obtain more general results by using the same Lyapunov function constructions as in the cited references. While the statements of our main results in discrete-time are very similar to continuous-time results of [1, 29], the proof technique is notably different and it requires a judicious use of the Mean Value Theorem (see the proof of Lemma 4.1).

Finally, we apply our results in a unified manner to several problems: (i) a Lasalle criterion for ISS of discrete-time systems (see also [1]); (ii) constructing ISS Lyapunov functions for time-varying discrete-time cascade systems (see also [13, 25, 29, 12]); (iii) testing ISS of discrete-time systems using positive semidefinite Lyapunov functions (see also [11, 4]); (iv) observer based input to state stabilization of discrete-time systems (see also [15, 16]). We emphasize that our results have potential for further important applications. Finally, we apply our results in a case study of a two link manipulator and illustrate the usefulness of our results via simulations.

The paper is organized as follows. In Section 2 we provide background and motivation for our approach and present a result from [26] that we will use in the case study. In Section 3 we introduce notation and definitions. Main results are stated in Section 4. Four applications of the main results and a case study are presented respectively in Section 5 and Section 6. The proofs of main results are provided in Section 7. Conclusions are presented in Section 8.

2 Background and motivation

In order to put our results in a better context, we revise some results in the literature that motivate our approach. In particular, we recall a result from [26] (see Theorem 2.1) that directly motivates results of the current paper. Moreover, Theorem 2.1 is used in Section 6 to design a controller for a two link manipulator based on its Euler approximate discrete-time model. For any unfamiliar notation refer to the next section.

Our work is motivated by the fact that most control systems are nowdays sampled-data in nature. Indeed, the controller is usually implemented digitally using a computer and it is inter-connected with a continuous-time plant via D/A and A/D converters. One possible approach for controller design in this case is to design a continuous-time controller and then discretize it for digital implementation. This approach, which is sometimes referred to as emulation, was pursued for instance in [21, 34]. The emulation approach often fails to yield satisfactory performance because it usually requires very fast sampling that may not be achievable due to the hardware limitations. Moreover, the emulation design ignores sampling at the controller design step and hence it is reasonable to expect that if sampling is taken into account in the design, then better performance can be achieved.

This has lead to approaches that use the discrete-time model of the plant for the controller design. However, in the case of nonlinear plants it is in general not possible to obtain the exact discrete-time model of the plant and an approximate discrete-time plant model has to be used instead. This approach was taken, for instance, in [8, 9, 19, 28] for several special classes of systems. Recently, a general unified framework for controller design based on approximate discrete-time models was presented in [23] for the stabilization problem and further generalized in [26] for the input-to-state stabilization problem. Advantages of this approach were illustrated in [24] where is was shown for that the Euler based backstepping controller may outperform the emulated backstepping controller.

It is the main purpose of the current paper to further contribute to the approach that was pursued in [23, 26]. In order to motivate better our contribution we present a result from [26] on input-to-state stabilization via approximate discrete-time models that is also needed in Section 6. Our interest in input-to-state stabilization is motivated by numerous applications of this robust stability property that have appeared in the literature [?, 12, 18].

Consider a continuous-time nonlinear plant with disturbances:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ are respectively the state, control input and exogenous disturbance. The control is taken to be a piecewise constant signal $u(t) = u(kT) =: u(k),\ \forall t \in [kT, (k+1)T),\ k \in \mathbb{N}$, where $T > 0$ is the sampling period. Suppose, for simplicity, that the disturbances $w(\cdot)$ are constant during sampling intervals, that is $w(t) = w(k), \forall t \in [kT, (k+1)T)$ (a more general situation when $w(\cdot)$ is an arbitrary measurable disturbance was considered in [26]). Also, we assume that some combination (output) or all of the states ($x(k) := x(kT)$) are available at sampling instant $kT, k \in \mathbb{N}$. The exact discrete-time model for the plant (1), which describes the plant behavior at sampling instants kT, is obtained by integrating the initial value problem

$$\dot{x}(t) = f(x(t), u(k), w(t)) \ , \tag{2}$$

with given $w(k)$, $u(k)$ and $x_0 = x(k)$, over the sampling interval $[kT, (k+1)T]$.

If we denote by $x(t)$ the solution of the initial value problem (2) at time t with given $x_0 = x(k)$, $u(k)$ and $w(k)$, then the exact discrete-time model of (1) can be written as:

$$x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(x(\tau), u(k), w(k)) d\tau =: F_T^e(x(k), u(k), w(k)) \ . \tag{3}$$

We emphasize that F_T^e is not known in most cases. Indeed, in order to compute F_T^e we have to solve the initial value problem (2) analytically and this is usually impossible since f in (1) is nonlinear. Hence, we will use an approximate discrete-time model of the plant to design a controller.

Different approximate discrete-time models can be obtained using different methods, such as a classical Runge-Kutta numerical integration scheme (such as Euler) for the initial value problem (2) [33, 20]. The approximate discrete-time model can be written as

$$x(k+1) = F_T^a(x(k), u(k), w(k)) \ . \tag{4}$$

For instance, the Euler approximate model is $x(k+1) = x(k) + Tf(x(k), u(k), w(k))$. The sampling period T is assumed to be a design parameter which can be arbitrarily assigned. Since we are dealing with a family of approximate discrete-time models F_T^a, parameterized by T, in order to achieve a certain objective we need in general to obtain a family of controllers, parameterized by T. We consider a family of dynamic feedback controllers

$$\begin{aligned} z(k+1) &= G_T(x(k), z(k)) \\ u(k) &= u_T(x(k), z(k)) \ , \end{aligned} \tag{5}$$

where $z \in \mathbb{R}^{n_z}$.

We emphasize that if the controller (5) input-to-state stabilizes the approximate model (4) for all small T, this does not guarantee that the same controller would approximately input-to-state stabilize the exact (3) model for all small T (for counter-examples see [23]). The following result that was proved in [26] gives sufficient conditions for input-to-state stabilization of (3) via controllers (5) that are designed using (4).

Theorem 2.1 *Suppose that there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\tilde{\gamma} \in \mathcal{K}$, and for any strictly positive real numbers $(\Delta_1, \Delta_2, \Delta_3, \nu)$ there exist $\rho \in \mathcal{K}_\infty$, strictly positive real numbers T^*, L and M such that for all $T \in (0, T^*)$ there exists a function $V_T : \mathbb{R}^{n_x+n_z} \to \mathbb{R}_{\geq 0}$ such that for all $|(x,z)| \leq \Delta_1$, $|u| \leq \Delta_2$, $|w| \leq \Delta_3$ and $T \in (0,T^*)$ we have:*

- **ISS Lyapunov conditions for approximate:**

$$\begin{aligned} \alpha_1(|(x,z)|) &\leq V_T(x,z) \leq \alpha_2(|(x,z)|) & (6)\\ V_T(F_T^a(x,u_T(x,z),w), G_T(x,z)) - V_T(x,z) &\leq T\Big(-\alpha_3(|(x,z)|) + \tilde{\gamma}(|w|) + \nu\Big) , & (7)\end{aligned}$$

and, moreover, for all x_1, x_2, z with $\max\{|(x_1,z)|, |(x_2,z)|\} \leq \Delta_1$ and all $T \in (0,T^*)$, we have

$$|V_T(x_1,z) - V_T(x_2,z)| \leq L\,|x_1 - x_2| \ . \tag{8}$$

- **consistency between F_T^a and F_T^e:**

$$|F_T^e(x,u,w) - F_T^a(x,u,w)| \leq T\rho(T) \ . \tag{9}$$

- **uniform local boundedness of u_T:**

$$|u_T(x,z)| \leq M \ . \tag{10}$$

Then, there exists $\beta \in \mathcal{KL}, \gamma \in \mathcal{G}$ such that for any strictly positive real numbers $(\widetilde{\Delta}_1, \widetilde{\Delta}_2, \widetilde{\nu})$ there exists $\widetilde{T} > 0$ such that for all $|(x(0),z(0))| \leq \widetilde{\Delta}_1$, $\|w\|_\infty \leq \widetilde{\Delta}_2$ and $T \in (0,\widetilde{T})$ the solutions of (3), (5) satisfy:

- **ISS of exact:**

$$|(x(k),z(k))| \leq \beta(|(x(0),z(0))|, kT) + \gamma(\|w\|_\infty) + \widetilde{\nu}, \quad \forall k \geq 0 \ . \tag{11}$$

■

We emphasize that the consistency condition in Theorem 2.1 is checkable although F_T^e is not known in general. This condition is commonly used in numerical analysis literature [33]. Hence, Theorem 2.1 presents sufficient conditions on the approximate model, controller and continuous-time plant model that guarantee that the controller which input-to-state stabilizes an approximate discrete-time plant model would also approximately input-to-state stabilize the exact discrete-time model. We emphasize that under very weak conditions this guarantees that the inter-sample behaviour of the sample-data closed-loop system would also be bounded (see [22]).

The conditions (6), (7) of Theorem 2.1 are hard to check in general and one of the main contributions of this paper is in presenting technical results that can be used to verify that conditions similar to (6), (7) hold for a family of paramtereized discrete-time systems. These technical conditions can be then used in conjunction with Theorem 2.1 to design input-to-state stabilizing controllers for sampled-data nonlinear plants via their approximate discrete-time models. This approach is illustrated in Section 6 where we consider input-to-state stabilization of a two link manipluator via its Euler approximate discrete-time model.

3 Preliminaries

The set of real numbers is denoted by $\mathbb{R}$. $\mathcal{SN}$ denotes the class of all smooth nondecreasing functions $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which satisfy $q(t) > 0$ for all $t > 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{G}$ if it is continuous, nondecreasing and zero at zero. It is of class $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing; and it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Functions of class $\mathcal{K}_\infty$ are invertible. Given two functions $\alpha(\cdot)$ and $\gamma(\cdot)$, we denote their composition and multiplication respectively as $\alpha \circ \gamma(\cdot)$ and $\alpha(\cdot) \cdot \gamma(\cdot)$. $|x|$ denotes the 1-norm of a vector $x \in \mathbb{R}^n$, where $|x| := \sum_{i=1}^{n} |x_i|$.

Motivated by the discussion of the previous section, we consider a parameterized family of discrete-time nonlinear systems of the following form:

$$\begin{aligned} x(k+1) &= F_T(x(k), u(k)) \\ y(k) &= h(x(k)) \end{aligned} \tag{12}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are respectively the state, input and output of the system. It is assumed that F_T is well defined for all x, u and sufficiently small T, $F_T(0,0) = 0$ for all T for which F_T is defined, $h(0) = 0$ and F_T and h are continuous. $T > 0$ is the sampling period, which parameterizes the system and can be arbitrarily assigned. Parameterized discrete-time systems (12) commonly arise when an approximate discrete-time model is used for designing a digital controller for a nonlinear sampled-data system (see [23, 26]). Non parameterized discrete-time systems are a special case of (12) when T is constant (for instance $T = 1$). We use the following definition.

Definition 3.1 *The system (12) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-semiglobally practically input-output to state stable ($(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-SP-IOSS) with measuring functions, if there exist functions $\underline{\alpha}$, $\overline{\alpha}$, $\alpha \in \mathcal{K}_\infty$, and λ, $\sigma \in \mathcal{G}$, functions $w_{\underline{\alpha}} : \mathbb{R}^n \to \mathbb{R}^{n_{\underline{\alpha}}}$, $w_{\overline{\alpha}} : \mathbb{R}^n \to \mathbb{R}^{n_{\overline{\alpha}}}$, $w_\alpha : \mathbb{R}^n \to \mathbb{R}^{n_\alpha}$, $w_\lambda : \mathbb{R}^n \to \mathbb{R}^{n_\lambda}$, $w_\sigma : \mathbb{R}^m \to \mathbb{R}^{n_\sigma}$, $w_x : \mathbb{R}^n \to \mathbb{R}^{n_x}$, $w_u : \mathbb{R}^m \to \mathbb{R}^{n_u}$, which are zero at zero[1], and for any triple of strictly positive real numbers Δ_x, Δ_u, ν, there exists $T^* > 0$ and for all $T \in (0, T^*)$ there exists a smooth function $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $|w_x(x)| \leq \Delta_x$, $|w_u(u)| \leq \Delta_u$ the following holds:*

$$\underline{\alpha}(|w_{\underline{\alpha}}(x)|) \leq V_T(x) \leq \overline{\alpha}(|w_{\overline{\alpha}}(x)|) \tag{13}$$

$$V_T(F_T(x,u)) - V_T(x) \leq -T\alpha(|w_\alpha(x)|) + T\lambda(|w_\lambda(x)|) + T\sigma(|w_\sigma(u)|) + T\nu \,. \tag{14}$$

The functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_λ, w_σ, w_x and w_u are called measuring functions; $\underline{\alpha}$, $\overline{\alpha}$, α, λ, σ are called bounding functions; α, λ, σ are called supply functions; and V_T is called a SP-IOSS Lyapunov function. If $T^* > 0$ exists such that (13) and (14), with $\nu = 0$, hold for all $T \in (0, T^*)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, the property holds globally and the system (12) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS with measuring functions. ■

Often, when all functions are clear from the context, we refer to the property defined in Definition 3.1 as SP-IOSS (or IOSS if the property holds globally). Moreover, if the system is SP-IOSS (respectively IOSS) with $\lambda = 0$ then we say that the system is SP-ISS (respectively ISS). SP-IOSS with measuring functions is quite a general notion that covers a range of different properties of nonlinear discrete-time systems, such as stability, detectability, output to state stability, etc. For example, by letting $\lambda = 0$, $\sigma = 0$ and $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = x$, we obtain the standard Lyapunov characterization for asymptotic stability of (12). By letting $\lambda = 0$, $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = x$, and $w_\sigma(u) = u$, we obtain a Lyapunov characterization for (semiglobal practical) ISS. The reason for introducing such a general property in Definition 3.1 is that we will apply our results to a range of its different special cases (see Section 4) for particular choices of λ, σ and the measuring functions. Hence, Definition 3.1 is a very compact way of defining various different properties to which our results apply.

The following two lemmas and remark are used in proving our main results (Theorems 4.1 and 4.2).

Lemma 3.1 *[29] Assume that the functions β, $\beta' \in \mathcal{K}$ are such that $\beta'(s) = O[\beta(s)]$ as $s \to 0^+$. Then there exists a function $q \in \mathcal{SN}$ so that $\beta'(s) \leq q(s)\beta(s)$, $\forall s \geq 0$.* ■

Lemma 3.2 *[29] Assume that the functions β, $\beta' \in \mathcal{K}$ are such that $\beta(r) = O[\beta'(r)]$ as $r \to +\infty$. Then there exists a function $q \in \mathcal{SN}$ so that $q(r)\beta(r) \leq \beta'(r)$, $\forall r \geq 0$.* ■

Remark 3.1 *Since for any $\alpha \in \mathcal{K}$ we have $\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2)$ for all $s_1 \geq 0, s_2 \geq 0$, then for any $\alpha_1, \alpha_2 \in \mathcal{K}$, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}$ such that the following holds:*

$$\underline{\alpha}(s_1 + s_2) \leq \alpha_1(s_1) + \alpha_2(s_2) \leq \overline{\alpha}(s_1 + s_2), \quad \forall s_1 \geq 0, s_2 \geq 0, \tag{15}$$

where $\underline{\alpha}(s) := \min\{\alpha_1(\frac{s}{2}), \alpha_2(\frac{s}{2})\}$ and $\overline{\alpha}(s) := \max\{2\alpha_1(s), 2\alpha_2(s)\}$. ■

[1] We need to check if it is better to assume this condition only in the application where we use it.

4 Main results

In this section, we state our main results, which consist of two main theorems (Theorems 4.1 and 4.2), where we show two partial constructions of a SP-IOSS Lyapunov function from two auxiliary Lyapunov functions. Some corollaries following from our main results are also presented. First, we discuss our approach with more detail.

When using the SP-IOSS property of Definition 3.1 to check if a certain property (such as, stability, input-to-state stability or some other special case of SP-IOSS property) holds, one usually needs to have that all bounding functions and the corresponding measuring functions satisfy appropriate conditions. For example, if we want to check global asymptotic stability of the origin of the input-free system (12) then we need to have:

$$\begin{aligned} \underline{\alpha}(\left|w_{\underline{\alpha}}(x)\right|) \leq V_{1T}(x) \leq \overline{\alpha}(|w_{\overline{\alpha}}(x)|) \\ V_{1T}(F_T(x,0)) - V_{1T}x) \leq -T\alpha(|w_\alpha(x)|) \ , \end{aligned} \tag{16}$$

for all $x \in \mathbb{R}^n$ and $T \in (0, T^*)$, for some $T^* > 0$; $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ and α is positive definite; $\left|w_{\underline{\alpha}}(x)\right|$ is positive definite and radially unbounded; and $|w_\alpha(x)|$ is positive definite.

However, it is often the case that some of the desired conditions are not satisfied by a candidate SP-IOSS Lyapunov function V_{1T}. For instance, in the above example it may happen that $\left|w_{\underline{\alpha}}(x)\right|$ and/or $|w_\alpha(x)|$ is positive semidefinite. If this happens, then one possibility is by trying to prove that the appropriate property holds with weaker conditions. For instance, in the above example when $\left|w_{\underline{\alpha}}(x)\right|$ is positive semidefinite one can use the results of [11] to check stability of the system. If, on the other hand, $|w_\alpha(x)|$ is positive semidefinite one can use the celebrated LaSalle's invariance principle to check stability. Another approach that is taken in this paper is to construct a new SP-IOSS Lyapunov function that satisfies all the required conditions. This construction is carried out by first introducing an auxiliary SP-IOSS Lyapunov function V_{2T} and then combining the two functions into a new SP-IOSS Lyapunov function V_T. We note that neither V_{1T} nor V_{2T} can be used alone to conclude the desired property, whereas their appropriate combination can.

We present in Theorems 4.1 and 4.2 two constructions that can be used (under different conditions on bounding and measuring functions) to construct a new SP-IOSS Lyapunov function V_T from two SP-IOSS Lyapunov functions V_{1T} and V_{2T}. In particular, in Theorem 4.1 we construct V_T from V_{1T} and V_{2T}, by using a scaling function $\rho \in \mathcal{K}_\infty$ of a special form, in the following way:

$$V_T = V_{1T} + \rho(V_{2T}) \ , \tag{17}$$

while in Theorem 4.2 we use two scaling functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$ of a special form to construct V_T as follow:

$$V_T = \rho_1(V_{1T}) + \rho_2(V_{2T}) \ . \tag{18}$$

In Theorem 4.1 we use weaker conditions on the measuring functions of V_{1T} than in Theorem 4.2. This leads to a less general construction (17) than (18). The important point to be made is that the measuring functions for the new function V_T are different from measuring functions of either V_{1T} or V_{2T}. It is this fact that allows us to conclude that the system has a property which was impossible to conclude by using either V_{1T} or V_{2T} alone.

We first present Lemma 4.1, which is instrumental in proving our main results. The lemma is a discrete-time version, as well as a generalization, of the lemma on changing supply rates for IOSS continuous-time systems in [1]. Lemma 4.1 also generalizes the result of [25] on changing supply rates for ISS discrete-time systems. We use the following construction that was introduced in [1, 29]. Given an arbitrary $q \in \mathcal{SN}$, we define:

$$\rho(s) := \int_0^s q(\tau)d\tau \ , \tag{19}$$

where it is easy to see that $\rho \in \mathcal{K}_\infty$ and ρ is smooth. Suppose that we have a SP-IOSS Lyapunov function V_T for a system, and then consider a new function $\rho(V_T)$. In Lemma 4.1, we state conditions under which the new function is also a SP-IOSS Lyapunov function for the system.

Lemma 4.1 *Let the following conditions be satisfied:*

1. *System (12) is* $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$*-SP-IOSS with measuring functions* $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_λ, w_σ, w_x *and* w_u.

2. *There exist* $\underline{\kappa}, \overline{\kappa} \in \mathcal{K}_\infty$ *such that* $\underline{\kappa}(|w_\alpha(x)|) \leq \left|w_{\underline{\alpha}}(x)\right|$ *and* $|w_{\overline{\alpha}}(x)| \leq \overline{\kappa}(|w_\alpha(x)|)$, $\forall x \in \mathbb{R}^n$.

3. *For any strictly positive real numbers* Δ_x, Δ_u *there exist strictly positive real numbers* M *and* T^* *such that*

$$\begin{aligned} &|w_x(x)| \leq \Delta_x, |w_u(u)| \leq \Delta_u, T \in (0, T^*) \Longrightarrow \\ &\qquad \max\{|w_{\overline{\alpha}}(F_T(x,u))|, |w_{\overline{\alpha}}(x)|, |w_\lambda(x)|, |w_\sigma(u)|\} \leq M \ . \end{aligned} \tag{20}$$

Then for any $q \in \mathcal{SN}$ *and* $\rho \in \mathcal{K}_\infty$ *defined by (19) there exist* $\underline{\alpha}'$, $\overline{\alpha}'$, α', λ', σ' *such that the system (12) is* $(\rho(V_T), \underline{\alpha}', \overline{\alpha}', \alpha', \lambda', \sigma')$*-SP-IOSS with the same measuring functions, where* $\underline{\alpha}'(s) = \rho \circ \underline{\alpha}(s)$, $\overline{\alpha}'(s) = \rho \circ \overline{\alpha}(s)$, $\alpha'(s) = \frac{1}{4} q \circ \frac{1}{2} \underline{\alpha} \circ \underline{\kappa}(s) \cdot \alpha(s)$, $\lambda'(s) = 2q \circ \theta_\lambda(s) \cdot \lambda(s)$, $\sigma'(s) = 2q \circ \theta_\sigma(s) \cdot \sigma(s)$, *and*

$$\theta_\sigma(s) := \overline{\alpha} \circ \overline{\kappa} \circ \alpha^{-1} \circ 4\sigma(s) + 2\sigma(s) \tag{21}$$

$$\theta_\lambda(s) := \overline{\alpha} \circ \overline{\kappa} \circ \alpha^{-1} \circ 4\lambda(s) + 2\lambda(s) \ . \tag{22}$$

■

Lemma 4.1 provides us with some flexibility when constructing a SP-IOSS Lyapunov function V_T from two Lyapunov functions using (17) and (18). We proved the result for semiglobal practical IOSS since this is a property that naturally arises when an approximate discrete-time model is used for controller design of a sampled-data nonlinear systems (see Example 6 in the next section). Some of the conditions of Lemma 4.1 are rather technical but they were considered in order to prove the result in a considerable generality.

Remark 4.1 *It is instructive to discuss the third condition of Lemma 4.1 since it appears to be the least intuitive. We consider its two special cases for stability of the origin and stability of arbitrary (not necessarily compact) sets.*

Let us first consider stability of the origin of the input-free system (12). In this case, the conditions (16) need to hold and we can assume without loss of generality that $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = w_x(x) = x$. *In this case the third condition of Lemma 4.1 holds if* $F_T(x, 0)$ *is bounded on compact sets, uniformly in* $T \in (0, T^*)$. *This holds if* $F_T(0,0) = 0$ *for all* $T \in (0, T^*)$ *and* $F_T(x, 0)$ *is continuous in* x, *unifromly in* $T \in (0, T^*)$. *This condition is rather natural to use and it is often assumed in the literature (see for instance [13]).*

Suppose now that (16) hold with $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = w_x(x) = |x|_{\mathcal{A}}$, *where* $\mathcal{A}$ *is a non-empty closed set. In this case, the condition 3 of Lemma 4.1 requires that for any* Δ_x *there exists* M *and* T^* *such that*

$$|x|_{\mathcal{A}} \leq \Delta_x, T \in (0, T^*) \Longrightarrow |F_T(x, 0)|_{\mathcal{A}} \leq M \ .$$

This condition also appears to be natural and similar conditions have been used in the literature []. ■

We can also state a similar result to Lemma 4.1, when the IOSS property holds globally, that is when the system (12) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS with measuring functions. It is interesting that in this case the third condition of Lemma 4.1 is not needed to prove the result.

Corollary 4.1 *Let the following conditions be satisfied:*

1. *System (12) is* $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$*-IOSS with measuring functions* $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_λ *and* w_σ.

2. *There exist* $\underline{\kappa}, \overline{\kappa} \in \mathcal{K}_\infty$ *such that* $\underline{\kappa}(|w_\alpha(x)|) \leq \left|w_{\underline{\alpha}}(x)\right|$ *and* $|w_{\overline{\alpha}}(x)| \leq \overline{\kappa}(|w_\alpha(x)|)$, $\forall x \in \mathbb{R}^n$.

Then for any $q \in \mathcal{SN}$ and $\rho \in \mathcal{K}_\infty$ defined by (19) there exist $\underline{\alpha}'$, $\overline{\alpha}'$, α', λ', σ' such that the system (12) is $(\rho(V_T), \underline{\alpha}', \overline{\alpha}', \alpha', \lambda', \sigma')$-IOSS with the same measuring functions, where $\underline{\alpha}', \overline{\alpha}', \alpha', \lambda', \sigma'$ are the same as in Lemma 4.1. ■

We present our main results below. Note that Theorem 4.1 is a discrete-time version, as well as generalization, of the continuous-time results in [1], whereas Theorem 4.2 has appeared in a simpler form in [25], which is a discrete-time version of [29], when $\lambda = 0$, $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = x$, $w_\sigma(u) = u$ and all properties hold globally.

Theorem 4.1 *Suppose that:*

1. *the system (12) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \sigma_1)$-SP-ISS with measuring functions $w_{\underline{\alpha}_1}$, $w_{\overline{\alpha}_1}$, w_{α_1}, w_{σ_1}, w_{x_1}, w_{u_1};*
2. *the system (12) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-SP-IOSS with measuring functions $w_{\underline{\alpha}_2}$, $w_{\overline{\alpha}_2}$, w_{α_2}, w_{λ_2}, w_{σ_2}, w_{x_2}, w_{u_2}, and there exist $\underline{\kappa}_2, \overline{\kappa}_2 \in \mathcal{K}_\infty$, such that the second and third conditions of Lemma 4.1 hold;*
3. *there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that $|w_{\lambda_2}(x)| \le \gamma_1(|w_{\alpha_1}(x)|)$, $|w_{x_2}(x)| \le \gamma_2(|w_{x_1}(x)|)$, $|w_{u_2}(u)| \le \gamma_3(|w_{u_1}(u)|)$ for all $x \in \mathbb{R}^n$. $u \in \mathbb{R}^m$;*
4. $\limsup_{s\to+\infty} \frac{\lambda_2(s)}{\alpha_1(s)} < +\infty$.

Then there exists $\rho \in \mathcal{K}_\infty$ such that the system (12) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \sigma)$-SP-ISS with new measuring functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_σ, w_x, w_u where

$$V_T = V_{1T} + \rho(V_{2T}) \ , \tag{23}$$

and the new measuring functions are

$$\begin{array}{lll} w_{\underline{\alpha}}(x) := \left|w_{\underline{\alpha}_1}(x)\right| + \left|w_{\underline{\alpha}_2}(x)\right| , & w_{\overline{\alpha}}(x) := |w_{\overline{\alpha}_1}(x)| + |w_{\overline{\alpha}_2}(x)| , & w_x(x) := w_{x_1}(x), \\ w_\alpha(x) := |w_{\alpha_2}(x)| , & w_\sigma(u) := |w_{\sigma_1}(u)| + |w_{\sigma_2}(u)| , & w_u(u) := w_{u_1}(u). \end{array} \tag{24}$$

■

Remark 4.2 *We note that in Theorems 4.1 and 4.2 we concentrate only on verifying conditions similar to (6), (7). However, we note that if the functions V_{1T} and V_{2T} satisfy the local Lipschitz condition (8), then the new Lyapunov function constructed using either (17) or (18) would also satisfy the local Lipschitz condition. Hence, results of Theorem 4.1 and 4.2 can be used to verify the first condition of Theorem 2.1.* ■

In the next result, we consider a stronger condition for the Lyapunov function V_{1T}, so that we can relax the condition 4 of the Theorem 4.1.

Theorem 4.2 *Suppose that:*

1. *the system (12) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \sigma_1)$-SP-ISS with measuring functions $w_{\underline{\alpha}_1}$, $w_{\overline{\alpha}_1}$, w_{α_1}, w_{σ_1}, w_{x_1}, w_{u_1} and there exist $\underline{\kappa}_1, \overline{\kappa}_1 \in \mathcal{K}_\infty$, such that the second and third conditions of Lemma 4.1 hold;*
2. *the system (12) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-SP-IOSS with measuring functions $w_{\underline{\alpha}_2}$, $w_{\overline{\alpha}_2}$, w_{α_2}, w_{λ_2}, w_{σ_2}, w_{x_2}, w_{u_2} and there exist $\underline{\kappa}_2, \overline{\kappa}_2 \in \mathcal{K}_\infty$, such that the second and third condition of Lemma 4.1 hold;*
3. *there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that $|w_{\lambda_2}(x)| \le \gamma_1(|w_{\alpha_1}(x)|)$, $|w_{x_2}(x)| \le \gamma_2(|w_{x_1}(x)|)$, $|w_{u_2}(u)| \le \gamma_3(|w_{u_1}(u)|)$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.*

Then there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that the system (12) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \sigma)$-SP-ISS and new measuring functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_σ, w_x, w_u, where

$$V_T = \rho_1(V_{1T}) + \rho_2(V_{2T}) \ , \tag{25}$$

and the new measuring functions are

$$\begin{array}{lll} w_{\underline{\alpha}}(x) := \left|w_{\underline{\alpha}_1}(x)\right| + \left|w_{\underline{\alpha}_2}(x)\right|, & w_{\overline{\alpha}}(x) = \left|w_{\overline{\alpha}_1}(x)\right| + \left|w_{\overline{\alpha}_2}(x)\right|, & w_x(x) := w_{x_1}(x), \\ w_\alpha(x) := \left|w_{\alpha_1}(x)\right| + \left|w_{\alpha_2}(x)\right|, & w_\sigma(u) := \left|w_{\sigma_1}(u)\right| + \left|w_{\sigma_2}(u)\right|, & w_u(u) := w_{u_1}(u). \end{array} \tag{26}$$

■

Note that the main difference between Theorems 4.1 and 4.2 is that in Theorem 4.1 we cannot apply Lemma 4.1 to the Lyapunov function V_{T1}, since the second and third conditions of the lemma do not hold. Consequently, we need an extra condition on the bounding functions (condition 4 in Theorem 4.1) and we use a less general construction (23) than in Theorem 4.2 where we use (25).

As a consequence of Corollary 4.1, we can also state global results of the Theorem 4.1 and 4.2, if V_{1T} and V_{2T} characterize IOSS property of system (12) in a global sense. The following corollary is derived from Theorem 4.1.

Corollary 4.2 *Suppose that:*

1. *the system (12) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \sigma_1)$-ISS with measuring functions $w_{\underline{\alpha}_1}$, $w_{\overline{\alpha}_1}$, w_{α_1}, w_{σ_1}, w_{x_1}, w_{u_1};*

2. *the system (12) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-IOSS with measuring functions $w_{\underline{\alpha}_2}$, $w_{\overline{\alpha}_2}$, w_{α_2}, w_{λ_2}, w_{σ_2}, w_{x_2}, w_{u_2} and there exist $\underline{\kappa}_2, \overline{\kappa}_2 \in \mathcal{K}_\infty$, such that the second condition of Corollary 4.1 holds;*

3. *there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that $|w_{\lambda_2}(x)| \leq \gamma_1(|w_{\alpha_1}(x)|)$, $|w_{x_2}(x)| \leq \gamma_2(|w_{x_1}(x)|)$, $|w_{u_2}(u)| \leq \gamma_3(|w_{u_1}(u)|)$ for all $x \in \mathbb{R}^n$, $x \in \mathbb{R}^n$.*

4. $\limsup_{s \to +\infty} \frac{\lambda_2(s)}{\alpha_1(s)} < +\infty$.

Then there exists $\rho \in \mathcal{K}_\infty$ such that the system (12) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \sigma)$-ISS with V_T is given by (23) and new measuring functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_σ, w_x, w_u are given by (24). ■

The next corollary, which is derived from Theorem 4.2 considers the same conditions as those of Theorem 4.2, when IOSS property holds globally for the system (12).

Corollary 4.3 *Suppose that:*

1. *the system (12) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \sigma_1)$-ISS with measuring functions $w_{\underline{\alpha}_1}$, $w_{\overline{\alpha}_1}$, w_{α_1}, w_{σ_1}, w_{x_1}, w_{u_1}, and there exist $\underline{\kappa}_1, \overline{\kappa}_1 \in \mathcal{K}_\infty$, such that the second condition of Corollary 4.1 holds;*

2. *the system (12) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-IOSS with measuring functions $w_{\underline{\alpha}_2}$, $w_{\overline{\alpha}_2}$, w_{α_2}, w_{λ_2}, w_{σ_2}, w_{x_2}, w_{u_2} and there exist $\underline{\kappa}_2, \overline{\kappa}_2 \in \mathcal{K}_\infty$, such that the second condition of Corollary 4.1 holds;*

3. *there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that $|w_{\lambda_2}(x)| \leq \gamma_1(|w_{\alpha_1}(x)|)$, $|w_{x_2}(x)| \leq \gamma_2(|w_{x_1}(x)|)$, $|w_{u_2}(u)| \leq \gamma_3(|w_{u_1}(u)|)$ for all $x \in \mathbb{R}^n$.*

Then there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that the system (12) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \sigma)$-ISS with V_T is given by (25) and new measuring functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_σ, w_x, w_u are given by (26). ■

5 Applications

In this section we show how our results can be specialized to deal with several important situations: (i) a LaSalle criterion for SP-ISS of parameterized discrete-time systems;; (ii) SP-ISS of parameterized time-varying discrete-time cascaded systems; (iii) SP-ISS via positive semidefinite Lyapunov functions for parameterized discrete-time systems; (iv) observer based ISS controller design for paramtereized discrete-time systems. We emphasize that our results are general enough for other applications to be also possible. This section also illustrates the generality of Definition 3.1, since we show that a range of properties considered in the literature are in fact special cases of the SP-IOSS property with measuring functions.

5.1 LaSalle criterion for SP-ISS

In this subsection, we present a novel result which is a discrete-time version of the continuous-time result presented in [1]. This result is a direct consequence of Theorem 4.1. We use this result in the next section to design a digital controller for a two link manipulator via its Euler approximate model.

It was shown in [1] that if for the continuous time system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{27}$$

there exist two Lyapunov functions V_1 and V_2, the functions $\underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$, and $\sigma_1, \lambda_2, \sigma_2 \in \mathcal{G}$ satisfying the following conditions for all x and u:

$$\begin{aligned} \underline{\alpha}_1(|x|) &\leq V_1(x) \leq \overline{\alpha}_1(|x|) \\ \tfrac{\partial V_1}{\partial x} f(x,u) &\leq -\alpha_1(|y|) + \sigma_1(|u|) \end{aligned} \tag{28}$$

and

$$\begin{aligned} \underline{\alpha}_2(|x|) &\leq V_2(x) \leq \overline{\alpha}_2(|x|) \\ \tfrac{\partial V_2}{\partial x} f(x,u) &\leq -\alpha_2(|x|) + \lambda_2(|y|) + \sigma_2(|u|) \end{aligned} \tag{29}$$

and, moreover,

$$\lim \sup_{s \to +\infty} \frac{\lambda_2(s)}{\alpha_1(s)} < +\infty, \tag{30}$$

then there exists an ISS Lyapunov function V, functions $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{G}$ satisfying the following conditions for all x and u:

$$\begin{aligned} \underline{\alpha}(|x|) &\leq V(x) \leq \overline{\alpha}(|x|) \\ \tfrac{\partial V}{\partial x} f(x,u) &\leq -\alpha(|x|) + \sigma(|u|) \end{aligned} .$$

In other words, the system (27) is ISS. In [1] the properties in (28) and (29) were respectively referred to as quasi input-to-state stability (qISS) and input-output-to-state stability (IOSS). Using Theorem 4.1 we can state a semiglobal practical version of this result for parameterized discrete-time systems (12). In particular, we show that semiglobal practical qISS, semiglobal practical IOSS and the condition (30) imply semiglobal practical ISS. We use the following assumption:

Assumption 5.1 *For any strictly positive real numbers Δ_x, Δ_u there exist strictly positive real number M, T^* such that $|x| \leq \Delta_x, |u| \leq \Delta_u, T \in (0, T^*)$ implies $|F_T(x,u)| \leq M$.* ■

We state now a discrete-time version of the result in [1].

Corollary 5.1 *Consider the system (12) and suppose that Assumption 5.1 holds. Suppose that there exist $\underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$, and $\sigma_1, \lambda_2, \sigma_2 \in \mathcal{G}$ such that:*

1. *for any triple of strictly positive real numbers $(\Delta_x, \Delta_u, \nu)$ there exists $T^* > 0$ and for any $T \in (0, T^*)$ there exist $V_{1T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $V_{2T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $T \in (0, T^*)$ we have the following:*

$$
\begin{aligned}
&SP - qISS \left\{ \begin{array}{rcl} \underline{\alpha}_1(|x|) & \leq & V_{1T}(x) \ \leq \ \overline{\alpha}_1(|x|) \\ V_{1T}(F_T(x,u)) - V_{1T}(x) & \leq & T\Big(- \alpha_1(|y|) + \sigma_1(|u|) + \nu \Big) \end{array} \right. \\
&SP - IOSS \left\{ \begin{array}{rcl} \underline{\alpha}_2(|x|) & \leq & V_{2T}(x) \ \leq \ \overline{\alpha}_2(|x|) \\ V_{2T}(F_T(x,u)) - V_{2T}(x) & \leq & T\Big(- \alpha_2(|x|) + \lambda_2(|y|) + \sigma_2(|u|) + \nu \Big) \end{array} \right.
\end{aligned}
\tag{31}
$$

2. *the condition (30) holds.*

Then, there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{G}$ such that for any triple of strictly positive real numbers $(\widetilde{\Delta}_x, \widetilde{\Delta}_u, \widetilde{\nu})$ there exists $\widetilde{T} > 0$ and for any $T \in (0, \widetilde{T})$ there exist $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $|x| \leq \widetilde{\Delta}_x$, $|u| \leq \widetilde{\Delta}_u$, $T \in (0, \widetilde{T})$ we have:

$$
SP - ISS \left\{ \begin{array}{rcl} \underline{\alpha}(|x|) & \leq & V_T(x) \ \leq \ \overline{\alpha}(|x|) \\ V_T(F_T(x,u)) - V_T(x) & \leq & T\Big(- \alpha(|x|) + \sigma(|u|) + \widetilde{\nu} \Big) \ . \end{array} \right.
\tag{32}
$$

■

Proof of Corollary 5.1: It can be seen immediately that all conditions of Theorem 4.1 hold, by noting that: (i) the system (12) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \sigma_1)$-SP-ISS with measuring functions $w_{\underline{\alpha}_1}(x) = w_{\overline{\alpha}_1}(x) = w_{x_1}(x) = x$, $w_{\alpha_1}(x) = h(x) = y$, $w_{\sigma_1}(u) = w_{u_1}(u) = u$; (ii) the system (12) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-SP-IOSS with measuring functions $w_{\underline{\alpha}_2}(x) = w_{\overline{\alpha}_2}(x) = w_{\alpha_2}(x) = w_{x_2}(x) = x$, $w_{\lambda_2}(x) = h(x) = y$ and $w_{\sigma_2}(u) = w_{u_2}(u) = u$; the second condition of Lemma 4.1 holds since $w_{\underline{\alpha}_2}(x) = w_{\overline{\alpha}_2}(x) = w_{\alpha_2}(x)$; from Assumption 5.1 and Remark 4.1 we have that the third condition of Lemma 4.1 holds; hence, the second condition of Theorem 4.1 holds; (iii) the third condition of Theorem 4.1 holds since $w_{\alpha_1}(x) = w_{\lambda_2}(x) = h(x) = y$, $w_{x_1}(x) = w_{x_2}(x) = x$ and $w_{u_1}(u) = w_{u_2}(u) = u$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; (iv) the fourth condition of Theorem 4.1 follows trivially from the second condition of the corollary.

Therefore, applying Theorem 4.1 and defining the new SP-ISS Lyapunov function V_T as in (23), we obtain that the system (12) is SP-ISS with measuring functions $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = w_x(x) = x$ and $w_\sigma(u) = w_u(u) = u$. ■

5.2 SP-ISS of time-varying cascade systems

A novel result on SP-ISS for time-varying discrete-time cascade-connected system is presented in this subsection. This result is a direct consequence of Theorem 4.2 and it generalizes the main result of [25] in two directions: (i) the result is stated for semiglobal practical ISS (only global stability was considered in [25]); (ii) the result is stated for time-varying cascade-connected systems (only time-invariant cascade-connected systems were considered in [25]). We note that similar non Lyapunov based proofs of the same result can be found in [13] for non paramtereized discrete-time systems.

Consider the time-varying discrete-time system:

$$
\begin{aligned}
x(k+1) &= F_T(k, x(k), z(k), u(k)) \\
z(k+1) &= G_T(k, z(k), u(k)) \ ,
\end{aligned}
\tag{33}
$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$ and $u \in \mathbb{R}^m$. The state of the overall system is denoted as $\tilde{x} := (x^T \ z^T)^T$, $\tilde{x} \in \mathbb{R}^n$, where $n := n_x + n_z$. We will assume the following:

Assumption 5.2 *For any strictly positive real numbers $\Delta_{\tilde{x}}, \Delta_u$ there exist strictly positive real numbers M and T^* such that*

$$
|\tilde{x}| \leq \Delta_{\tilde{x}}, |u| \leq \Delta_u, T \in (0, T^*), k \geq 0 \Longrightarrow \max\{|F_T(k,x,z,u)|\,, |G_T(k,z,u)|\} \leq M \ . \tag{34}
$$

■

The family of systems (33) is not in the form (12) which is time invariant. However, we can still apply results of our paper in the following way. The time-varying system (33) can be written as an augmented *time-invariant* system in the following way:

$$\begin{aligned} x(k+1) &= F_T(p(k), x(k), z(k), u(k)) \\ z(k+1) &= G_T(p(k), z(k), u(k)) \\ p(k+1) &= p(k) + 1 \ , \end{aligned} \tag{35}$$

where $p \in \mathbb{R}$ is a new state variable. Then it is standard to show that semiglobal practical uniform ISS of the time-varying system (33) with respect to the origin $(x, z) = (0, 0)$ can be deduced from semiglobal practical ISS of the time-invariant system (35) with respect to a noncompact set $\mathcal{A} := \{(\tilde{x}, p) : \tilde{x} = 0\}$. Note also that we can write $|\tilde{x}| = |(\tilde{x}, p)|_{\mathcal{A}}$.

In the next result we show that SP-ISS Lyapunov function for the overall system (35) can be constructed from Lyapunov functions for individual subsystems in (35). In particular, we can state the following:

Corollary 5.2 *Consider the system (33) and suppose that Assumption 5.2 holds. Suppose that there exist* $\underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \ \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$, *and* $\sigma_1, \lambda_1, \sigma_2 \in \mathcal{G}$ *such that for any triple of strictly positive real numbers* $(\Delta_{\tilde{x}}, \Delta_u, \nu)$ *there exists* $T^* > 0$ *and for any* $T \in (0, T^*)$ *there exist* $V_{1T} : \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ *and* $V_{2T} : \mathbb{R} \times \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$ *such that for all* $|\tilde{x}| \leq \Delta_{\tilde{x}}$, $|u| \leq \Delta_u$, $p \geq 0$, $T \in (0, T^*)$ *we have the following:*

$$\begin{aligned} \underline{\alpha}_1(|x|) \ &\leq \ V_{1T}(p, x) \ \leq \ \overline{\alpha}_1(|x|) \\ V_{1T}(p+1, F_T(p, x, z)) - V_{1T}(p, x) \ &\leq \ T\Big(-\alpha_1(|x|) + \lambda_1(|z|) + \sigma_1(|u|) + \nu \Big) \\ \underline{\alpha}_2(|z|) \ &\leq \ V_{2T}(p, z) \ \leq \ \overline{\alpha}_2(|z|) \\ V_{2T}(p+1, G_T(p, z, u)) - V_{2T}(p, z) \ &\leq \ T\Big(-\alpha_2(|z|) + \sigma_2(|u|) + \nu \Big) \ . \end{aligned} \tag{36}$$

Then, there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ *and* $\sigma \in \mathcal{G}$ *such that for any triple of strictly positive real numbers* $(\widetilde{\Delta}_{\tilde{x}}, \widetilde{\Delta}_u, \widetilde{\nu})$ *there exists* $\widetilde{T} > 0$ *and for any* $T \in (0, \widetilde{T})$ *there exist* $V_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that for all* $|\tilde{x}| \leq \widetilde{\Delta}_{\tilde{x}}$, $|u| \leq \widetilde{\Delta}_u$, $p \geq 0$, $T \in (0, \widetilde{T})$ *we have:*

$$SP-ISS \left\{ \begin{array}{c} \underline{\alpha}(|\tilde{x}|) \leq V_T(p, x, z) \leq \overline{\alpha}(|\tilde{x}|) \\ V_T(p+1, F_T(p, x, z, u), G_T(p, z, u)) - V_T(p, x, z) \leq T\Big(-\alpha(|\tilde{x}|) + \sigma(|u|) + \widetilde{\nu} \Big) \ . \end{array} \right. \tag{37}$$

■

Proof of Corollary 5.2: It follows directly from Assumption 5.2 and conditions of the corollary that all conditions of Theorem 4.2 hold. Indeed, we have that: (i) the system is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \sigma_1, \lambda_1)$-SP-IOSS with measuring functions $w_{\underline{\alpha}_1}(\tilde{x}, p) = w_{\overline{\alpha}_1}(\tilde{x}, p) = w_{\alpha_1}(\tilde{x}, p) = x$, $w_{\lambda_1}(\tilde{x}, p) = z$, $w_{x_1}(\tilde{x}, p) = \tilde{x}$, $w_{\sigma_1}(u) = w_{u_1}(u) = u$, so that $\underline{\kappa}_1$, $\overline{\kappa}_1$ exist; morover, from Assumption 5.2 we have that the third condition of Lemma 4.1 holds; hence, condition 2 of Theorem 4.2 holds; (ii) the system is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \sigma_2)$-SP-ISS with measuring functions $w_{\underline{\alpha}_2}(\tilde{x}, p) = w_{\overline{\alpha}_2}(\tilde{x}, p) = w_{\alpha_2}(\tilde{x}, p) = z$, $w_{x_2}(\tilde{x}, p) = \tilde{x}$, $w_{\sigma_2}(u) = w_{u_2}(u) = u$ and $\lambda_2 = 0$, so that $\underline{\kappa}_2$, $\overline{\kappa}_2$ exist; moreover, from Assumption 5.2 we have that the third condition of Lemma 4.1 holds; hence, condition 1 of Theorem 4.2 holds; (iii) $|w_{\lambda_1}(\tilde{x}, p)| = |w_{\alpha_2}(\tilde{x}, p)| = |z|$, $|w_{x_1}(\tilde{x}, p)| = |w_{x_2}(\tilde{x}, p)| = |\tilde{x}|$ and $|w_{u_1}(u)| = |w_{u_2}(u)| = |u|$ for all $\tilde{x} \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; hence, condition 3 of Theorem 4.2 holds.

Therefore, applying Theorem 4.2 and defining the new SP-ISS Lyapunov function V_T as in (25), we obtain that the system (33) is SP-ISS with measuring functions $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = w_x(x) = \tilde{x}$ and $w_\sigma(u) = w_u(u) = u$. ■

5.3 SP-ISS via positive semidefinite Lyapunov functions

The problem of checking stability using positive semidefinite Lyapunov functions has been considered in [4] for continuous-time systems and in [11] for discrete-time systems. The idea is to use a Lyapunov function $V(x)$, which is positive semidefinite, to check stability of a system. An approach taken in [4, 11] was to use a trajectory based proof to prove stability of the origin of the system. In particular, besides appropriate conditions on the Lyapunov function it was required in [4, 11] that all trajectories in the maximal invariant subset of the set $Z := \{x : V(x) = 0\}$ satisfy the $\epsilon-\delta$ definition of asymptotic stability (this property was referred to as conditional stability to the set Z).

We note that the results on stability of cascade-connected systems in [25, 29] and the previous section can be interpreted as a special case of testing ISS using positive semidefinite Lyapunov functions. However, this approach is different from the one in [4, 11] since an ISS Lyapunov function is constructed explicitly from ISS and IOSS Lyapunov funcitons of subsystems. The advantage of the approach of [25, 29] is that it leads to a construction of a Lyapunov function for the overall system. A disadvantage is that it requires usually stronger conditions and it appears to apply only to a special class of cascade-connected systems. However, we show here that the same approach can be used with little modifications to test semiglobal practical ISS of general parameterized discrete-time systems (12) that are not in the cascade form. In particular, we can state:

Corollary 5.3 *Consider the family of systems (12) and suppose that Assumption 5.1 holds. Suppose that there exist* $\underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$, $\sigma_1, \lambda_1, \sigma_2 \in \mathcal{G}$ *and positive semidefinite functions* $W_1 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and* $W_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *whose sum* $W_1(x) + W_2(x)$ *is positive definite and radially unbounded, such that for any triple of strictly positive real numbers* $(\Delta_x, \Delta_u, \nu)$ *there exists* $T^* > 0$ *and for any* $T \in (0, T^*)$ *there exist* $V_{1T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and* $V_{2T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that for all* $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $T \in (0, T^*)$ *we have the following:*

$$\begin{array}{rcl} \underline{\alpha}_1(W_1(x)) & \leq & V_{1T}(x) \ \leq \ \overline{\alpha}_1(W_1(x)) \\ V_{1T}(F_T(x,u)) - V_{1T}(x) & \leq & T\Big(-\alpha_1(W_1(x)) + \lambda_1(W_2(x)) + \sigma_1(|u|) + \nu \Big) \\ \underline{\alpha}_2(W_2(x)) & \leq & V_{2T}(x) \ \leq \ \overline{\alpha}_2(W_2(x)) \\ V_{2T}(F_T(x,u)) - V_{2T}(x) & \leq & T\Big(-\alpha_2(W_2(x)) + \sigma_2(|u|) + \nu \Big) \end{array} \tag{38}$$

Then, there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ *and* $\sigma \in \mathcal{G}$ *such that for any triple of strictly positive real numbers* $(\widetilde{\Delta}_x, \widetilde{\Delta}_u, \widetilde{\nu})$ *there exists* $\widetilde{T} > 0$ *and for any* $T \in (0, \widetilde{T})$ *there exist* $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that for all* $|x| \leq \widetilde{\Delta}_x$, $|u| \leq \widetilde{\Delta}_u$, $T \in (0, \widetilde{T})$ *we have:*

$$SP-ISS \left\{ \begin{array}{rcl} \underline{\alpha}(|x|) \ \leq \ V_T(x) & \leq & \overline{\alpha}(|x|) \\ V_T(F_T(x,u)) - V_T(x) & \leq & T\Big(-\alpha(|x|) + \sigma(|u|) + \widetilde{\nu} \Big) \ . \end{array} \right. \tag{39}$$

■

Proof of Corollary 5.3: It can be seen immediately that all conditions of Theorem 4.1 hold, by noting that: (i) the system (12) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$-SP-IOSS with measuring functions $w_{\underline{\alpha}_1}(x) = w_{\overline{\alpha}_1}(x) = w_{\alpha_1}(x) = W_1(x)$, $w_{x_1}(x) = x$, $w_{\lambda_1}(x) = W_2(x)$, $w_{\sigma_1}(u) = w_{u_1}(u) = u$, so that $\underline{\kappa}_1$, $\overline{\kappa}_1$ exist; moreover, from Assumption 5.1 and Remark 4.1 we have that the third condition of Lemma 4.1 holds; hence, condition 2 of Theorem 4.2 holds; (ii) the system (12) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \sigma_2)$-SP-ISS with measuring functions $w_{\underline{\alpha}_2}(x) = w_{\overline{\alpha}_2}(x) = w_{\alpha_2}(x) = W_2(x)$, $w_{x_2}(x) = x$ and $w_{\sigma_2}(u) = w_{u_2}(u) = u$, so that $\underline{\kappa}_2$, $\overline{\kappa}_2$ exist; moreover, from Assumption 5.1 and Remark 4.1 we have that the third condition of Lemma 4.1 holds; hence, condition 1 of Theorem 4.2 holds; (iii) the third condition of Theorem 4.2 holds since $w_{\alpha_2}(x) = w_{\lambda_1}(x) = W_2(x)$, $w_{x_1}(x) = w_{x_2}(x) = x$ and $w_{u_1}(u) = w_{u_2}(u) = u$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$;

Then, applying Theorem 4.2 and defining the new SP-ISS Lyapunov function V_T as in (25), we obtain that the system (12) is SP-ISS with measuring functions $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = W_1(x) + W_2(x)$, $w_x(x) = x$ and $w_\sigma(u) = w_u(u) = u$. The conclusion follows from the fact that $W_1(x)+W_2(x)$ is a positive definite and radially unbounded and hence there exist $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathcal{K}_\infty$ such that $\widetilde{\alpha}_1(|x|) \leq W_1(x)+W_2(x) \leq \widetilde{\alpha}_2(|x|)$ for all $x \in \mathbb{R}^n$. ■

5.4 Observer based input-to-state stabilization of discrete-time systems

Observer based stabilization of discrete-time nonlinear systems that was considered in [15, 16] uses a very similar construction to the ones considered in this paper. It was shown in [15, 16] that if a discrete-time plant can be robustly stabilized with full state feedback (in an ISS sense) and there exists an observer for the system satisfying appropriate Lyapunov conditions (that is, the system is weakly detectable), then the plant is also stabilized using the controller/observer pair where the controller uses the state estimate obtained from the observer. Both local and global results were considered in [15, 16].

In this section, we show that our results, particularly Theorem 4.2, can be used to generalize results of [15, 16] in two directions: (i) we present results on observer based input-to-state stabilization of discrete-time systems (in [15, 16] only stabilization was considered); (ii) results on semiglobal practical ISS of parameterized systems (12) are presented (in [15, 16] only global and local stabilization of non-parameterized discrete-time systems were considered).

In this section we consider the parameterized family of plants:

$$\begin{aligned} x(k+1) &= F_T(x(k), u(k), v(k)) \\ y(k) &= h(x(k)) \ , \end{aligned} \tag{40}$$

where u and v are respectively the control and exogeneous inputs, with the following observer

$$z(k+1) = G_T(z(k), h(x(k)), u(k), v(k)) \ , \tag{41}$$

and controller

$$u(k) = \phi_T(z(k)) \tag{42}$$

that are defined for sufficiently small T. Let $\tilde{x} := (x^T \ z^T)^T$, annd we assume the following:

Assumption 5.3 *For any strictly positive real numbers $\Delta_{\tilde{x}}, \Delta_u, \Delta_v$ there exist strictly positive real numbers M and T^* such that*

$$|\tilde{x}| \leq \Delta_{\tilde{x}}, |u| \leq \Delta_u, |v| \leq \Delta_v, T \in (0, T^*) \Longrightarrow \max\{|F_T(x,u,v)|\,, |G_T(x,z,u,v)|\,, |\phi_T(z)|\} \leq M \ . \tag{43}$$

■

Then, we can state the following result:

Corollary 5.4 *Consider the family of systems (40), (41) and (42) and suppose that Assumption 5.3 holds. Suppose that there exist, $\underline{\alpha}_1, \overline{\alpha}_1, \alpha_1$, $\underline{\alpha}_2, \overline{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$, $\sigma_1, \lambda_1, \sigma_2 \in \mathcal{G}$, such that for any triple of strictly positive real numbers $(\Delta_{\tilde{x}}, \Delta_v, \nu)$ there exists $T^* > 0$ and for any $T \in (0, T^*)$ there exist $V_{1T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $V_{2T} : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ such that for all $|\tilde{x}| \leq \Delta_{\tilde{x}}$, $|v| \leq \Delta_v$, $T \in (0, T^*)$ we have the following:*

$$\begin{aligned} \underline{\alpha}_1(|x|) &\leq V_{1T}(x) \leq \overline{\alpha}_1(|x|) \\ V_{1T}(F_T(x, \phi_T(z), v)) - V_{1T}(x) &\leq T\Big(-\alpha_1(|x|) + \lambda_1(|x-z|) + \sigma_1(|v|) + \nu \Big) \\ \underline{\alpha}_2(|x-z|) &\leq V_{2T}(x,z) \leq \overline{\alpha}_2(|x-z|) \\ V_{2T}(F_T(x, \phi_T(z), v), G_T(z, h(x), \phi_T(z), v)) - V_{2T}(x,z) &\leq T\Big(-\alpha_2(|x-z|) + \sigma_2(|v|) + \nu \Big) \end{aligned} \tag{44}$$

Then, there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{G}$ such that for any triple of strictly positive real numbers $(\widetilde{\Delta}_{\tilde{x}}, \widetilde{\Delta}_v, \widetilde{\nu})$ there exists $\widetilde{T} > 0$ and for any $T \in (0, \widetilde{T})$ there exist $V_T : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ such that for all $|\tilde{x}| \leq \widetilde{\Delta}_{\tilde{x}}$, $|v| \leq \widetilde{\Delta}_v$, $T \in (0, \widetilde{T})$ we have:

$$SP-ISS \begin{cases} \underline{\alpha}(|\tilde{x}|) \leq V_T(x,z) \leq \overline{\alpha}(|\tilde{x}|) \\ V_T(F_T(x, \phi_T(z), v), G_T(z, h(x), \phi_T(z), v)) - V_T(x,z) \leq T\Big(-\alpha(|\tilde{x}|) + \sigma(|v|) + \widetilde{\nu} \Big) \end{cases} \tag{45}$$

■

Proof of Corollary 5.4: It can be seen immediately that all conditions of Theorem 4.2 hold, by noting that: (i) the systems (40), (42) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$-SP-IOSS with measuring functions

$w_{\underline{\alpha}_1}(\tilde{x}) = w_{\overline{\alpha}_1}(\tilde{x}) = w_{\alpha_1}(\tilde{x}) = x$, $w_{x_1}(\tilde{x}) = \tilde{x}$, $w_{\lambda_1}(\tilde{x}) = x - z$, $w_{\sigma_1}(v) = w_{u_1}(v) = v$, so that $\underline{\kappa}_1$, $\overline{\kappa}_1$ exist; moreover from Assumption 5.3 and Remark 4.1 we have that the third condition of Lemma 4.1 holds; hence, condition 2 of Theorem 4.2 holds; (ii) the systems (40), (41) and (42) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \sigma_2)$-SP-ISS with measuring functions $w_{\underline{\alpha}_2}(\tilde{x}) = w_{\overline{\alpha}_2}(\tilde{x}) = w_{\alpha_2}(\tilde{x}) = x - z$, $w_{x_2}(\tilde{x}) = \tilde{x}$ and $w_{\sigma_2}(v) = w_{u_2}(v) = v$, so that $\underline{\kappa}_2$, $\overline{\kappa}_2$ exist; moreover, from Assumption 5.3 and Remark 4.1 we have that the third condition of Lemma 4.1 holds; hence, condition 1 of Theorem 4.2 holds; (iii) the third condition of Theorem 4.2 holds since $w_{\alpha_2}(\tilde{x}) = w_{\lambda_1}(\tilde{x}) = x - z$, $w_{x_1}(\tilde{x}) = w_{x_2}(\tilde{x}) = \tilde{x}$ and $w_{u_1}(v) = w_{u_2}(v) = v$ for all $x \in \mathbb{R}^n$, $v \in \mathbb{R}^m$;

Then, applying Theorem 4.2 and defining the new SP-ISS Lyapunov function V_T as in (25), we obtain that the system (40), (41) and (42) is SP-ISS with measuring functions $w_{\underline{\alpha}}(\tilde{x}) = w_{\overline{\alpha}}(\tilde{x}) = w_{\alpha}(\tilde{x}) = |x| + |x - z|$, $w_x(\tilde{x}) = \tilde{x}$ and $w_\sigma(v) = w_u(v) = v$. The proof follows from the fact that $\frac{1}{2}|\tilde{x}| \leq |x| + |x - z| \leq 2|\tilde{x}|$ ■

Remark 5.1 *There are many variations of conditions in (44) that could be used to state similar results (for more details see [15, 16]). Also, there is a small discrepancy between the way we write conditions (44) and conditions used in [15, 16]. However, it is not hard to show that these conditions are equivalent. For example, we note that instead of the second inequality in (44) we could use:*

$$V_{1T}(F_T(x, \phi_T(x+d), v)) - V_{1T}(x) \leq T\Big(- \alpha_1(|x|) + \lambda_1(|d|) + \sigma_1(|v|) + \nu \Big) ,$$

where d is a "new disturbance" (similar conditions were used in [15, 16]). This condition states that the full state feedback controller $u = \phi_T(x)$ robustly stabilizes the plant (40) in an ISS sense. Since for the controller that uses the estimates state we can write $\phi_T(z) = \phi_T(x + (z - x))$ and let $d = x - z$, we can see that this is the same condition as the one we used in (44).

6 Case study: two link manipulator

We now revisit the problem of controlling a two link manipulator considered in [1]. In particular, we illustrate how Theorem 2.1 and Corollary 5.1 may be used to obtain a controller based on the Euler approximate discrete-time model of the manipulator. We emphasize that our results provide a rigorous framework for achieving ISS via approximate discrete-time models. Finally to illustrate advantages of out approach, we compare the performace of this controller with the discretized continuous-time controller obtained in [1].

Consider a two link manipulator shown in Fig.1, with mass of the arm M and length L, and the gripper with mass m. We denote the angle of the link and the position of the gripper respectively as θ and r. The continuous time model of the manipulator is:

$$\begin{aligned} (mr^2 + ML^2/3)\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= \tau \\ m\ddot{r} - mr\dot{\theta}^2 &= F \end{aligned} \tag{46}$$

We denote the state vector $(\theta\ r\ \dot{\theta}\ \dot{r})^T$ as $x := (q_1\ q_2\ z_1\ z_2)^T$ and then write the model in a state space form:

$$\begin{aligned} \dot{q}_1 &= z_1 \\ \dot{q}_2 &= z_2 \\ \dot{z}_1 &= -\frac{2mq_2z_1z_2}{mq_2^2 + ML^2/3} + \frac{\tau}{mq_2^2 + ML^2/3} \\ \dot{z}_2 &= q_2z_1^2 + \frac{F}{m} , \end{aligned} \tag{47}$$

and the output equations are

$$\begin{aligned} y_1 &= z_1 \\ y_2 &= z_2 . \end{aligned} \tag{48}$$

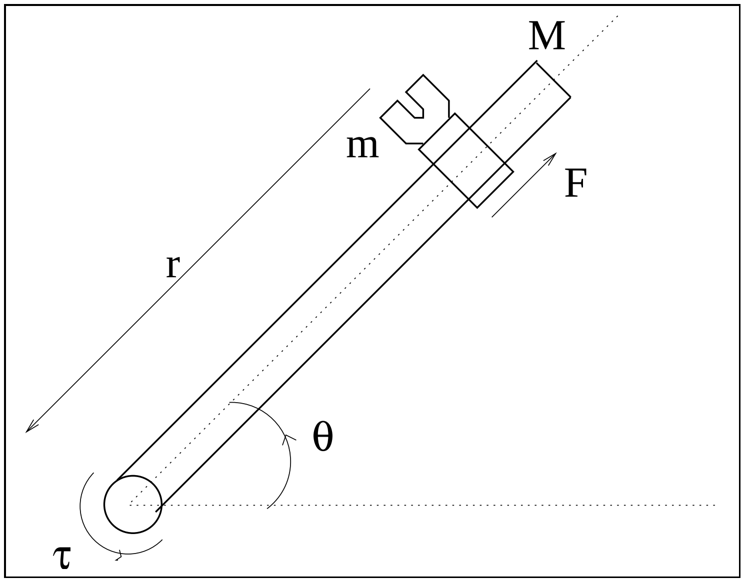

Figure 1: A two link manipulator

The physical parameters of the manipulator are given in Table 1. A continuous-time ISS controller was designed for the system (47) in [1]:

$$\begin{aligned} \tau_c &= -k_{d_1} z_1 - k_{p_1}(q_1 - q_{1d}) \\ F_c &= -k_{d_2} z_2 - k_{p_2}(q_2 - q_{2d}) - k_{nl}(q_2^3 - q_{2d}^3) \ , \end{aligned} \tag{49}$$

where $w := (q_{1d}\ q_{2d})^T$ and the value of the controller gains are given in Table 1.

Suppose now that the manipulator is controlled digitally using sample and zero order hold devices. That is, F and τ are constant during sampling intervals and the state x is measured at sampling instants kT, where T is the sampling period. In this case one may simply discretize the controller (49) in the following way:

Table 1: Manipulator's and controller's parameters

Parameter	Value	Parameter	Value
m	1	ML^2	3
k_{p_1}	2	k_{p_2}	1
k_{d_1}	2	k_{d_2}	1
k_{e_1}	2	k_{e_2}	2
-	-	k_{nl}	1

$$\begin{aligned} \tau_c(x(k), w(k)) &= -k_{d_1} z_1(k) - k_{p_1}(q_1(k) - q_{1d}(k)) \\ F_c(x(k), w(k)) &= -k_{d_2} z_2(k) - k_{p_2}(q_2(k) - q_{2d}(k)) - k_{nl}(q_2^3(k) - q_{2d}^3(k)) \ , \end{aligned}$$

and implement the emulated controller digitally. It was proved in [21, 34] that the sampled-data closed-loop system with this controlled would be semiglobally practically ISS.

However, as will be shown below, it may be better if one takes the sampling into account when designing a controller by using a discrete time model of the plant. Since it is very hard to obtain the exact discrete time model of the manipulator, we use instead the Euler approximate discrete time model for controler design. The Euler approximate model of the manipulator with sampling period T, when we substitute values of the physical parameters is:

$$\left.\begin{aligned} q_1(k+1) &= q_1(k) + Tz_1(k) \\ q_2(k+1) &= q_2(k) + Tz_2(k) \\ z_1(k+1) &= z_1(k) + T\left[-\tfrac{2q_2(k)z_1(k)z_2(k)}{q_2(k)^2+1} + \tfrac{\tau(k)}{q_2(k)^2+1}\right] \\ z_2(k+1) &= z_2(k) + T\left[q_2(k)z_1(k)^2 + F(k)\right] \ . \end{aligned}\right\} =: \widetilde{F}_T^a(x(k), \tau(k), F(k)) \ , \tag{50}$$

where In order to guarantee that the controller that achieves ISS for system (50) would also achieve SP-ISS of the sampled-data system, we need to use the results Theorem 2.1. In particular, consistency condition of Theorem 2.1 holds since we are using the Euler approximate model. We have supposed that the controller has the following form

$$\begin{aligned} \tau_T^{Euler} &= \tau_c + Tg_1(x) \\ F_T^{Euler} &= F_c + Tg_2(x) \ , \end{aligned} \tag{51}$$

where g_1 and g_2 are functions that need to be designed based on (50) to for the Euler closed loop system to satisfy (32). We formally let the control input to be $u := (g_1 \ g_2)^T$ and using (49), (50) and (51) we can write the approximate model as follows:

$$x(k+1) = \widetilde{F}_T^a(x(k), \tau(x(k), w(k)) + Tg_1(k), F(x(k), w(k)) + Tg_2(k)) =: F_T^a(x(k), u(k), w(k)) \ ,$$

which has the desirable form. If g_1, g_2 are bounded on compact sets we can conclude that the controller (51) is locally uniformly bounded and hence the third condition of Theorem 2.1 holds. Although other controller structures are possible, our choice is guided by the fact that we want to have that the continuous time and the Euler based controllers coincide for $T = 0$, so that it makes sense to compare their performance. Systematic controller design procedure based on these ideas are an interesting topic for further research.

It remains to design g_1 and g_2 so that the ISS Lyapunov conditions for approximate model in Theorem 2.1 hold. In order to do this we use Corollary 5.1 and Remark 4.2. Let K and P be the kinetic and potential energy of the system.

$$K = \frac{(1+q_2^2)z_1^2}{2} + \frac{1}{2}z_2^2 \tag{52}$$

$$P = q_1^2 + \frac{1}{2}q_2^2 + \frac{1}{4}q_2^4 \ . \tag{53}$$

The same as in [1], we let the Lyapunov function V_{1T} to be the energy function of the system:

$$V_{1T} = K + P = \frac{(1+q_2^2)z_1^2}{2} + \frac{1}{2}z_2^2 + q_1^2 + \frac{1}{2}q_2^2 + \frac{1}{4}q_2^4 \ , \tag{54}$$

We next consider the first difference for V_{1T} to compute g_1 and g_2, and we write

$$\begin{aligned} \Delta V_{1T} &= V_{1T}(F_T) - V_{1T}(x) \\ &= T(-2z_1^2 - z_2^2 + 2z_1q_{1d} + z_2q_{2d} + z_2q_{2d}^3) + T^2\Big(z_1\big(g_1 + 3\frac{z_1}{q_2^2+1} + 0.5z_1^3q_2^2\big) \\ &\quad + z_2\big(g_2 + 0.5\frac{z_2z_1^2}{q2^2+1} + z_2 + 1.5z_2q_2^2\big) + f(q, z, q_d)\Big) + O(T^3) \ , \end{aligned} \tag{55}$$

where $z := (z_1 \ z_2)^T$, $q := (q_1 \ q_2)^T$. g_1 and g_2 are designed to reduce the positivity of the $O(T^2)$ term on the right-hand side of (55) and we choose the following:

$$\begin{aligned} g_1(x) &= -k_{e_1}\left(3\frac{z_1}{q_2^2+1} + 0.5z_1^3q_2^2\right) \\ g_2(x) &= -k_{e_2}\left(0.5\frac{z_2z_1^2}{q2^2+1} + z_2 + 1.5z_2q_2^2\right) \ , \end{aligned} \tag{56}$$

where the values of k_{e_1}, k_{e_2} are listed in Table 1. With the Euler based controller (51), (56), we obtain the dissipation inequality:

$$
\begin{aligned}
\Delta V_{1T} &\leq T(-2z_1^2 - z_2^2 + 2z_1 q_{1d} + z_2 q_{2d} + z_2 q_{2d}^3) + T^2\Big(- 3\frac{z_1^2}{q_2^2+1} - 0.5z_1^4 q_2^2 - 0.5\frac{z_2^2 z_1^2}{q2^2+1} - z_2^2 \\
&\quad - 1.5 z_2^2 q_2^2\Big) + T^2 f(q,z,q_d) + O(T^3) \\
&\leq T(-\frac{1}{2}|z|^2 + a_1|q|^2 + a_2|q|^6) + T^2\Big(- 3\frac{z_1^2}{q_2^2+1} - 0.5z_1^4 q_2^2 - 0.5\frac{z_2^2 z_1^2}{q2^2+1} - z_2^2 \\
&\quad - 1.5 z_2^2 q_2^2\Big) + T^2 f(q,z,q_d) + O(T^3) \ ,
\end{aligned}
\tag{57}
$$

where a_1 and a_2 are sufficiently large positive numbers. The system is SP-qISS and hence the first part of condition 1 of Corollary 5.1 holds.

Define another Lyapunov function V_{2T} in the following form:

$$
V_{2T} = K + P + \varepsilon\frac{q_2 z_2 + q_1(1+q_2^2)z_1}{(1+q_2^4+q_1^2)^{3/4}} \ , \tag{58}
$$

where $\varepsilon > 0$ is a sufficiently small constant (to guarantee that V_{2T} positive definite). We can write

$$
\begin{aligned}
\Delta V_{2T} &= V_{2T}(F_T) - V_{2T}(x) \\
&= T\Big[- 2z_1^2 - z_1^2 + 2z_1 q_{1d} + z_2 q_{2d} + z_2 q_2 d^3 \\
&\quad + \epsilon\frac{z_2^2 + 2q_2^2 z_2^2 + z_1^2 + q_2(F_c + Tg_2) + q_1(\tau_c + Tg_1)}{(1+q_2^4+q_1^2)^{3/4}} \\
&\quad + \frac{3}{4}\epsilon\frac{4q_2^3 z_2 + 2q_1 z_1}{(1+q_2^4+q_1^2)^{7/4}}(q_2 z_2 + q_1(1+q_2^2)z_1)\Big] + O(T^2) \\
&\leq T\Big[M_1(q_{1d}^2 + q_{2d}^2 + q_{2d}^6) - M_2|z|^2 + M_3|z|^2 + \epsilon\frac{q_2 F_c + q_1\tau_c}{(1+q_2^4+q_1^2)^{3/4}}\Big] + O(T^2)
\end{aligned}
\tag{59}
$$

for a sufficiently small T, ϵ and M_2 and sufficiently large M_1 and M_3. Substituting the controller τ_T^{Euler} and F_T^{Euler}, we can write the dissipation inequality as

$$
\Delta V_{2T} \leq T\{M_1(q_{1d}^2 + q_{2d}^2 + q_{2d}^6) - M_2|z|^2 + M_3|z|^2 + \epsilon\frac{q_2 F_c + q_1\tau_c}{(1+q_2^4+q_1^2)^{3/4}}\} + O(T^2) \tag{60}
$$

for a sufficiently small T, $\tilde{\epsilon}$ and $\tilde{M}_2$ and sufficiently large $\tilde{M}_1$ and $\tilde{M}_3$. The system is SP-IOSS and hence the second part of condition 1 of Corollary 5.1 holds. Finally, since $\alpha_1(s) = \frac{s^2}{2}$ and $\lambda_2(s) = \tilde{M}_3 s^2$, we have that condition 2 of Corollary 5.1 holds. Hence, from Corollary 5.1 and Remark 4.2 we have that the first condition of Theorem 2.1 holds and it follows from Theorem 2.1 that the exact discrete-time closed-loop system is SP-ISS. Finally, using results of [22] we can conclude that the closed-loop sampled data system is SP-ISS.

We show some simulation results using SIMULINK to illustrate the performance of the system when we apply the Euler based controller (51), and comparing it with the emulation controller (49). Fig. 2 shows the simulation results with simulation parameters given in Table 2.

Fig. 2(a) shows the reference signal θ_d and the actual angular position of the arm θ, while Fig. 2(b) shows the position of the gripper r obtained when applying the Euler based controller, while Fig. 2(c) and (d) are respectively showing the response of the corresponding variables with emulation controller. By computation it has been shown that applying the emulation controller also renders SP-ISS property to the closed-loop system. However, it is shown by simulation results that the Euler based controller (51) performs significantly better than the emulation controller (49) for the corresponding simulation setup.

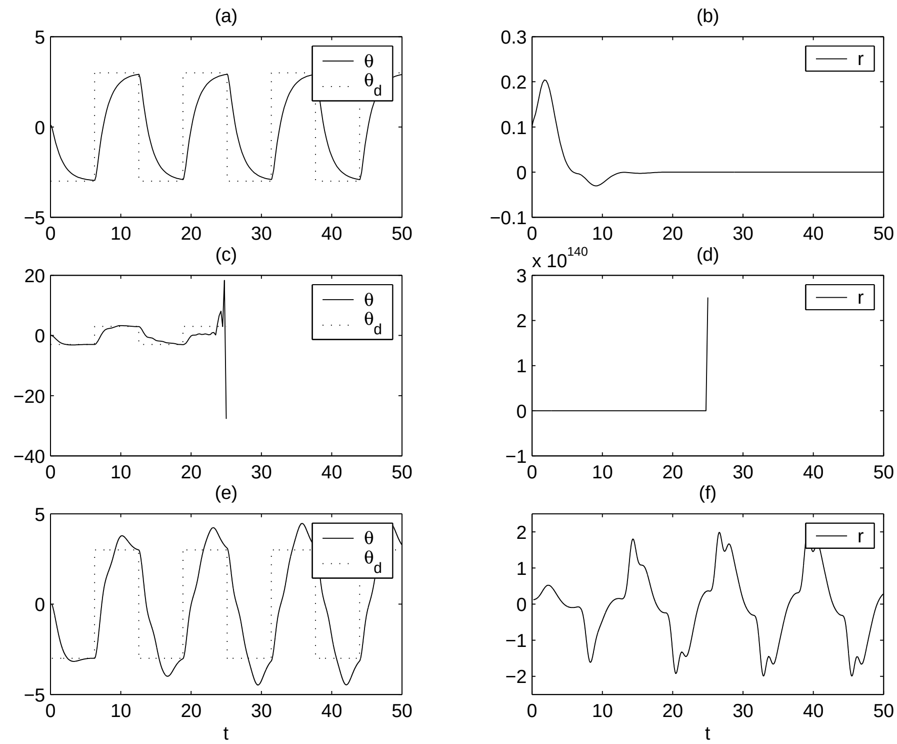

Figure 2: Responses with Euler based controller (a,b) and emulation controller (c,d), for T=0.1.

Table 2: Simulation parameters

Parameter	Value
Sampling period (T)	$0.1s$
Initial state	$(0.1\ 0.1\ 0.1\ 0.1)^T$
θ_d	0
r_d	$3\,\mathrm{sign}(\dot{\theta})$

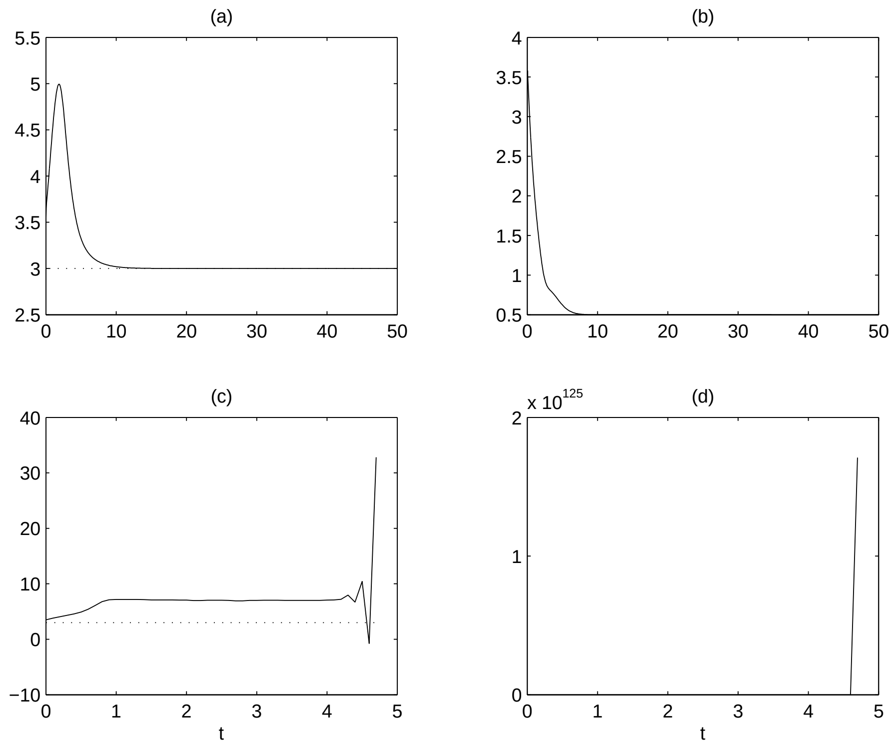

Figure 3: Responses with Euler based controller (a,b) and emulation controller (c,d), for T=0.01.

Another set of simulations has been done to obtain a similar response for both controllers. The simulations are carried out using the same parameter as given in Table 2, except now we need to reduce the sampling period to $T = 0.01s$. The simulation results are shown in Fig. 3.

From its formulation, the Euler based controller approximate the emulation controller with order $O(T)$, and both controllers coincide at $T = 0$. It is shown by the second simulation set that reducing T makes both controllers to have similar performance. However, since reducing T means reducing the effect of the $O(T)$ term of the Euler based controller, it degrades the performance of the controller, while the contrast happens to the emulation controller. We remark that since the Euler based controller (51)id designed by taking the advantage of the term that multiplied by T, we may expect that controllers which take this form will always outperform the emulation controller for relatively high sampling period.

7 Proofs of main results

In this section, we provide proofs of our main results. The proofs of corollaries are omitted since they follow directly from the main results. We use a proof technique, which is similar to the one used in [1] and [25]. The proof of Lemma 4.1 is the main difference between the continuous-time results in [1] and our discrete-time results.

Proof of Lemma 4.1
We denote $V_T(F_T) := V_T(F_T(x,u))$ and $V_T := V_T(x)$. Suppose that all conditions in Lemma 4.1 are satisfied. Fix an arbitrary $q \in \mathcal{SN}$ and let ρ be defined using (19). We prove next that $\rho(V_T)$ is a SP-IOSS Lyapunov function for the system with appropriate bounding and measuring functions stated in the lemma.

First, note that from the Mean Value Theorem and the fact that $q(\cdot) = \frac{d\rho}{ds}(\cdot)$ is nondecreasing, it follows that

$$\rho(a) - \rho(b) \le q(a)[a-b] \qquad \forall a \ge 0,\ b \ge 0\ . \tag{61}$$

Let arbitrary strictly positive real numbers $(\Delta'_x, \Delta'_u, \nu')$ be given. Let the numbers Δ'_x, Δ'_u generate numbers M, T_1^* via the third condition of the lemma, so that (20) holds. Let ν_1 be such that

$$\max\{\lambda(M), \sigma(M)\}[q(s+\nu_1) - q(s)] \le \frac{\nu'}{2}\ , \forall s \in [0, \overline{\alpha}(M) + 2\max\{\lambda(M), \sigma(M)\}]\ .$$

Such ν_1 always exists since $q(\cdot)$ is continuous. We define:

$$\Delta_x := \Delta'_x, \qquad \Delta_u := \Delta'_u, \qquad \nu := \min\left\{\frac{\nu'}{2q\circ\overline{\alpha}(M)}, \nu_1\right\} \tag{62}$$

Let $(\Delta_x, \Delta_u, \nu)$ determine $T_2^* > 0$ and V_T using the first condition of the lemma, such that for all $T \in (0, T_2^*)$ and all $|\omega_x(x)| \le \Delta_x$, $|\omega_u(u)| \le \Delta_u$ the inequalities (13) and (14) hold. Fix $T^* := \min\{T_1^*, T_2^*, 1\}$. In the rest of the proof we always consider arbitrary $T \in (0, T^*)$, $|\omega_x(x)| \le \Delta_x$ and $|\omega_u(u)| \le \Delta_u$.

Note that a direct consequence of condition 1 of the lemma and the fact that $T^* \le 1$ is:

$$V_T \ge \max\{\underline{\alpha}(|w_{\underline{\alpha}}(x)|), T\alpha(|w_\alpha(x)|) - T\lambda(|w_\lambda(x)|) - T\sigma(|w_\sigma(u)|) - T\nu\} \tag{63}$$

$$V_T(F_T) \le \overline{\alpha}(|w_{\overline{\alpha}}(x)|) + \sigma(|w_\sigma(u)|) + \lambda(|w_\lambda(x)|) + \nu\ . \tag{64}$$

Note first that

$$\rho\circ\underline{\alpha}(|w_{\underline{\alpha}}(x)|) \le \rho(V_T) \le \rho\circ\overline{\alpha}(|w_{\overline{\alpha}}(x)|)\ ,$$

which shows that (13) holds with the new bounding functions $\underline{\alpha}'(s) = \rho\circ\underline{\alpha}(s)$ and $\overline{\alpha}'(s) = \rho\circ\overline{\alpha}(s)$ and the same measuring functions. Now we prove that (14) holds for $\rho(V_T)$ with the new bounding functions and the same measuring functions. The following two preliminary cases are first considered:

1. $\underline{V_T(F_T) \le \frac{1}{2}V_T}$
   Using the inequalities (61) and (63) and the definition of M and ν we obtain

$$\begin{aligned}
\rho(V_T(F_T)) - \rho(V_T) &\le \rho\left(\frac{1}{2}V_T\right) - \rho(V_T)\\
&\le q\left(\frac{1}{2}V_T\right)\left[-\frac{1}{2}V_T\right]\\
&\le \frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot(-\alpha(|w_\alpha(x)|) + \sigma(|w_\sigma(u)|) + \lambda(|w_\lambda(x)|) + \nu)\\
&\le \frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot(-\alpha(|w_\alpha(x)|) + \sigma(|w_\sigma(u)|) + \lambda(|w_\lambda(x)|)) + T\frac{q\circ\overline{\alpha}(M)}{2}\nu\\
&\le \frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot(-\alpha(|w_\alpha(x)|) + \sigma(|w_\sigma(u)|) + \lambda(|w_\lambda(x)|)) + T\frac{\nu'}{4}
\end{aligned} \tag{65}$$

2. $\underline{V_T(F_T) > \frac{1}{2}V_T}$
   Using the inequalities (61) and (14) and the definition of M and ν we obtain

$$\begin{aligned}
\rho(V_T(F_T)) - \rho(V_T) &\le q\left(V_T(F_T)\right)\left[V_T(F_T) - V_T\right]\\
&\le Tq\left(V_T(F_T)\right)\cdot(-\alpha(|w_\alpha(x)|) + \sigma(|w_\sigma(u)|) + \lambda(|w_\lambda(x)|) + \nu)\\
&\le Tq\left(V_T(F_T)\right)\cdot(-\alpha(|w_\alpha(x)|) + \sigma(|w_\sigma(u)|) + \lambda(|w_\lambda(x)|)) + Tq\circ\overline{\alpha}(M)\nu\\
&\le Tq\left(V_T(F_T)\right)\cdot(-\alpha(|w_\alpha(x)|) + \sigma(|w_\sigma(u)|) + \lambda(|w_\lambda(x)|)) + T\frac{\nu'}{2}\ .
\end{aligned} \tag{66}$$

The proof is completed by considering the following three cases.
Case 1: $\underline{\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) \le \frac{1}{2}\alpha(|w_\alpha(x)|)}$

- $\underline{V_T(F_T) \leq \frac{1}{2}V_T}$
  We use (65) to write:

$$\begin{aligned}\rho(V_T(F_T)) - \rho(V_T) &\leq \frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot(-\frac{1}{2}\alpha(|w_\alpha(x)|)) + T\frac{\nu'}{4} \\ &\leq -\frac{T}{4}q\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) + T\frac{\nu'}{4}\end{aligned} \tag{67}$$

- $\underline{V_T(F_T) > \frac{1}{2}V_T}$
  We use (66) and the fact that q is nondecreasing to write:

$$\begin{aligned}\rho(V_T(F_T)) - \rho(V_T) &\leq Tq\left(V_T(F_T)\right)\cdot(-\frac{1}{2}\alpha(|w_\alpha(x)|)) + T\frac{\nu'}{2} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) + T\frac{\nu'}{2} \\ &\leq -\frac{T}{4}q\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) + T\frac{\nu'}{2}\end{aligned} \tag{68}$$

Since q is nondecreasing, using (63) and the second condition of the lemma, the following always holds for Case 1:

$$\rho(V_T(F_T)) - \rho(V_T) \leq -\frac{T}{4}q\left(\frac{1}{2}\underline{\alpha}\circ\underline{\kappa}(|w_\alpha(x)|)\right)\cdot\alpha(|w_\alpha(x)|) + T\frac{\nu'}{2} \tag{69}$$

<u>Case 2: $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) > \frac{1}{2}\alpha(|w_\alpha(x)|)$, $\lambda(|w_\lambda(x)|) \geq \sigma(|w_\sigma(u)|)$</u>

- $\underline{V_T(F_T) \leq \frac{1}{2}V_T}$
  We use (65), (13), the fact that q is nondecreasing, $T^* \leq 1$ and the choice of ν_1 to write:

$$\begin{aligned}\rho(V_T(F_T)) - \rho(V_T) &\leq \frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot(-\alpha(|w_\alpha(x)|) + 2\lambda(|w_\lambda(x)|)) + T\frac{\nu'}{4} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) + Tq\left(\frac{1}{2}\overline{\alpha}(|w_{\overline{\alpha}}(x)|)\right)\cdot\lambda(|w_\lambda(x)|) + T\frac{\nu'}{4} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) \\ &\quad + Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|) + \nu_1\right)\cdot\lambda(|w_\lambda(x)|) + T\frac{\nu'}{4} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) \\ &\quad + Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|)\right)\cdot\lambda(|w_\lambda(x)|) + T\frac{\nu'}{2} + T\frac{\nu'}{4}\end{aligned} \tag{70}$$

- $\underline{V_T(F_T) > \frac{1}{2}V_T}$
  We use (66), (13), the fact that q is nondecreasing, $T^* \leq 1$ and the choice of ν_1 to write:

$$\begin{aligned}\rho(V_T(F_T)) - \rho(V_T) &\leq Tq\left(V_T(F_T)\right)\cdot(-\alpha(|w_\alpha(x)|) + 2\lambda(|w_\lambda(x)|)) + T\frac{\nu'}{2} \\ &\leq -Tq\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) \\ &\quad + 2Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|) + \nu_1\right)\cdot\lambda(|w_\lambda(x)|) + T\frac{\nu'}{2} \\ &\leq -Tq\left(\frac{1}{2}V_T\right)\cdot\alpha(|w_\alpha(x)|) \\ &\quad + 2Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|)\right)\cdot\lambda(|w_\lambda(x)|) + T\frac{\nu'}{2} + T\frac{\nu'}{2}\end{aligned} \tag{71}$$

Since q is nondecreasing, using (63), (70), (71), the second condition of the lemma, the condition that $\lambda(|w_\lambda(x)|) > \frac{1}{4}\alpha(|w_\alpha(x)|)$ and the definition of θ_λ given by (22), the following always holds for Case 2:

$$\rho(V_T(F_T)) - \rho(V_T) \leq -\frac{T}{2} q\left(\frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|)\right) \cdot \alpha(|w_\alpha(x)|) + 2Tq \circ \theta_\lambda(|w_\lambda(u)|) \cdot \lambda(|w_\lambda(u)|) + T\nu' \quad (72)$$

Case 3: $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) > \frac{1}{2}\alpha(|w_\alpha(x)|)$, $\lambda(|w_\lambda(x)|) < \sigma(|w_\sigma(u)|)$

- $V_T(F_T) \leq \frac{1}{2}V_T$
  We use (65), (13), the fact that q is nondecreasing, $T^* \leq 1$ and the choice of ν_1 to write:

$$\begin{aligned}
\rho(V_T(F_T)) - \rho(V_T) &\leq -\frac{T}{2} q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + Tq\left(\frac{1}{2}V_T\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{4} \\
&\leq -\frac{T}{2} q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + Tq\left(\frac{1}{2}\overline{\alpha}(|w_{\overline{\alpha}}(x)|)\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{4} \\
&\leq -\frac{T}{2} q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
&\quad + Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\sigma(|w_\sigma(u)|) + \nu_1\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{4} \\
&\leq -\frac{T}{2} q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
&\quad + Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\sigma(|w_\sigma(u)|)\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} + T\frac{\nu'}{4}
\end{aligned} \quad (73)$$

- $V_T(F_T) > \frac{1}{2}V_T$
  We use (66), (13), the fact that q is nondecreasing, $T^* \leq 1$ and the choice of ν_1 to write:

$$\begin{aligned}
\rho(V_T(F_T)) - \rho(V_T) &\leq Tq\left(V_T(F_T)\right) \cdot \left(-\alpha(|w_\alpha(x)|) + 2\sigma(|w_\sigma(u)|)\right) + T\frac{\nu'}{2} \\
&\leq -Tq\left(V_T(F_T)\right) \cdot \alpha(|w_\alpha(x)|) + 2Tq\left(V_T(F_T)\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} \\
&\leq -Tq\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
&\quad + 2Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\sigma(|w_\sigma(x)|) + \nu_1\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} \\
&\leq -Tq\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
&\quad + 2Tq\left(\overline{\alpha}(|w_{\overline{\alpha}}(x)|) + 2\sigma(|w_\sigma(x)|)\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} + T\frac{\nu'}{2}
\end{aligned} \quad (74)$$

Since q is nondecreasing, using (63), (73), (74), the second condition of the lemma, the condition that $\sigma(|w_\sigma(u)|) > \frac{1}{4}\alpha(|w_\alpha(x)|)$ and the definition of θ_σ given by (21), the following always holds for Case 3:

$$\rho(V_T(F_T)) - \rho(V_T) \leq -\frac{T}{2} q\left(\frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|)\right) \cdot \alpha(|w_\alpha(x)|) + 2Tq \circ \theta_\sigma(|w_\sigma(x)|) \cdot \sigma(|w_\sigma(x)|) + T\nu' \quad (75)$$

We have shown through these three cases that the following holds:

$$\rho(V_T(F_T(x,u))) - \rho(V_T(x)) \leq T\Big[2q \circ \theta_\sigma(|w_\sigma(u)|) \cdot \sigma(|w_\sigma(u)|) + 2q \circ \theta_\lambda(|w_\lambda(x)|) \cdot \lambda(|w_\lambda(x)|) - \frac{1}{4} q \circ \frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|) \cdot \alpha(|w_\alpha(x)|) + \nu'\Big], \quad (76)$$

which completes the proof of Lemma 4.1. ∎

Proof of Theorem 4.1 Suppose that all conditions of the theorem be satisfied. Let $\underline{\alpha}_1$, $\overline{\alpha}_1$, α_1, σ_1 come from the condition 1 and $\underline{\alpha}_2$, $\overline{\alpha}_2$, α_2, λ_2, σ_2 come from the condition 2. Define $\tilde{q}$ as:

$$\tilde{q}(r) := \inf_{r\leq s} \frac{\alpha_1(s)}{2(1+\lambda_2(s))} \ . \tag{77}$$

Notice that $\tilde{q}$ is by definition a nondecreasing function. Condition 4 of the theorem implies $\tilde{q}(r) > 0$ for all $r > 0$. Let $q(s) := \tilde{q}\circ\gamma_1^{-1}\circ\theta_{\lambda_2}^{-1}(s)$, where θ_{λ_2} is defined in (22) and γ_1 comes from the third condition of the theorem. Using $q(\cdot)$ we define $\rho(\cdot)$ via (19). Let ρ generate via Lemma 4.1 the new bounding functions $\underline{\alpha}'_2$, $\overline{\alpha}'_2$, α'_2, λ'_2, σ'_2.

Let arbitrary strictly positive real numbers $(\Delta_x, \Delta_u, \nu)$ be given. Let $(\Delta_x, \Delta_u, \frac{\nu}{2})$ generate via condition 1 the number T_1^* and V_{1T}. Let $(\gamma_2(\Delta_x), \gamma_3(\Delta_u), \frac{\nu}{2})$ generate via condition 2 and Lemma 4.1 the number T_2^* and $\rho(V_{2T})$. Let $T^* = \min\{T_1^*, T_2^*\}$ and define now V_T as:

$$V_T := V_{1T} + \rho(V_{2T}) \ . \tag{78}$$

Let $w_x(x) := w_{x_1}(x)$ and $w_u(u) := w_{u_1}(u)$. We consider now arbitrary $|w_x(x)| \leq \Delta_x$, $|w_u(u)| \leq \Delta_u$ and $T \in (0, T^*)$. Note that this implies via condition 3 of the theorem that $w_{x_2}(x) \leq \gamma_2(\Delta_x)$ and $w_{u_2}(x) \leq \gamma_3(\Delta_u)$.

First, it follows from the definition of V_T that

$$\underline{\alpha}_1(|w_{\underline{\alpha}_1}(x)|) + \rho\circ\underline{\alpha}_2(|w_{\underline{\alpha}_2}(x)|) \leq V_T(x) \leq \overline{\alpha}_1(|w_{\overline{\alpha}_1}(x)|) + \rho\circ\overline{\alpha}_2(|w_{\overline{\alpha}_2}(x)|) \ . \tag{79}$$

Then by Remark 3.1, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|) \leq V_T(x) \leq \overline{\alpha}(|w_{\overline{\alpha}_1}(x)| + |w_{\overline{\alpha}_2}(x)|) \ . \tag{80}$$

Using condition 4 of the theorem, the dissipation inequality for V_T can be written as:

$$\begin{aligned} V_T(F_T(x,u)) - V_T(x) &= V_{1T}(F_T) - V_{1T} + \rho(V_{2T}(F_T)) - \rho(V_{2T}) \\ &\leq T\Big[\sigma_1(|w_{\sigma_1}(u)|) + \sigma'_2(|w_{\sigma_2}(u)|) + \frac{\nu}{2} - \alpha_1(|w_{\alpha_1}(x)|) \\ &\quad + \lambda'_2\circ\gamma_1(|w_{\alpha_1}(x)|) - \alpha'_2(|w_{\alpha_2}(x)|) + \frac{\nu}{2}\Big] \\ &\leq T\Big[\sigma_1(|w_{\sigma_1}(u)|) + \sigma'_2(|w_{\sigma_2}(u)|) + \frac{\nu}{2} - \alpha_1(|w_{\alpha_1}(x)|) \\ &\quad + \frac{\alpha_1(|w_{\alpha_1}(x)|)\lambda_2(|w_{\alpha_1}(x)|)}{2(1+\lambda_2(|w_{\alpha_1}(x)|))} - \alpha'_2(|w_{\alpha_2}(x)|) + \frac{\nu}{2}\Big] \ . \end{aligned} \tag{81}$$

Since

$$\frac{\lambda_2(s)}{1+\lambda_2(s)} \leq 1 \ , \ \forall s \geq 0 \ ,$$

by monotonicity of $q(\cdot)$ and using Remark 3.1, there exist $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ so that we can write

$$V_T(F_T(x,u)) - V_T(x) \leq -T\alpha(|w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|) + T\sigma(|w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|) + T\nu \ . \tag{82}$$

This completes the proof of Theorem 4.1. ∎

Proof of Theorem 4.2 Suppose that all conditions of the theorem are satisfied. Let $\underline{\alpha}_1$, $\overline{\alpha}_1$, α_1, σ_1 come from the condition 1 and $\underline{\alpha}_2$, $\overline{\alpha}_2$, α_2, λ_2, σ_2 come from the condition 2. Define a function $\alpha'_1 \in \mathcal{K}_\infty$ as follows

$$\alpha'_1(s) := \begin{cases} \alpha_1(s) & \text{for small } s, \\ \lambda_2(s) & \text{for large } s. \end{cases} \tag{83}$$

It is clear that $\alpha_1'(s) = O[\alpha_1(s)]$ for $s \to 0^+$. Hence, by Lemma 3.1 there exists $\tilde{q}_1 \in \mathcal{SN}$ such that $\tilde{q}_1(s) \cdot \alpha_1(s) \geq \alpha_1'(s)$. Further, define a function $\lambda_2' \in \mathcal{K}$:

$$\lambda_2'(s) := \frac{1}{2}\alpha_1'(s) \ , \tag{84}$$

and it is clear that $\lambda_2(s) = O[\lambda_2'(s)]$ for $s \to +\infty$. Then by Remark 3.1, there exists $\tilde{q}_2 \in \mathcal{SN}$ such that $\tilde{q}_2(s) \cdot \lambda_2(s) \leq \lambda_2(s)$. Let

$$q_1(s) := 4\tilde{q}_1 \circ \underline{\kappa}_1^{-1} \circ \underline{\alpha}_1^{-1}(2s) \tag{85}$$

$$q_2(s) := \frac{1}{2}\tilde{q}_2 \circ \gamma_1^{-1} \circ \theta_{\lambda_2}^{-1}(s) \ , \tag{86}$$

where θ_{λ_2} is given in (22) and γ_1 comes from the third condition of the theorem. We use q_1 and q_2 respectively to define ρ_1 and ρ_2, and then let (q_1, ρ_1) and (q_2, ρ_2) respectively generate via Lemma 4.1 new bounding functions $\underline{\alpha}_1'$, $\overline{\alpha}_1'$, α_1', σ_1' and $\underline{\alpha}_2'$, $\overline{\alpha}_2'$, α_2', σ_2'.

Let arbitrary strictly positive real numbers $(\Delta_x, \Delta_u, \nu)$ be given. Let $(\Delta_x, \Delta_u, \frac{\nu}{2})$ generate via item 1 of the theorem and Lemma 4.1 T_1^* and $\rho_1(V_{1T})$ and let $(\gamma_2(\Delta_x), \gamma_3(\Delta_u), \frac{\nu}{2})$ generate via item 2 of the theorem and Lemma 4.1 T_2^* and $\rho_2(V_{2T})$. Let $T^* := \min\{T_1^*, T_2^*\}$. We now define V_T as:

$$V_T := \rho_1(V_{1T}) + \rho_2(V_{2T}) \ , \tag{87}$$

Let $w_x(x) := w_{x_1}(x)$ and $w_u(u) := w_{u_1}(u)$. In all calculations below we consider arbitrary $|w_x(x)| \leq \Delta_x$, $|w_u(u)| \leq \Delta_u$ and $T \in (0, T^*)$. Note that this implies $|w_{x_2}(x)| \leq \gamma_2(\Delta_x)$ and $|w_{u_2}(x)| \leq \gamma_3(\Delta_u)$.

It follows from the definition of V_T that

$$\rho_1 \circ \underline{\alpha}_1(|w_{\underline{\alpha}_1}(x)|) + \rho_2 \circ \underline{\alpha}_2(|w_{\underline{\alpha}_2}(x)|) \leq V_T(x) \leq \rho_1 \circ \overline{\alpha}_1(|w_{\overline{\alpha}_1}(x)|) + \rho_2 \circ \overline{\alpha}_2(|w_{\overline{\alpha}_2}(x)|) \ . \tag{88}$$

Then by Remark 3.1, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|) \leq V_T(x) \leq \overline{\alpha}(|w_{\overline{\alpha}_1}(x)| + |w_{\overline{\alpha}_2}(x)|) \ . \tag{89}$$

Using condition 3 of the theorem and (84), we have:

$$\begin{aligned} V_T(F_T(x,u)) - V_T(x) &= \rho_1(V_{1T}(F_T)) - \rho_1(V_{1T}) + \rho_2(V_{2T}(F_T)) - \rho_2(V_{2T}) \\ &\leq T\Big[-\alpha_1'(|w_{\alpha_1}(x)|) + \sigma_1'(|w_{\sigma_1}(u)|) + \frac{\nu}{2} - \alpha_2'(|w_{\alpha_2}(x)|) + \lambda_2' \circ \gamma_1(|w_{\alpha_1}(x)|) \\ &\quad + \sigma_2'(|w_{\sigma_2}(u)|) + \frac{\nu}{2}\Big] \\ &\leq T\Big[-\alpha_2'(|w_{\alpha_2}(x)|) - \frac{1}{2}\alpha_1'(|w_{\alpha_1}(x)|) + \sigma_1'(|w_{\sigma_1}(u)|) + \sigma_2'(|w_{\sigma_2}(u)|) \\ &\quad + \frac{\nu}{2} + \frac{\nu}{2}\Big] \ . \end{aligned} \tag{90}$$

Finally, using Remark 3.1, there exist $\sigma \in \mathcal{K}$ and $\alpha \in \mathcal{K}_\infty$ that

$$V_T(F_T(x,u)) - V_T(x) \leq T\Big[\sigma(|w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|) - \alpha(|w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|) + \nu\Big] \ . \tag{91}$$

This completes the proof of Theorem 4.2. ■

8 Conclusions

We have presented results on changing supply rates for discrete-time SP-IOSS systems that allow for a partial construction of Lyapunov functions. Our results apply to investigation of different semiglobal practical stability properties of discrete-time parameterized systems that arise when an approximate discrete-time model is used for controller design of a sampled-data nonlinear system. We have applied our results to several problems, such as the LaSalle criterion for SP-ISS of discrete-time systems. We emphasize that there is a great potential for further applications of our results. We have illustrated this approach by an example where a discrete-time SP-ISS controller was designed for a manipulator based on its Euler approximate model. Using simulations, we compared the performance of our controller with the performance of the discretized continuous-time controller obtained in [1] for the same problem and it was shown that our controller yielded better performance. This strongly motivates a development of systematic controller design procedures for sampled-data nonlinear systems based on their approximate discrete-time models where results the present paper could play an important role.

References

[1] D. Angeli, "Input-to-state stability of PD controlled robotic systems," *Automatica*, vol. 35, pp. 1285-1290, 1999.

[2] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Trans. Automat. Contr.*, submitted for publication, 2001.

[3] M. Arcak, D. Angeli and E.D. Sontag, "A unifying integral ISS framework for stability of nonlinear cascades", to appear in *SIAM J. Contr. Optimiz.*, 2001.

[4] M. Bensoubaya, A. Ferfera and A. Iggidr, "Stabilization of nonlinear systems by use of semidefinite Lyapunov functions," *Appl. Math. Letters*, vol. 12, pp. 11-17, 1999.

[5] B. Castillo, S. Di Gennaro, S. Monaco and D. Normand-Cyrot, *On regulation under sampling*, IEEE Trans. Automat. Contr., 42 (1997), pp. 864-868.

[6] T. Chen and B. A. Francis, Optimal sampled-data control systems. Springer-Verlag: London, 1995.

[7] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag and A. I. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Contr., 42 (1997), pp. 1394-1407.

[8] D. Dochain and G. Bastin, *Adaptive identification and control algorithms for nonlinear bacterial growth systems*, Automatica, 20 (1984), pp. 621-634.

[9] G. C. Goodwin, B. McInnis and R. S. Long, *Adaptive control algorithm for waste water treatment and pH neutralization*, Optimal Contr. Applic. Meth., 3 (1982), pp. 443-459.

[10] L. Grüne, *Asymptotic controllability and exponential stabilization of nonlinear control systems at singular points*, SIAM J. Contr. Optimiz., 36 (1998), pp. 1485-1503.

[11] J.W. Grizzle and J.-M. Kang, "Discrete-time control design with positive semidefinite Lyapunov functions", *Syst. Contr. Lett.*, vol. 43, pp. 287-292, 2001.

[12] A. Isidori, *Nonlinear Control Systems II*, Springer, 1999.

[13] Z.P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857-869, 2001.

[14] Z.P. Jiang, I.M.Y. Mareels and Y. Wang, "A Lypunov formulation of the nonlinear small gain theorem for interconnected ISS systems", *Automatica*, vol. 32, pp. 1211-1215, 1996.

[15] D. Kazakos and J. Tsinias, "Stabilization of nonlinear discrete-time systems using state detection,"*IEEE Trans. Automat. Contr.*, vol. 38, pp. 1398-1400, 1993.

[16] D. Kazakos and J. Tsinias, "The input to state stability condition and global stabilization of discrete-time systems,"*IEEE Trans. Automat. Contr.*, vol. 39, no. 10, pp. 2111-2113, 1994.

[17] H. K. Khalil, *Nonlinear Systems 2nd Ed.*, PHI, 1996.

[18] M. Krstić, I. Kanellakopoulos and P. V. Kokotović, *Nonlinear and adaptive control design.* J. Wiley & Sons: New York, 1995.

[19] I. M. Y. Mareels, H. B. Penfold and R. J. Evans, *Controlling nonlinear time-varying systems via Euler approximations*, Automatica, 28 (1992), pp. 681-696.

[20] S. Monaco and D. Normand-Cyrot, *Sampling of a linear analytic control system*, Proc. CDC'85, pp. 1457-1462.

[21] D.S. Laila, D. Nešić and A. R. Teel, "On preservation of dissipation inequalities under sampling", *IEEE Conf. Decis. Contr.*, Sydney, Australia, pp. 2472-2477, 2000.

[22] D. Nešić, A. R. Teel and E.D. Sontag, "Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems", *Syst. Contr. Lett.*, vol. 38, pp. 49-60, 1999.

[23] D. Nešić, A. R. Teel and P. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations," *Syst. Contr. Lett.*, vol. 38, pp. 259-270, 1999.

[24] D. Nešić and A. R. Teel, "Backstepping on the Euler approximate model for stabilization of sampled-data nonlinear systems", to appear in Proc. IEEE Conf. Decis. Contr., Orlando, Florida, 2001.

[25] D. Nešić and A. R. Teel, "Changing supply functions in input to state stable systems: the discrete-time case," *IEEE Trans. Automat. Contr.*, vol. 45, no. 6, pp. 960-962, 2001.

[26] D. Nešić and D. S. Laila, "A note on input-to-state stabilization for nonlinear sampled-data systems," *to appear in IEEE Trans. Automat. Contr., 2001.*

[27] N. Rouche, P. Habets and M. Laloy, *Stability Theory by Lyapunov's Direct Method*, Springer, 1977.

[28] E. Skafidas, A. Fradkov, R. J. Evans and I. M. Y. Mareels, *Trajectory based adaptive control for nonlinear systems under matching conditions*, Automatica, 34 (1998), pp. 287-299.

[29] E. D. Sontag and A. R. Teel, "Changing supply functions in input/state stable systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1476-1478, 1995.

[30] E. D. Sontag and Y. Wang, "Detectability of nonlinear systems," *Proc. Conf. on Information Science and Systems (CISS'96)*, pp. 1031-1036, Princeton, NJ, 1996.

[31] E. D. Sontag and Y. Wang, "Output to state stability and detectability of nonlinear systems," *Systems and Control Letters*, vol. 29, pp. 279-290, 1997.

[32] E. D. Sontag and Y. Wang, "Lyapunov characterization of input to output stability," *SIAM J. Control and Optimization*, vol. 39, pp. 226-249, 2001.

[33] A. M. Stuart and A. R. Humphries, Dynamical systems and numerical analysis. Cambridge University Press: New York, 1996.

[34] A.R. Teel, D. Nešić and P. Kokotović, "A note on ISS of nonlinear sampled-data systems", *IEEE Conf. Decis. Contr.*, Tampa, Florida, pp. 2473-2478, 1998.

[35] A. R. Teel, L. Moreau and D. Nešić, *A unification of time-scale methods for systems with disturbances*, submitted, 2001.

[36] L. Zaccarian, A.R. Teel and D. Nešić, "On finite gain L_p stability of nonlinear sampled-data systems", submitted for publication, 2001.