Open and closed loop dissipation inequalities under sampling and controller emulation

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Abstract

We present a general and unified framework for the design of nonlinear digital controllers using the emulation method for nonlinear systems with disturbances. It is shown that if a (dynamic) continuous-time controller, which is designed so that the continuous-time closed-loop system satisfies a certain dissipation inequality, is appropriately discretized and implemented using sample and zero-order-hold, then the discrete-time model of the closed-loop sampled-data system satisfies a similar dissipation inequality in a semiglobal practical sense (sampling period is the parameter that we can adjust). We consider two different forms of dissipation inequalities for the discrete-time model: the "weak" form and the "strong" form. The results are also applicable for open-loop systems.

Keywords: Discrete-time; dissipation; emulation design; nonlinear; sampled-data.

1 Introduction

Emulation is a well-established method to design digital controllers for continuous-time plants (see, for instance [2, 7, 10]). The first step in the emulation method is to design a continuous-time controller for a continuous-time plant using a certain known continuous-time design method; sampling is completely ignored at this stage. Then, in the second step, the continuous-time controller is discretized and implemented using sample and hold devices. Digital controllers designed using emulation have been proved to perform well for a number of control problems under sufficiently fast sampling. The following problems have been addressed in the literature: stability for linear [6] and nonlinear [3, 17, 26, 28, 35] plants, $\mathcal{L}_p$ stability of linear systems [6], input-to-state stability (ISS) of nonlinear systems [29, 33] and adaptive stabilization of nonlinear systems [11]. Also, ideas similar to emulation were exploited in [27], where the dissipativity property of continuous-time nonlinear systems is investigated using discrete observation of its storage function. For more details on dissipation inequalities see [12, 16, 19, 23, 24, 29, 33, 34] and references therein.

In this paper we generalize and unify the known results on emulation design in the literature, by considering preservation of general dissipation inequalities under sampling in the context of emulation design of dynamic state feedback controllers (preliminary results of this paper can be found in the conference papers [15, 21]). The nonlinear plants and dynamic state feedback controllers that we consider

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only need to satisfy a local Lipschitz condition. Static state feedback and open-loop results follow as corollaries from the dynamic state feedback case. Moreover, the dissipation property we consider is rather general and its special cases are dissipation inequalities used to investigate stability, L_p stability, passivity, input-to-state stability, integral input-to-state stability, forward completeness, detectability, etc. (see for instance [12, 30, 34]). Applications of our results to investigation of input-to-state stability and passivity properties are presented in this paper to illustrate the generality of our approach.

Since, in general, the exact discretization of a dynamic controller can not be computed exactly, we use an approximate discrete-time model of the controller. In order to obtain a valid approximate model, the discretization of the dynamic controller should be carried out carefully. We introduce properties that the discretized controller should satisfy in order to have preservation of the dissipation inequality under sampling. These properties, which are called one-step strong and weak consistency, are specified in Definitions 2.4 and 2.5 and sufficient conditions for these properties to hold are given in Lemmas 2.1 and 2.2 respectively, and are proved in the Appendix.

In our main results we explore two types of dissipation inequalities for the discrete-time model of the closed-loop sampled-data system: the weak and strong form. In Definition 2.6 and 2.7, we introduce properties associated to the weak and strong dissipation inequalities. A relationship among the properties is given in Theorem 2.1. For the weak dissipation result to hold, the discretized controller needs to satisfy the one-step weak consistency condition (Definition 2.4) and the disturbances need to be uniformly Lipschitz (Theorem 3.1). It is shown in Proposition 3.2 that uniformly Lipschitz disturbances can be obtained by filtering bounded measurable disturbances through a strictly proper input-to-state stable (ISS) filter. The strong dissipation inequality holds if the discretized controller satisfies the one-step strong consistency condition (Definition 2.5) and in this case disturbances are allowed to be only measurable (see Theorem 3.3). In general, strong and weak dissipation inequalities do not imply each other and this is illustrated by Example 3.1. Similar results then follow for the static feedback and open loop cases. The generality of our approach is illustrated by two applications of our results to investigation of input-to-state stability of sampled-data systems with emulated controllers and results on preservation of passivity under sampling. A special case of the input-to-state stability results is a result on preservation of stability under sampling, which is proved for a much general situation than any of the results in the literature that we are aware of (see [3, 6, 26, 35]).

Our main results are semiglobal and practical in nature and their important feature is that the required sampling period can be computed using our method, although it may be conservative (smaller than necessary) which is a consequence of the conservative Lipschitz bounds that we are using in the proofs. This is a common problem in numerical analysis literature [32] and the emulation design in sampled-data systems [11, 35].

The paper is organized as follows. In Section 2 we present preliminaries. Main results are stated and discussed in Section 3. Proofs of the main results and their applications are presented in Section 4 and Section 5 respectively. Finally the conclusions are given in the last section. Sufficient conditions for one-step weak and strong consistency properties are proved in the Appendix.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing; it is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. For a given function $d(\cdot)$, we use the following notation $d[t_1, t_2] := \{d(t) : t \in [t_1, t_2]\}$. If $t_1 = kT, t_2 = (k+1)T$, we use the shorter notation $d[k]$, and take the norm of $d[k]$ to be the supremum of $d(\cdot)$ over $[kT, (k+1)T]$, that is $\|d[k]\|_\infty = \operatorname{ess\,sup}_{\tau \in [kT,(k+1)T]} |d(\tau)|$.

Consider the continuous-time nonlinear plant model:

$$\dot{x} = f(x, u, d_c, d_s) \tag{1}$$

$$y = h(x, u, d_c, d_s) \ , \tag{2}$$

with the dynamic state feedback controller:

$$\begin{array}{rcl} \dot{z} & = & g(x, z, d_c, d_s) \\ u & = & u(x, z, d_c, d_s) \ , \end{array} \tag{3}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state of the plant, state of the controller, control input and output of the plant. $d_c \in \mathbb{R}^{n_c}$ and $d_s \in \mathbb{R}^{n_s}$ are disturbance inputs to the system. It is assumed that f, g, h and u are locally Lipschitz. We also assume that $f(0,0,0,0) = 0$, $g(0,0,0,0) = 0$, $h(0,0,0,0) = 0$ and $u(0,0,0,0) = 0$. The controller (3) covers the case of dynamic output feedback:

$$\begin{array}{rcl} \dot{z} & = & \tilde{g}(y, z, d_c, d_s) =: g(x, z, d_c, d_s) \\ u & = & \tilde{u}(y, z, d_c, d_s) =: u(x, z, d_c, d_s) \ , \end{array} \tag{4}$$

where we assume that the feedback system (1), (2), (3) is Lipschitz well posed, that is the equations:

$$\begin{array}{l} y = h(x, u(y, z, d_c, d_s), d_c, d_s) \\ u = \tilde{u}(h(x, u, d_c, d_s), z, d_c, d_s) \end{array}$$

have unique solutions $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ so that (1), (2) and (4) can be written in the form $\dot{\eta} = \mathcal{F}(\eta, d_c, d_s)$, $\psi = \mathcal{H}(\eta, d_c, d_s)$ where $\eta := (x^T \ \ z^T)^T$, $\psi := (y^T \ \ u^T)^T$ and $\mathcal{F}$ and $\mathcal{H}$ are locally Lipschitz.

The following definitions are used in the sequel.

Definition 2.1 *The system (1), (2), (3) is said to be (V, w)-dissipative if there exist a continuously differentiable function $V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}$, called the storage function, and a continuous function $w : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \to \mathbb{R}$, called the dissipation rate, such that for all $x \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, d_c \in \mathbb{R}^{n_c}, d_s \in \mathbb{R}^{n_s}$ the following holds:*

$$\frac{\partial V}{\partial x} f(x, u(x, z, d_c, d_s), d_c, d_s) + \frac{\partial V}{\partial z} g(x, z, d_c, d_s) \leq w(x, z, d_c, d_s) \ . \tag{5}$$

■

Remark 2.1 *Dissipation inequality is sometimes expressed in terms of an integral, the result of integrating (5) along the solutions (see, for instance [34]), which takes the following form:*

$$V(x(t), z(t)) - V(x(t_\circ), z(t_\circ)) \leq \int_{t_\circ}^{t} w(x(\tau), z(\tau), d_c(\tau), d_s(\tau)) d\tau \ . \tag{6}$$

In this form, no differentiability assumptions are imposed on V (see, for instance, [34]). We will concentrate mainly on the differential form of dissipation inequalities in this paper, but the same proof technique can be used to prove our main results using the integral form (6). We also note that it is usually assumed in the literature that V is positive semidefinite or positive definite. We do not use these conditions on V in Definition 2.1 since they are not needed for the proofs. ■

Definition 2.2 *The system $\dot{x} = f(x)$ is globally asymptotically stable (GAS) if there exists $\beta \in \mathcal{KL}$ such that the solutions of the system satisfy $|x(t)| \leq \beta(|x_\circ|, t), \quad \forall x_\circ \in \mathbb{R}^n, \forall t \geq 0$.* ■

Definition 2.3 *The system $\dot{x} = f(x, d)$ is input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x_\circ \in \mathbb{R}^n$ and all $d \in \mathcal{L}_\infty$, the solutions of the system satisfy:*

$$|x(t)| \leq \beta(|x_\circ|, t) + \gamma(\|d\|_\infty), \quad \forall t \geq 0 \ . \tag{7}$$

■

Emulation procedure: Suppose that, as a first step in the emulation design, we designed a controller (3) for the plant (1), (2) in the continuous-time domain, so that the closed-loop continuous-time system is (V, w)-dissipative.

As a second step, we discretize the controller and implement it using sample and zero order hold devices. The discretization of the controller is carried out as follows. First, we consider an auxiliary system where the state measurements are assumed to be constant during sampling intervals $x(t) = x(kT) =: x(k)$ and $d_s(t) = d_s(kT) =: d_s(k)$ for all $t \in [kT, (k+1)T)$ in the differential equation (3), where $T > 0$ is the sampling period. Consider the following initial value problem:

$$\dot{z}(t) = g(x(k), z(t), d_c(t), d_s(k)) \ , \qquad z_\circ = z(k) \tag{8}$$

where $x(k), z(k), d_c[k], d_s(k)$ are given. Denote the solution of the initial value problem (8) as $z(t)$, and then we obtain the exact discretization of the controller (3) (see also [6]):

$$\begin{aligned} z(k+1) &= z(k) + \int_{kT}^{(k+1)T} g(x(k), z(\tau), d_c(\tau), d_s(k)) d\tau =: G_T^e(x(k), z(k), d_c[k], d_s(k)) \\ u(k) &= u(x(k), z(k), d_c(k), d_s(k)) \ . \end{aligned} \tag{9}$$

Note that in general the discretization (9) can not be implemented directly since G_T^e in (9) is usually impossible to compute exactly (since we need to solve the nonlinear initial value problem (8) explicitly over one sampling interval), so we need to use instead an approximate discrete-time model of the controller:

$$\begin{aligned} z(k+1) &= G_T^a(x(k), z(k), d_c(k), d_s(k)) \\ u(k) &= u(x(k), z(k), d_c(k), d_s(k)) \ , \end{aligned} \tag{10}$$

which is obtained from (8) using one of the numerical integration methods (e.g. Runge-Kutta). For instance, if we use the forward Euler method, we obtain $G_T^a(x, z, d_c, d_s) := x + Tg(x, z, d_c, d_s)$. It is obvious that in general we will have to use a sufficiently small sampling period T, since the approximate discrete-time model (10) is usually a good approximation of the exact discrete-time model (9) typically only for small T.

The sampled-data closed-loop system consists of the continuous-time plant (1), (2) and the controller (10), which is between a sample and zero order hold device. In the sequel, we use the discrete-time model of this sampled-data system, which consists of (10) and the exact discrete-time model of the plant, which is obtained as follows. We assume that $u(t) = u(kT) =: u(k)$, $d_s(t) = d_s(kT) =: d_s(k)$ for all $t \in [kT, (k+1)T]$ and consider the initial value problem

$$\dot{x}(t) = f(x(t), u(k), d_c(t), d_s(k)) \ , \qquad x_\circ = x(k) \tag{11}$$

where $x(k)$, $u(k)$, $d_c[k]$ and $d_s(k)$ are given. The output y is measured only at sampling instants kT, $k \geq 0$. Denote the solution of the initial value problem (11) as $x(t)$. Then the exact discrete-time model of the plant can be written as:

$$\begin{aligned} x(k+1) &= x(k) + \int_{kT}^{(k+1)T} f(x(\tau), u(k), d_c(\tau), d_s(k)) d\tau =: F_T(x(k), u(k), d_c[k], d_s(k)) \\ y(k) &= h(x(k), u(k), d_c(k), d_s(k)) \ . \end{aligned} \tag{12}$$

The discrete-time model of the sampled-data closed-loop system consists of (10) and (12).

The sampling period T is assumed to be a design parameter which can be arbitrarily assigned. In practice, the sampling period T is fixed and our results could be used to determine if it is suitably small. We emphasize that F_T in (12) is not known in most cases, and G_T^e in (9) can not be computed exactly, so we need to use G_T^a in (10) instead. Similarly to [24] we will think of F_T, G_T^e and G_T^a as being defined globally for all small T, even though the initial value problem (11) and (8) may exhibit finite escape times. We do this by defining F_T and G_T^e arbitrarily for $(x(k), z(k), d_c[k], d_s(k))$ corresponding to the finite escapes and noting that such points correspond only to states and inputs of arbitrarily large norm as $T \to 0$, since f and g are assumed locally Lipschitz (and hence locally bounded). So, the behavior of F_T and G_T^e will reflect the behavior of (11) and (8) respectively, as long as $(x(k), z(k), d_c[k], d_s(k))$ remain bounded with a bound that is allowed to grow as $T \to 0$. This is consistent with our main results that guarantee semiglobal dissipativity properties in the sampling period, that is as $T \to 0$ the set of states and inputs for which a dissipation inequality for the discrete-time model (10), (12) holds is guaranteed to contain an arbitrary large neighborhood of the origin.

In order to prove our main results, we need to guarantee that the mismatch between the exact discrete-time model of the controller (9) and its approximation (10) is small in some sense. We define two consistency properties that are used to limit the mismatch. Different forms of the consistency property are used in numerical analysis literature (see Definition 2 [22], Definition 1 [24] and Definition 3.4.2 [32]).

Definition 2.4 (One-step weak consistency) *The family G_T^a is said to be one-step weakly consistent with G_T^e if given any quintuple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$, there exist a function $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that, for all $T \in (0, T^*)$, $|x| \le \Delta_x$, $|z| \le \Delta_z$, $|d_s| \le \Delta_{d_s}$ and functions $d_c(\cdot)$ that are uniformly Lipschitz and satisfy $\|d_c[0]\|_\infty \le \Delta_{d_c}$ and $\left\|\dot{d}_c[0]\right\|_\infty \le \Delta_{\dot{d}_c}$, we have*

$$|G_T^e - G_T^a| \le T\rho(T) \ . \tag{13}$$

latex ■

A sufficient condition for one-step weak consistency is the following (the proof is given in the Appendix):

Lemma 2.1 *Consider G_T^e and G_T^a of the controller (3). If*

1. *G_T^a is one-step weakly consistent with G_T^{Euler} where $G_T^{Euler} := z + Tg(x, z, d_c, d_s)$,*
2. *given any strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{d_s})$, there exist $\rho_1 \in \mathcal{K}_\infty$, $\rho_2 \in \mathcal{K}_\infty$, $M > 0$, $T^* > 0$, such that, for all $T \in (0, T^*)$ and for all $|x| \le \Delta_x$, $|z| \le \Delta_z$, $|d_c| \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$,*

   (a) $\max\limits_{\{|x| \le \Delta_x, |z| \le \Delta_z, |d_c| \le \Delta_{d_c}, |d_s| \le \Delta_{d_s}\}} |g(x, z, d_c, d_s)| \le M$,

   (b) $|g(x, z_1, d_{c1}, d_s) - g(x, z_2, d_{c2}, d_s)| \le \rho_1(|z_1 - z_2|) + \rho_2(|d_{c1} - d_{c2}|)$,

then G_T^a is one-step weakly consistent with G_T^e. ■

Definition 2.5 (One-step strong consistency) *The family G_T^a is said to be one-step strongly consistent with G_T^e if given any quadruple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{d_s})$, there exists a function $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that, for all $T \in (0, T^*)$, $|x| \le \Delta_x$, $|z| \le \Delta_z$, $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$, we have*

$$|G_T^e - G_T^a| \le T\rho(T) \ . \tag{14}$$

■

A sufficient condition for one-step strong consistency is the following (the proof is given in the Appendix):

Lemma 2.2 *Consider G_T^e and G_T^a of the controller (21). If*

1. *G_T^a is one-step strongly consistent with G_T^{Euler} where $G_T^{Euler} := z + Tg(x, z, d_s)$,*
2. *given any triple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_s})$, there exist $\rho_1 \in \mathcal{K}_\infty$, $M > 0$, $T^* > 0$, such that, for all $T \in (0, T^*)$ and for all $|x| \le \Delta_x$, $|z| \le \Delta_z$, $|d_s| \le \Delta_{d_s}$,*

   (a) $\max\limits_{\{|x| \le \Delta_x, |z| \le \Delta_z, |d_s| \le \Delta_{d_s}\}} |g(x, z, d_s)| \le M$,

   (b) $|g(x, z_1, d_s) - g(x, z_2, d_s)| \le \rho_1(|z_1 - z_2|)$.

then G_T^a is one-step strongly consistent with G_T^e. ∎

Remark 2.2 *Consistency properties specify how the controller should be discretized for the emulation procedure to yield desired results. Lemmas 2.1 and 2.2 present general checkable conditions under which one-step weak and strong consistency properties hold. It is important to emphasize that if the exact discrete-time model of the controller can be obtained, then we do not have to use an approximate discrete-time model of the controller and consistency definitions become superfluous, i.e., they hold automatically. Two important such cases were considered in the literature: emulation for linear systems was considered in [6] and emulation for static state feedback controllers was considered in [21]. However in linear system case, although the exact discrete-time model is computable, one may still implement its approximation. Finally, note that the weak and strong consistency definitions become equivalent when G_T^e and G_T^a are independent of d_c.* ∎

Remark 2.3 *Note that the Euler approximation is one-step (weakly or strongly) consistent whenever the second condition in Lemma 2.1 or 2.2 is satisfied, since the first condition automatically holds. Also, if we want to implement the Euler approximate model of the controller, that is $G_T^a = z + Tg(x, z, d_s)$, then we can regard the closed-loop system (1), (2) and (3) as an augmented plant of the form*

$$\begin{aligned} \dot{x} &= f(x, u, d_c, d_s) \\ \dot{z} &= v \end{aligned}$$

controlled by the static state feedback controller of the form:

$$\begin{aligned} u &= u(x, z, d_s) \\ v &= g(x, z, d_s) \end{aligned}$$

which is implemented between the sample and zero order hold device(s). Note that, this form is valid only when g is independent of d_c. In this case, one can use results in [21] on emulation for static state feedback controllers. However, if we want to use an approximate discretization G_T^a other than Euler, this method is not applicable and we need to use results proved in this paper that use the notion of consistency for general discretizations. ∎

Remark 2.4 *There is a strong motivation to consider controller discretizations other than Euler, although even the simple Euler discretization may sometimes yield satisfactory performance (see for instance [9, 24]). Indeed, a number of studies have shown that the Euler approximation of the controller dynamics is not always appropriate to use. For instance, the Euler approximation is, in general, not recommended to use for singularly perturbed systems that exhibit two-time scale behavior (see [20] and [4]). Using a comparative study in [8], the authors showed that the Tustin (bilinear) approximation is superior to Euler for the particular application. Moreover, even for linear systems, some examples in [1, 13] indicate that if the sampling period is given and fixed, then most of the classical discretization methods (such as Euler) might fail to yield acceptable performance or even stability. For linear systems, this has led to more advanced techniques for controller discretization that obtain the approximate model as a solution of an optimization problem (see [1] for more details). Similar results for nonlinear systems are yet to be proved.*

The consistency properties that we use provide a general and unified framework for investigation of a range of different controller discretizations. Moreover, they generalize in a natural way the consistency definitions commonly found in the numerical analysis literature that apply to ordinary differential equations without inputs (see for instance Definition 3.4.2 in [32]). A range of different consistent discretization can be defined using the results in [18]. Indeed, if the controller dynamics do not depend on d_c then the results in [18] can be used to write the solution of the initial value problem (8) as a series expansion in the sampling period T. Finite truncations of these expansions give a range of approximate discretization of the controller that are one step consistent. Moreover, classical Runge-Kutta integration schemes can also be used to obtain one step consistent approximations (see for instance [32]). ∎

In order to precisely state the main results, we introduce the following properties. In the sequel, we use the following notation $x = x(0)$ in order to shorten the notation.

Definition 2.6 *Let V be continuously differentiable and w be continuous. The system (10), (12) is said to have Property P1 (respectively, have Property P2) if given any 6-tuple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and all $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $|d_s| \leq \Delta_{d_s}$ and for all disturbances $d_c(\cdot)$ that satisfy $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$ the following holds:*

$$\frac{V(F_T(x, u(x, z, d_c, d_s), d_c[0], d_s), G_T^a(x, z, d_c, d_s)) - V(x, z)}{T} \leq \frac{1}{T}\int_0^T w(x, z, d_c(\tau), d_s)d\tau + \nu \ , \qquad (15)$$

(respectively the following holds for the system (10), (12):

$$\frac{V(F_T(x, u(x, z, d_c, d_s), d_c[0], d_s), G_T^a(x, z, d_c, d_s)) - V(x, z)}{T} \leq w(x, z, d_c, d_s) + \nu \ ). \qquad (16)$$

■

Definition 2.7 *Let V be continuously differentiable and w be continuous. The system (10), (12) is said to have Property P3 if given any quintuple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{d_s}, \nu)$, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and all $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$ the inequality (15) holds.* ■

Remark 2.5 *We defined several different properties (P1, P2 and P3) since each of them may be useful in a particular situation. For instance, properties P1 or P2 are useful when the input d_c is filtered through an input-to-state stable filter (see Proposition 3.2) or when all inputs are constant during the sampling intervals (see application of our results to preservation of passivity under sampling in Section 5). On the other hand, Property P3 is useful when the disturbance d_c is only assumed to be a measurable function of time, which is important, for instance, in investigation of input-to-state stability (see Section 5).* ■

The following preliminary result that is proved in Section 4 shows that Properties P1 and P2 in Definition 2.6 are equivalent.

Theorem 2.1 *The system (10), (12) has Property P1 if and only if it has Property P2.* ■

The main difference between the Properties P1 and P3 (or P2 and P3, since Properties P1 and P2 are equivalent) is that Property P1 requires the disturbances d_c to be Lipschitz, uniformly in T, for the inequality (15) to hold, whereas the inequality (15) in Property P3 must hold for non-uniformly Lipschitz disturbances as well. The dissipation inequalities in Properties P1 and P2 (since they are equivalent) are said to have the "weak" form (since they hold for a smaller class of disturbances) and the dissipation inequality in Property P3 is said to have the "strong" form (since it holds for a larger class of disturbances).

3 Main results

In this section we state the main results (Theorem 3.1 and 3.3) which assume that the continuous-time system is (V, w)-dissipative. Theorem 3.1 states that if one-step weak consistency holds and disturbances $d_c(\cdot)$ are uniformly Lipschitz, then the (equivalent) Properties P1 and P2 hold for discrete-time model of the sampled-data system. Since in most cases we do not know whether the disturbances are uniformly Lipschitz or not, in Proposition 3.2 we prove that if we filter a bounded measurable signal using a strictly proper input-to-state stable filter, we obtain a filtered signal which is bounded and uniformly Lipschitz. If disturbances are only measurable (but not uniformly Lipschitz) then the inequality (16) may not hold in a semiglobal practical sense while the inequality (15) still holds (see Example 3.1). In Theorem 3.3 we show that for a smaller class of controllers, if $d_c(\cdot)$ are measurable (but not uniformly Lipschitz) and one-step strong consistency holds then the discrete-time model has Property P3.

Theorem 3.1 *(Weak form of dissipativity) If the system (1), (2), (3) is (V, w)-dissipative, then given any approximate discrete-time model of the controller G^a_T (10) that is one-step weakly consistent with the exact discrete-time model of the controller G^e_T (9) the system (10), (12) has Property P1 (equivalently, Property P2).* ■

Note that Properties P1 and P2 require $d_c(\cdot)$ to be uniformly Lipschitz. The following example shows that indeed the uniformly Lipschitz condition on $d_c(\cdot)$ is necessary, since the inequality (16) may not hold if $d_c(\cdot)$ is not uniformly Lipschitz.

Example 3.1 *[21] Consider the continuous-time system $\dot{x} = u(x) + d_c = -x + d_c$, where $x, d_c \in \mathbb{R}$. Using the storage function $V = \frac{1}{2}x^2$, the derivative of V is $\dot{V} = -x^2 + x d_c \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2$, and the system is ISS. It was shown in [21] that if a family of bounded disturbances $d_c(t) = \cos\left(\frac{t+2T}{T}\right)$ is considered, then the inequality $\frac{\Delta V}{T} \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2 + \nu$ does not hold in a semiglobal practical sense, which implies that Property P2 does not hold! This is due to the fact that the family of disturbances is not Lipschitz, uniformly in T, since $\left\|\dot{d}_c\right\|_\infty = 1/T$. This illustrates that, in general, the Lipschitz condition, uniform in T, on $d_c(\cdot)$ in Theorem 3.1 is necessary for the result to hold.* ▲

The following result shows that if we can filter any bounded measurable disturbances using a strictly proper input-to-state stable filter, then the filtered disturbances are bounded and uniformly Lipschitz. This further motivates Theorems 2.1 and 3.1 that require disturbances to be uniformly Lipschitz.

Proposition 3.2 *Consider any nonlinear filter:*

$$\dot{\xi} = f(\xi, d_c) \tag{17}$$

$$\upsilon = h(\xi) \ , \tag{18}$$

which is input-to-state stable with respect to input d_c and where f and h are locally Lipschitz. Then, given any $d_c(\cdot) \in \mathcal{L}_\infty$ and any $\xi_\circ \in \mathbb{R}^{n_\xi}$ we have that the output $\upsilon(\cdot)$ is bounded, that is $\upsilon(\cdot) \in \mathcal{L}_\infty$. Moreover, $\dot{\upsilon}(\cdot) \in \mathcal{L}_\infty$, which implies that υ is uniformly Lipschitz. ■

The use of filters in sampled-data systems is standard (see for instance [6]). In particular, filters that are strictly proper, stable, linear and time invariant:

$$\dot{\xi} = A\xi + Bd_c \tag{19}$$

$$\upsilon = C\xi \ , \tag{20}$$

were considered in [6] in the context of $\mathcal{L}_p$ stability of linear sampled-data systems. In this case, we have that the filter satisfies all conditions of Proposition 3.2 and consequently for any $\xi_\circ$ and $d_c \in \mathcal{L}_\infty$ we have that $\upsilon, \dot{\upsilon} \in \mathcal{L}_\infty$.

Example 3.1 showed that if disturbances $d_c(\cdot)$ are not uniformly Lipschitz, then Properties P2 may not hold. It is of interest to investigate conditions, under which Property P3 still holds, for the case when $d_c(\cdot)$ are not uniformly Lipschitz. To prove a general result for this case it is necessary to restrict our attention to the controllers of the following form (see Example 3.2 below):

$$\begin{aligned} \dot{z} &= g(x, z, d_s) \\ u &= u(x, z, d_s) \ . \end{aligned} \tag{21}$$

We assume that g and u are locally Lipschitz, $g(0,0,0) = 0$ and $u(0,0,0) = 0$. In a similar manner as for controller (3), we define the exact discrete-time model of the controller (21) as:

$$\begin{aligned} z(k+1) &= z(k) + \int_{kT}^{(k+1)T} g(x(k), z(\tau), d_s(k)) d\tau =: G^e_T(x(k), z(k), d_s(k)) \\ u(k) &= u(x(k), z(k), d_s(k)) \ , \end{aligned} \tag{22}$$

and its approximate discrete-time model:

$$\begin{aligned} z(k+1) &= G_T^a(x(k), z(k), d_s(k)) \\ u(k) &= u(x(k), z(k), d_s(k)) \ . \end{aligned} \tag{23}$$

Note that the controller (21) does not have $d_c(\cdot)$ as its input and the following example shows that this is necessary in general if we want to prove that the discrete-time model of the sampled-data system has Property P3.

Example 3.2 *[21] Consider the system $\dot{x} = u$, where $u = -d_c$, where $d_c(0) = 0$ and $d_c(t) = 1$, $\forall t > 0$. The storage function that we consider is $V(x) = x$, so that the derivative: $\frac{\partial V}{\partial x}(-d_c) = -d_c$, and hence the dissipation rate is $w(x, d_c, d_s) = -d_c$. Since u is sampled and $d_c(0) = 0$, we have that $x(t) = 0, \forall t \in [0, T]$ and so $\Delta V/T = 0$. On the other hand $\int_0^T w(d_c(\tau))d\tau = -T$. Hence, if Property P3 was hold, then we would obtain $0 \leq -1 + \nu$, which is not true for small ν.* ▲

Compared to Theorem 3.1, the following result on strong form of dissipativity considers a larger class of measurable disturbances d_c.

Theorem 3.3 *(Strong form of dissipativity) If the system (1), (2), (21) is (V, w)-dissipative, then given any approximate discrete-time model of the controller G_T^a (23) that is one-step strongly consistent with the exact discrete-time model of the controller G_T^e (22) the system (12), (23) has Property 3.* ■

Two important special cases of our main results are the static state feedback and open-loop system. All of the results given below follow directly from the more general case of dynamic state feedback and we describe below the connections.

3.1 Static state feedback results

The static state feedback:

$$u = u(x, d_c, d_s) \tag{24}$$

is a special case of (3), where $n_z = 0$. Similarly, the controller:

$$u = u(x, d_s) \tag{25}$$

is a special case of the controller (21). Obvious changes are introduced in definitions of Properties P1, P2 and P3 to cover the static state feedback case and we list them below for ease of reference. The inequality (5) in the (V, w)-dissipativity property is replaced by

$$\frac{\partial V}{\partial x} f(x, u(x, d_c, d_s), d_c, d_s) \leq w(x, d_c, d_s) \ . \tag{26}$$

The discretized controllers of (24) and (25) take respectively the following forms:

$$u(k) = u(x(k), d_c(k), d_s(k)), \quad k \geq 0 \ , \tag{27}$$
$$u(k) = u(x(k), d_s(k)), \quad k \geq 0 \ , \tag{28}$$

and they are implemented using a sample and zero order hold. As already indicated in Remark 2.2, the consistency properties are always satisfied since the controller has no dynamics. Since $n_z = 0$, we omit all conditions on z variable in Properties P1, P2 and P3. Consequently, the inequalities (15) and (16) are respectively replaced by the following inequalities:

$$\frac{V(F_T(x, u(x, d_c, d_s), d_c[0], d_s)) - V(x)}{T} \leq \frac{1}{T} \int_0^T w(x, d_c(\tau), d_s) d\tau \ + \ \nu \ , \tag{29}$$

and

$$\frac{V(F_T(x, u(x, d_c, d_s), d_c[0], d_s)) - V(x)}{T} \leq w(x, d_c, d_s) + \nu \ . \tag{30}$$

Direct consequences of Theorems 3.1 and 3.3 are the following corollaries.

Corollary 3.1 *If the system (1), (2), (24) is (V, w)-dissipative, then the exact discrete-time model (12), (27) of the system has Property P1 (equivalently, Property P2).* ■

Corollary 3.2 *If the system (1), (2), (25) is (V, w)-dissipative, then the exact discrete-time model (12), (28) of the system has Property P3.* ■

Example 3.1 (cont'd) *Note that since the state feedback of the system in Example 3.1 is static and it does not depend on d_c, all conditions of Corollary 3.2 are satisfied and the exact discrete-time model has Property P3.* ▲

3.2 Open-loop configuration results

Besides the static feedback results, the results on preservation of dissipation inequalities under sampling for open-loop systems are also a direct consequence of our main results on dynamics state feedback controllers. Indeed, the open-loop systems can be viewed as a special case of "closed-loop" systems, with $m = 0$ and $n_z = 0$. The continuous-time system (1), (2) can be rewritten as

$$\dot{x} = \tilde{f}(x, d_c, \tilde{d}_s) := f(x, u, d_c, d_s) \tag{31}$$

$$y = \tilde{h}(x, d_c, \tilde{d}_s) := h(x, u, d_c, d_s) \ , \tag{32}$$

where $\tilde{d}_s := (u^T \ \ d_s^T)^T$ and the control u can be treated in the same way as the disturbance d_s.

For ease of reference we list the changes needed in Properties P1, P2 and P3 to cover the open-loop case. We replace (5) of the (V, w)-dissipativity property with

$$\frac{\partial V}{\partial x} f(x, u, d_c, d_s) \le w(x, u, d_c, d_s) \ . \tag{33}$$

Since there is no controller in this case, the consistency properties are superfluous. The exact discrete-time model of the open-loop system is given by (12). The statements of Properties P1, P2 and P3 are changed in the following way: "... given any quintuple of strictly positive numbers $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{d_s}, \nu)$ there exists $T^* > 0$ such that ...". The inequalities (15) and (16) are respectively replaced by the following inequalities:

$$\frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \le \frac{1}{T} \int_0^T w(x, u, d_c(\tau), d_s) d\tau \ + \ \nu \ , \tag{34}$$

and

$$\frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \le w(x, u, d_c, d_s) + \nu \ . \tag{35}$$

The following results are direct consequences of our main results.

Corollary 3.3 *If the system (31), (32) is (V, w)-dissipative, then the exact discrete-time model (12) of the system has Property P1 (equivalently, Property P2).* ■

Under slightly stronger conditions we can prove a stronger result that is useful in some situations:

Proposition 3.4 *If the system (31), (32) is (V, w)-dissipative, with $\frac{\partial V}{\partial x}$ being locally Lipschitz and $\frac{\partial V}{\partial x}(0) = 0$, then given any quintuple of strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$, there exist $T^* > 0$ and positive constants K_1, K_2, K_3, K_4, K_5 such that for all $T \in (0, T^*)$ and all $|x| \le \Delta_x$, $|u| \le \Delta_u$, $|d_s| \le \Delta_{d_s}$ and functions $d_c(\cdot)$ that are uniformly Lipschitz and satisfy $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \le \Delta_{\dot{d}_c}$, we have for the exact discrete-time model (12) of the system:*

$$\frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \le w(x, u, d_c, d_s) + T\left(K_1|x|^2 + K_2|u|^2 + K_3|d_s|^2 + K_4\|d_c[0]\|_\infty^2 + K_5\left\|\dot{d}_c[0]\right\|_\infty^2\right) \ . \tag{36}$$

■

Analogous to Theorem 3.1, we need the uniformly Lipschitz condition on $d_c(\cdot)$ for Corollary 3.3 and Proposition 3.4 to hold. For the case when $d_c(\cdot)$ is not uniformly Lipschitz, results similar to Theorem 3.3 are stated in the following. Note that in this open-loop case, for either the weak or strong dissipativity result, there is no dependency of control on d_c, since the control is an external input.

Corollary 3.4 *If the system (31), (32) is (V, w)-dissipative, whereas $d_c(\cdot)$ is measurable but not necessarily uniformly Lipschitz, then the exact discrete-time model (12) of the system has Property P3.* ■

Proposition 3.5 *If the system (31), (32) is (V, w)-dissipative, with $\frac{\partial V}{\partial x}$ being locally Lipschitz and $\frac{\partial V}{\partial x}(0) = 0$, then given any quadruple of strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{d_s})$ there exist $T^* > 0$ and positive constants K_1, K_2, K_3, K_4 such that for all $T \in (0, T^*)$ and all $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $|d_s| \leq \Delta_{d_s}$ we have for the exact discrete-time model (12) of the system:*

$$\frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \leq \frac{1}{T}\int_0^T w(x, u, d_c(\tau), d_s)d\tau + T\left(K_1|x|^2 + K_2|u|^2 + K_3\|d_c[0]\|_\infty^2 + K_4|d_s|^2\right).$$

■

4 Proofs of main results

Proof of Theorem 2.1:
(P1) $\Longrightarrow$ **(P2)** Suppose that Property P1 holds. Let $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu_w)$ be given and let $T_s^* > 0$ (from Property P1) be such that for all $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, $|d_s| \leq \Delta_{d_s}$ and all $T \in (0, T_s^*)$ the following holds:

$$\begin{aligned}\frac{\Delta V}{T} &\leq \frac{1}{T}\int_0^T w(x, z, d_c(\tau), d_s)d\tau + \frac{\nu_w}{2} \\ &\leq w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{1}{T}\int_0^T |w(x, z, d_c(\tau), d_s) - w(x, z, d_c, d_s)|\, d\tau\ ,\end{aligned} \tag{37}$$

where the second inequality was obtained by adding and subtracting $w(x, z, d_c, d_s)$. Since $d_c(\cdot)$ is uniformly Lipschitz with Lipschitz constant $\Delta_{\dot{d}_c}$, we can write $|d_c(\tau) - d_c| \leq \Delta_{\dot{d}_c}\tau$. Moreover, since w is continuous, it is uniformly continuous on compact sets, and given any $\varepsilon > 0$ there exists $T_s > 0$ such that for any $\tau \in [0, T_s]$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, $|d_s| \leq \Delta_{d_s}$ we have that $|w(x, z, d_c(\tau), d_s) - w(x, z, d_c, d_s)| \leq \varepsilon$. Let $\varepsilon = \frac{\nu_w}{2}$ and let this fix T_s. Let $T_w^* = \min\{T_s, T_s^*\}$. Then using (37) we have that for all $T \in (0, T_w^*)$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, $|d_s| \leq \Delta_{d_s}$:

$$\frac{\Delta V}{T} \leq w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{1}{T}\int_0^T \frac{\nu_w}{2}d\tau = w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{\nu_w}{2}\ , \tag{38}$$

which shows that Property P2 holds.
(P2) $\Longrightarrow$ **(P1)** follows a similar way as the proof for **(P1)** $\Longrightarrow$ **(P2)**, to show that if Property P2 holds, then Property P1 holds. ■

Proof of Theorem 3.1: To shorten the notation we define $u := u(x, z, d_c, d_s)$, $f := f(x, u, d_c, d_s)$, $g := g(x, z, d_c, d_s)$, $F_T := F_T(x, u, d_c[0], d_s)$, $G_T^e := G_T^e(x, z, d_c[0], d_s)$ and $G_T^a := G_T^a(x, z, d_c, d_s)$.

Definition of T^*: Suppose that the continuous-time system (1), (2), (3) is (V, w)-dissipative, that is for all $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, $d_c \in \mathbb{R}^{n_c}$, $d_s \in \mathbb{R}^{n_s}$, the inequality (5) holds. Let G_T^a be one-step weakly consistent with G_T^e, and let a 6-tuple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$ be given. Let these data generate $\rho \in \mathcal{K}_\infty$ from the definition of one-step weak consistency. Define $R_x := \Delta_x + 1$ and $R_z := \Delta_z + 1$. Let $L > 0$ be the Lipschitz constant of f and g on the sets where $|x| \leq R_x$, $|z| \leq R_z$, $|d_c| \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$, and let $b > 0$ be a number that satisfies $\max\left\{\left|\frac{\partial V}{\partial x}\right|, \left|\frac{\partial V}{\partial z}\right|, |f|, |g|\right\} \leq b$ for all $|x| \leq R_x$, $|z| \leq R_z$, $|d_c| \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$. Define $\Delta := \Delta_x + \Delta_z + \Delta_{d_c} + \Delta_{d_s}$.

We assume without loss of generality that $\nu \leq 1$ and $b \geq 1$ and define

$$T_1^* := \min\left\{\frac{1}{2b}, \rho^{-1}\left(\frac{\nu}{2b}\right)\right\} \ . \tag{39}$$

Note that $T_1^* \leq \frac{1}{2b} \leq \frac{1}{2} < 1$. Let $T_2^* > 0$ be such that the following holds:

$$bL\left[(\Delta+1)\frac{\exp(LT) - 1 - LT}{LT} + \frac{1}{2}\Delta_{\dot{d}_c} T\right] \leq \frac{\nu}{8} \ , \ \forall T \in (0, T_2^*) \ . \tag{40}$$

It is easy to see that such a T_2^* always exists. Let $x_1 := x + \theta_1 T f$ and $z_1 := z + \theta_2 T g$ where $\theta_1, \theta_2 \in (0,1)$. Let $T_3^* > 0$ be such that:

$$b\left|\left.\frac{\partial V}{\partial x}\right|_{(x_1, z+Tg)} - \left.\frac{\partial V}{\partial x}\right|_{(x,z)}\right| \leq \frac{\nu}{8} \ , \tag{41}$$

for all $T \in (0, T_3^*)$, $|x| \leq R_x$, $|z| \leq R_z$, $|d_s| \leq \Delta_{d_s}$, and $d_c(\cdot)$ such that $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$. The required T_3^* always exists, which can be proved as follows. From the continuity of $\frac{\partial V}{\partial x}$, which implies that $\frac{\partial V}{\partial x}$ is uniformly continuous on the compact sets, and since $|x_1 - x| \leq T|f| \leq Tb$ and $|(z + Tg) - z| = T|g| \leq Tb$, it follows that given any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $\left|\left.\frac{\partial V}{\partial x}\right|_{(x_1, z+Tg)} - \left.\frac{\partial V}{\partial x}\right|_{(x,z)}\right| \leq \epsilon$, $\forall T \in (0, T_\epsilon)$, $|x| \leq R_x$, $|z| \leq R_z$, $|d_c| \leq \Delta_{d_c}$ and $|d_s| \leq \Delta_{d_s}$. Hence, we can choose $\epsilon^* := \nu/(8b)$ and let this fix the desired $T_3^* := T_{\epsilon^*}$ for which (41) holds.

In exactly the same way we choose $T_4^* > 0$ such that

$$b\left|\left.\frac{\partial V}{\partial z}\right|_{(x, z_1)} - \left.\frac{\partial V}{\partial z}\right|_{(x,z)}\right| \leq \frac{\nu}{8} \ , \tag{42}$$

for all $T \in (0, T_4^*)$, $|x| \leq R_x$, $|z| \leq R_z$, $|d_s| \leq \Delta_{d_s}$, and $d_c(\cdot)$ such that $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$. Finally, we define

$$T^* := \min\{T_1^*, T_2^*, T_3^*, T_4^*\} \ . \tag{43}$$

Proof that Property P1 (P2) holds: We will show first, that Property P2 holds. Consider arbitrary $T \in (0, T^*)$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $|d_s| \leq \Delta_{d_s}$, and $d_c(\cdot)$ such that $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$.

Since $T < T^* \leq \frac{1}{2b}$, the solutions $x(t)$ and $z(t)$ of the initial value problems (11) and (8) exist and $|x(t)| \leq \Delta_x + \frac{1}{2}$, $|z(t)| \leq \Delta_z + \frac{1}{2}$, $\forall t \in [0, T]$, which implies

$$\begin{aligned} |F_T| &\leq \Delta_x + \frac{1}{2} \ < \ R_x \ , \\ |G_T^e| &\leq \Delta_z + \frac{1}{2} \ < \ R_z \ . \end{aligned} \tag{44}$$

From the second inequality in (44), one-step weak consistency and the choice of T_1^* we have:

$$\begin{aligned} |G_T^a| &\leq |G_T^e| + |G_T^a - G_T^e| \\ &< \Delta_z + \frac{1}{2} + \rho(T_1^*) \\ &\leq \Delta_z + \frac{1}{2} + \frac{1}{2} \\ &= R_z \ . \end{aligned} \tag{45}$$

From the local Lipschitz properties of f and g and the fact that they are zero at zero, we can write

$$
\begin{aligned}
|x(\tau) - x| &\leq (\Delta + 1)[\exp(L\tau) - 1] \ , \ \forall \tau \in [0, T] && (46)\\
|z(\tau) - z| &\leq (\Delta + 1)[\exp(L\tau) - 1] \ , \ \forall \tau \in [0, T] && (47)
\end{aligned}
$$

and since $d_c(\cdot)$ is uniformly Lipschitz, with Lipschitz constant $\Delta_{\dot{d}_c}$, we can write that for all τ

$$
|d_c(\tau) - d_c| = |d_c(\tau) - d_c(0)| \leq \Delta_{\dot{d}_c} \tau \ . \tag{48}
$$

We consider

$$
\begin{aligned}
\frac{\Delta V}{T} &= \frac{V(F_T, G_T^a) - V(x, z)}{T} \\
&= \underbrace{\left.\frac{\partial V}{\partial x}\right|_{(x,z)} f + \left.\frac{\partial V}{\partial z}\right|_{(x,z)} g}_{\mathbf{1}} + \underbrace{\frac{1}{T}\Big\{V(F_T, G_T^a) - V(x + Tf, z + Tg)\Big\}}_{\mathbf{2}} \\
&\quad + \underbrace{\frac{1}{T}\Big\{V(x + Tf, z + Tg) - V(x, z) - \left.\frac{\partial V}{\partial x}\right|_{(x,z)} Tf - \left.\frac{\partial V}{\partial z}\right|_{(x,z)} Tg\Big\}}_{\mathbf{3}} \ ,
\end{aligned} \tag{49}
$$

where the second equality holds since we just added and subtracted $\frac{1}{T}V(x+Tf, z+Tg)$, $\left.\frac{\partial V}{\partial x}\right|_{(x,z)} f$ and $\left.\frac{\partial V}{\partial z}\right|_{(x,z)} g$. Now we bound each term in (49).

Term 1: It follows from (V, w)-dissipativity of the continuous-time system (1), (2), (3) that:

$$
\left.\frac{\partial V}{\partial x}\right|_{(x,z)} f + \left.\frac{\partial V}{\partial z}\right|_{(x,z)} g \leq w(x, z, d_c, d_s) \ . \tag{50}
$$

Term 2: Applying the Mean Value Theorem to the Term **2**, we have by adding and subtracting $\frac{1}{T}V(x + Tf, G_T^a)$:

$$
\begin{aligned}
&\frac{1}{T}\Big\{V(F_T, G_T^a) - V(x + Tf, z + Tg)\Big\} \\
&\qquad \leq \underbrace{\frac{1}{T}\left|\left.\frac{\partial V}{\partial x}\right|_{(x_2, G_T^a)}\right| |F_T - (x + Tf)|}_{\mathbf{2a}} + \underbrace{\frac{1}{T}\left|\left.\frac{\partial V}{\partial z}\right|_{(x+Tf, z_2)}\right| |G_T^a - (z + Tg)|}_{\mathbf{2b}} \ ,
\end{aligned} \tag{51}
$$

where $x_2 = \theta_3 F_T + (1 - \theta_3)(x + Tf)$ and $z_2 = \theta_4 G_T^a + (1 - \theta_4)(z + Tg)$ and $\theta_3, \theta_4 \in (0, 1)$.

Since $\max\{|F_T|, |x+Tf|\} \leq R_x$ (see (44)), then $|x_2| \leq R_x$. Moreover, since $\max\{|G_T^a|, |z+Tg|\} \leq R_z$ (see (44) and (45)), this implies $|z_2| \leq R_z$. Hence, we have that $\left|\left.\frac{\partial V}{\partial x}\right|_{(x_2, G_T^a)}\right| \leq b$ and $\left|\left.\frac{\partial V}{\partial z}\right|_{(x+Tf, z_2)}\right| \leq b$.

Term **2a**: Since $\left|\frac{\partial V}{\partial x}\big|_{(x_2, G_T^a)}\right| \le b$ and f is locally Lipschitz, we can write

$$\begin{aligned}
\frac{1}{T}\left|\frac{\partial V}{\partial x}\bigg|_{(x_2,G_T^a)}\right| |F_T - (x+Tf)| &\le \frac{b}{T}|F_T - (x+Tf)| \\
&= \frac{b}{T}\left|\int_0^T f(x(\tau),u,d_c(\tau),d_s)d\tau - \int_0^T f(x,u,d_c,d_s)d\tau\right| \\
&\le \frac{b}{T}\left\{L\int_0^T |x(\tau)-x|\,d\tau + L\int_0^T |d_c(\tau)-d_c|\,d\tau\right\} \\
&\le \frac{bL}{T}\left\{(\Delta+1)\int_0^T[\exp(L\tau)-1]d\tau + \Delta_{\dot{d}_c}\int_0^T \tau d\tau\right\} \\
&= bL\left\{(\Delta+1)\frac{\exp(LT)-1-LT}{LT} + \frac{1}{2}\Delta_{\dot{d}_c}T\right\} \\
&\le \frac{\nu}{8}\,, \qquad (52)
\end{aligned}$$

where we first added and subtracted $\frac{b}{T}\int_0^T f(x,u,d_c(\tau),d_s)d\tau$, then used the local Lipschitz property of f, then used bounds (46) and (48) and finally exploited the definition of T_2^*.

Term **2b**: We use the fact that $\left|\frac{\partial V}{\partial z}\big|_{(x+Tf,z_2)}\right| \le b$, then add and subtract G_T^e to the last factor of Term **2b** to obtain:

$$\begin{aligned}
\frac{1}{T}\left|\frac{\partial V}{\partial z}\bigg|_{(x+Tf,z_2)}\right| |G_T^a - (z+Tg)| &\le \frac{b}{T}|G_T^a - z - Tg| \\
&\le \frac{b}{T}|G_T^a - G_T^e| + \frac{b}{T}|G_T^e - z - Tg| \\
&\le b\rho(T) + \frac{b}{T}\left|\int_0^T g(x,z(\tau),d_c(\tau),d_s)d\tau - Tg(x,z,d_c,d_s)\right| \\
&\le b\rho(T) + \frac{b}{T}\int_0^T L\,|z(\tau)-z|\,d\tau + \frac{b}{T}\int_0^T L\,|d_c(\tau)-d_c|\,d\tau \\
&\le b\rho(T) + bL\left[(\Delta+1)\frac{\exp(LT)-1-LT}{LT} + \frac{1}{2}\Delta_{\dot{d}_c}T\right] \\
&\le \frac{\nu}{2} + \frac{\nu}{8}\,, \qquad (53)
\end{aligned}$$

where we first used one-step weak consistency and definition of T_1^*, then the local Lipschitz property of g, then inequalities (47) and (48) and finally the definition of T_2^*.

Term 3: From the differentiability of V, we apply the Mean Value Theorem to Term **3** (where x_1 and

z_1 are defined just before (41)) to obtain:

$$
\begin{aligned}
\frac{1}{T}\Bigg\{V(x+Tf, z+Tg) - V(x,z) - \frac{\partial V}{\partial x}\bigg|_{(x,z)} Tf - \frac{\partial V}{\partial z}\bigg|_{(x,z)} Tg\Bigg\}& \\
\le \frac{\partial V}{\partial x}\bigg|_{(x_1, z+Tg)} f + \frac{\partial V}{\partial z}\bigg|_{(x,z_1)} g - \frac{\partial V}{\partial x}\bigg|_{(x,z)} f - \frac{\partial V}{\partial z}\bigg|_{(x,z)} g& \\
\le |f| \cdot \left| \frac{\partial V}{\partial x}\bigg|_{(x_1,z+Tg)} - \frac{\partial V}{\partial x}\bigg|_{(x,z)} \right| + |g| \cdot \left| \frac{\partial V}{\partial z}\bigg|_{(x,z_1)} - \frac{\partial V}{\partial z}\bigg|_{(x,z)} \right|& \\
\le b \left| \frac{\partial V}{\partial x}\bigg|_{(x_1,z+Tg)} - \frac{\partial V}{\partial x}\bigg|_{(x,z)} \right| + b \left| \frac{\partial V}{\partial z}\bigg|_{(x,z_1)} - \frac{\partial V}{\partial z}\bigg|_{(x,z)} \right|& \\
\le \frac{\nu}{8} + \frac{\nu}{8} \, .&
\end{aligned} \tag{54}
$$

In deriving (54) we first used the definition of b and then definitions of T_3^* and T_4^*. Combining (49), (50), (52), (53) and (54) complete the proof that Property P2 holds. The proof for Property P1 to hold follows directly from Theorem 2.1. ∎

Proof of Proposition 3.2: It is trivial; since $d_c \in \mathcal{L}_\infty$ and (17) is ISS, then $\xi \in \mathcal{L}_\infty$. Since f and h are continuous, then $\dot{\xi} \in \mathcal{L}_\infty$ and $v \in \mathcal{L}_\infty$. Finally, since h is locally Lipschitz, then

$$
\begin{aligned}
|\dot{v}| &= \left| \lim_{\delta \to 0} \frac{h(\xi(t+\delta)) - h(\xi(t))}{\delta} \right| \\
&\le L \lim_{\delta \to 0} \left| \frac{\xi(t+\delta) - \xi(t)}{\delta} \right| \\
&\le L \left| \dot{\xi} \right| \, ,
\end{aligned}
$$

which implies $\dot{v} \in \mathcal{L}_\infty$. ∎

Proof of Proposition 3.4:

The proof of the proposition follows the same steps as the proof of Theorem 3.1. Using the idea from the theorem, we first take any number $\nu > 0$, and do the computation of T^* in the same way as we have done in the proof of Theorem 3.1. Then, we show how we can further reduce T^* to obtain K_1, K_2, K_3, K_4, K_5 so that the desired bound holds.

We arrive at the following, which comes from (49) after some changes to match the open-loop case:

$$
\begin{aligned}
\frac{\Delta V}{T} &= \frac{V(F_T) - V(x)}{T} \\
&= \underbrace{\frac{\partial V}{\partial x}\bigg|_x f}_{\mathbf{1}} + \underbrace{\frac{1}{T}\{V(F_T) - V(x+Tf)\}}_{\mathbf{2}} + \underbrace{\frac{1}{T}\left\{V(x+Tf) - V(x) - T\,\frac{\partial V}{\partial x}\bigg|_x f\right\}}_{\mathbf{3}} \, ,
\end{aligned} \tag{55}
$$

where the second equality holds since we just added and subtracted $V(x+Tf)/T$ and $\frac{\partial V}{\partial x}\big|_x f$ to $\Delta V/T$. The following changes are then used in the proof. Since $\frac{\partial V}{\partial x}$ is locally Lipschitz and $\frac{\partial V}{\partial x}(0) = 0$, we can write for all $|x| \le \Delta_x + 1$, $|u| \le \Delta_u$, $|d_c| \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$ that $\left|\frac{\partial V}{\partial x}\right| \le L\,|x|$. Also, since f is locally Lipschitz and $f(0,0,0,0) = 0$, we can write for all $|x| \le \Delta_x + 1$, $|u| \le \Delta_u$, $|d_c| \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$:

$$
|f(x, u, d_c, d_s)| \le L(|x| + |u| + |d_c| + |d_s|) \, . \tag{56}
$$

Since $\left| \frac{\partial V}{\partial x}\big|_{x_2} \right| \le L\,|x_2|$, where $x_2 = \theta_3 F_T + (1-\theta_3)(x+Tf)$, $\theta_3 \in (0,1)$, then we have that Term **2** in

(55) can be bounded as:

$$\begin{aligned}
\frac{1}{T}\left\{V\left(F_T\right)-V\left(x+Tf\right)\right\} &\leq \frac{1}{T}\left|\left.\frac{\partial V}{\partial x}\right|_{x_2}\right|\left|F_T-(x+Tf)\right| \\
&= \frac{1}{T}L\left|x_2\right|\left|\int_0^T f(x(\tau),u,d_c(\tau),d_s)d\tau-\int_0^T f(x,u,d_c,d_s)d\tau\right| \\
&\leq \frac{1}{T}L\left|x_2\right|\left\{L\int_0^T\left|x(\tau)-x\right|d\tau+L\int_0^T\left|d_c(\tau)-d_c\right|d\tau\right\} \\
&\leq \frac{1}{T}L^2\left|x_2\right|\left\{D_o\int_0^T[\exp(L\tau)-1]d\tau+\left\|\dot{d}_c[0]\right\|_\infty\int_0^T\tau d\tau\right\} \\
&= L^2\left|x_2\right|\left\{D_o\frac{\exp(LT)-1-LT}{LT}+\frac{1}{2}\left\|\dot{d}_c[0]\right\|_\infty T\right\} \\
&\leq TL^2\left|x_2\right|\left[D_oK+\frac{1}{2}\left\|\dot{d}_c[0]\right\|_\infty\right] ,
\end{aligned} \tag{57}$$

for some $K\geq\frac{\exp(LT)-1-LT}{LT^2}$, $\forall T\in(0,T^*)$, where $D_o:=|x|+|u|+\|d_c[0]\|_\infty+|d_s|$. We can write

$$|x_2|\leq|x|+L\left(\int_0^T|x(\tau)-x|\,d\tau+\int_0^T|d_c(\tau)-d_c|\,d\tau\right)+T\left|f(x,u,d_c,d_s)\right| . \tag{58}$$

Using calculations similar to (46) and (48), we obtain:

$$\begin{aligned}
\int_0^T|x(\tau)-x|\,d\tau+\int_0^T|d_c(\tau)-d_c|\,d\tau &\leq \int_0^T\left(D_o(\exp(L\tau)-1)+\left\|\dot{d}_c[0]\right\|_\infty\tau\right)d\tau \\
&\leq D_o\frac{\exp(LT)-1-LT}{L}+\frac{T^2}{2}\left\|\dot{d}_c[0]\right\|_\infty \\
&\leq T^2\left[D_oK+\frac{1}{2}\left\|\dot{d}_c[0]\right\|_\infty\right] ,
\end{aligned} \tag{59}$$

and substitute (56) and (59) into (58) to obtain

$$\begin{aligned}
|x_2| &\leq |x|+LT^2\left[D_oK+\frac{1}{2}\left\|\dot{d}_c[0]\right\|_\infty\right] \\
&\leq |x|+LTD_o(TK+1)+\frac{1}{2}LT^2\left\|\dot{d}_c[0]\right\|_\infty .
\end{aligned} \tag{60}$$

Hence, there exists $\bar{K}>0$ such that for all sufficiently small T we can write:

$$|x_2|\leq(1+\bar{K})\,|x|+\bar{K}\left(|u|+\|d_c[0]\|_\infty+\left\|\dot{d}_c[0]\right\|_\infty+|d_s|\right) .$$

Since $x_1=x+\theta_1Tf$, where $\theta_1\in(0,1)$, then $|x_1-x|\leq T\left|f(x,u,d_c,d_s)\right|$. By referring to (54), Term **3** in (55) can be bounded by:

$$L\left|x_1-x\right|\left|f(x,u,d_c,d_s)\right|\leq TL^3\left(|x|+|u|+\|d_c[0]\|_\infty+|d_s|\right)^2 .$$

Direct but lengthy calculations show the existence of K_1,K_2,K_3,K_4,K_5. ∎

The proof of Theorem 3.3 is omitted, since it follows the same steps as that of Theorem 3.1. The only difference is that instead of using one-step weak consistency, we use one-step strong consistency. Corollaries 3.1 and 3.3 follow directly from Theorem 3.1 and Remark 2.2. The proofs for Corollaries 3.2 and 3.4 and Proposition 3.5 are carried out similarly as the proofs of Corollaries 3.1 and 3.3 and Proposition 3.4 respectively, by using Theorem 3.3.

5 Applications

We present now two applications of our results. First, we consider ISS with respect to non-sampled inputs. It is interesting to see that we have to use strong dissipation inequalities in this case, since the use of weak dissipation inequalities would yield a weaker conclusion. Second, we consider preservation of passivity under sampling where the inputs are assumed to be controls that are constant during the sampling intervals. In the first and second applications we apply our results on, respectively, the dynamic feedback case and open-loop case. An asymptotic stability result is stated as a special case of the ISS result (see [21]). Further applications of our results to L_p stability, integral ISS, etc. are possible and are left for later exposition.

5.1 Input-to-state stability

It was shown in [33] that if an ISS controller is emulated then the ISS property is preserved in a semiglobal practical sense for the sampled-data system. Detailed proofs were given in [33] only for the case when Euler method was used to find the approximate discrete-time model of the controller (see Remark 2.3), while the case of higher order approximation was only commented on. Below we use the main results of this paper to provide a sketch of proof for the case of emulation of dynamic ISS controllers, when any one-step strongly consistent approximation is used. Suppose that the nonlinear plant

$$\dot{x} = f(x, u, d_c) \tag{61}$$

can be rendered ISS using the dynamic feedback controller

$$\begin{aligned} \dot{z} &= g(x, z) \\ u &= u(x, z) \ , \end{aligned} \tag{62}$$

where f, g, and u are locally Lipschitz. Suppose that the dynamic feedback controller is emulated and then implemented digitally using a sample and zero order hold, where we use an approximation of the dynamic controller, so that:

$$\begin{aligned} z(k+1) &= G_T^a(x(k), z(k)) \\ u(k) &= u(x(k), z(k)) \ , \end{aligned} \tag{63}$$

Assume that the approximate discrete-time model of the dynamic controller G_T^a is one-step strongly consistent with the exact discrete-time model G_T^e (see Definition 2.5 and Lemma 2.2). Motivated by discussions in [6, 25] we introduce the state of the sampled-data system $\chi(t) := (x^T(t)\ x^T(k)\ z^T(k))^T$ for $t \in [kT, (k+1)T)$. We write (x, z) to denote the vector $(x^T\ z^T)^T$. We also assume that:

Assumption 5.1 *There exists $\gamma_g \in \mathcal{K}_\infty$ such that given any $\Delta > 0$ there exists $T^* > 0$ such that for all $|(x,z)| \leq \Delta$ and $T \in (0, T^*)$ we have:*

$$|G_T^a(x, z)| \leq \gamma_g(|(x, z)|) \ . \tag{64}$$

■

Remark 5.1 *Note that since f and g are assumed to be locally Lipschitz and zero at zero, if we let $L > 0$ be the Lipschitz constant on the set $|(x,z)| \leq 2\Delta$, then we can write that for all $|(x,z)| \leq \Delta$ and all $T \in (0, \frac{\ln(2)}{L})$ that*

$$|G_T^e(x, z)| \leq 2\,|(x, z)| \ .$$

If, in addition, a slightly stronger consistency holds in the following sense: given any $\Delta > 0$ there exist $T^* > 0$ and $\gamma_1 \in \mathcal{K}_\infty$ such that for all $|(x,z)| \leq \Delta$ and $T \in (0, T^*)$ we have:

$$|G_T^e(x, z) - G_T^a(x, z)| \leq \gamma_1(|(x, z)|) \ ,$$

then Assumption 5.1 holds (just apply the triangular inequality). This stronger form of consistency is known to hold for a large class of Runge-Kutta methods (see for instance Theorem 4.6.7 in [32]). ■

Remark 5.2 *Since f and u are locally Lipschitz and zero at zero, and Assumption 5.1 holds, the following is true: there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that given any strictly positive numbers Δ_1, Δ_2, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and $t_\circ \geq 0$ the solutions of the sampled-data system (61), (63) satisfy:*

$$|\chi(t)| \leq \gamma_1(|\chi(t_\circ)|) + \gamma_2(\|d_c\|_\infty), \ \forall t \in [t_\circ, t_\circ + T] \ ,$$

whenever $|\chi(t_\circ)| \leq \Delta_1$ and $\|d_c\|_\infty \leq \Delta_2$. This conditions is referred to as uniform boundedness over T (UBT) in [25]. ■

We can state and prove the following result using Theorem 3.3:

Corollary 5.1 *If the continuous time system (61), (62) with f, g and u locally Lipschitz is ISS, then given any approximate discrete-time model G_T^a of the dynamic controller which satisfies Assumption 5.1 and is one-step strongly consistent with the exact discrete-time model of the dynamic controller G_T^e, there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ such that given any triple of strictly positive real numbers $(\Delta_\chi, \Delta_{d_c}, \nu)$, there exists $T^* > 0$ such that $\forall T \in (0, T^*)$, $|\chi(t_0)| \leq \Delta_\chi$, $\|d_c\|_\infty \leq \Delta_{d_c}$, the solutions of the sampled-data system (61), (63) satisfy:*

$$|\chi(t)| \leq \beta(|\chi(t_\circ)|, t - t_\circ) + \gamma(\|d_c\|_\infty) + \nu, \qquad \forall t \geq t_\circ \geq 0 \ . \tag{65}$$

■

Sketch of proof of Corollary 5.1: Since the continuous time system (61), (62) is ISS, it implies (see Theorem 1 in [31]) that the system (61), (62) is (V, w)-dissipative, where V is smooth and there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty, \gamma_1 \in \mathcal{K}$ such that

$$\begin{aligned} \alpha_1(|(x,z)|) \leq V(x,z) &\leq \alpha_2(|(x,z)|) \\ w(x,z,d_c) &= -\alpha_3(|(x,z)|) + \gamma_1(|d_c|) \\ \left|\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial z}\right)\right| &\leq \alpha_4(|(x,z)|) \ . \end{aligned} \tag{66}$$

Then it follows from Theorem 3.3, that given any G_T^a which is one-step strongly consistent with G_T^e, and given any $(\Delta_1, \Delta_2, \Delta_3, \nu_1)$ there exists $T_1^* > 0$ such that for all $T \in (0, T_1^*)$ and $|x| \leq \Delta_1, |z| \leq \Delta_2, \|d_c[0]\|_\infty \leq \Delta_3$, the discrete-time model of (61), (63) satisfies:

$$\begin{aligned} \frac{\Delta V}{T} &\leq \frac{1}{T}\int_0^T [-\alpha_3(|(x,z)|) + \gamma_1(|d_c(\tau)|)]\, d\tau + \nu_1 \\ &\leq -\alpha_3(|(x,z)|) + \gamma_1(\|d_c[0]\|_\infty) + \nu_1 \ . \end{aligned} \tag{67}$$

This implies (see Lemma 4 of [23]) that there exists $\beta_2 \in \mathcal{KL}, \gamma_2 \in \mathcal{K}$ such that if all the assumptions on G_T^a hold and given any $(\Delta_4, \Delta_5, \Delta_6, \nu_2)$ there exists $T_2^* > 0$ such that for all $T \in (0, T_2^*)$ and $|x(0)| \leq \Delta_4$, $|z(0)| \leq \Delta_5$, $\|d_c\|_\infty \leq \Delta_6$, the discrete-time model of (61), (63) satisfies:

$$|(x(k), z(k))| \leq \beta_2(|(x(0), z(0))|, kT) + \gamma_2(\|d_c\|_\infty) + \nu_2, \ \ \forall k \geq 0 \ . \tag{68}$$

From Lemma 2 in [25] it follows that there exist $\beta_3 \in \mathcal{KL}$ and $\gamma_3 \in \mathcal{K}$ such that given any strictly positive $(\Delta_7, \Delta_8, \nu_3)$ there exists $T_3^* > 0$ such that for all $T \in (0, T_3^*)$ and $|\chi(0)| \leq \Delta_7$, $\|d_c\|_\infty \leq \Delta_8$, the solutions of the sampled-data system satisfy:

$$|\chi(k)| \leq \beta_3(|\chi(0)|, kT) + \gamma_3(\|d_c\|_\infty) + \nu_3, \ \ \forall k \geq 0 \ . \tag{69}$$

Finally, from Assumption 5.1 it follows that solutions of the sampled-data system are UBT (see Remark 5.2 and Definition 2 in [25]) and then using results in Section 3 in [25], there exists $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ such that given any G_T^a which is one-step strongly consistent with G_T^a and any $(\Delta_\chi, \Delta_{d_c}, \nu)$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and $|\chi(t_\circ)| \leq \Delta_\chi$, $\|d_c\|_\infty \leq \Delta_{d_c}$, the solutions of (61), (63) satisfy:

$$|\chi(t)| \ \ \leq \ \ \beta(|\chi(t_\circ)|, t - t_\circ) + \gamma(\|d_c\|_\infty) + \nu, \ \ \ \forall t \geq t_\circ \geq 0 \ , \tag{70}$$

which completes the proof. ■

It is important to note that we can not use Theorem 3.1 instead of Theorem 3.3 to prove semiglobal practical ISS of the sampled-data system in Corollary 5.1. Indeed, Theorem 3.1 requires us to impose an additional condition on disturbances to be uniformly Lipschitz and hence the bound (70) would hold for a smaller set of disturbances (bounded and uniformly Lipschitz) than measurable bounded disturbances for which the ISS property is defined.

A direct consequence of the ISS result is a result on semiglobal practical asymptotic stability, which is stated in the following corollary. Note that since we will consider the systems which has no external input or disturbances, by Remark 2.2, one step weak and strong consistency are the same.

Corollary 5.2 *If the origin of the continuous time system*

$$\begin{aligned} \dot{x} &= f(x, u(x, z)) \\ \dot{z} &= g(x, z) \end{aligned} \tag{71}$$

is GAS, then given any approximate discrete-time model G^a_T of the dynamic controller which satisfies Assumption 5.1 and is one-step weakly/strongly consistent with the exact discrete-time model of the dynamic controller G^e_T, there exists $\beta \in \mathcal{KL}$ such that given any pair of strictly positive numbers (Δ_χ, ν), there exists $T^* > 0$ such that $\forall T \in (0, T^*)$, $|\chi(t_\circ)| \leq \Delta_\chi$, the solutions of the sampled-data system satisfy:

$$|\chi(t)| \leq \beta(|\chi(t_\circ)|, t - t_\circ) + \nu, \qquad \forall t \geq t_\circ \geq 0. \tag{72}$$

■

5.2 Passivity

Consider the continuous time system with outputs

$$\dot{x} = f(x, u), \quad y = h(x, u), \tag{73}$$

where $x \in \mathbb{R}^n, y, u \in \mathbb{R}^m$ and assume that the system is passive, that is (V, w)-dissipative, where $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $w = y^T u$. We can apply either results of Theorem 3.1 or 3.3 since u is a piecewise constant input, to obtain that the discrete-time model satisfies: for any $(\Delta_x, \Delta_u, \nu)$ there exists $T^* > 0$ such that $\forall T \in (0, T^*)$, $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ we have:

$$\frac{\Delta V}{T} \leq y^T u + \nu. \tag{74}$$

In ISS applications, adding ν in the dissipation inequality deteriorated the property, but the deterioration was gradual. However, in (74) ν acts as an infinite energy storage (finite power source) and hence it contradicts the definition of a passive system as one that can not generate power internally. As a result, conditions which guarantee that ν in (74) can be set to zero are very important. These conditions are spelled out in the next corollary:

Corollary 5.3 *Suppose that the system (73) is strictly input and state passive in the following sense: the dissipation rate can be taken as $w(x, y, u) = y^T u - \psi_1(x) - \psi_2(u)$, where ψ_1 and ψ_2 are positive definite functions that are locally quadratic. Then given any pair of strictly positive numbers (Δ_x, Δ_u) there exists $T^* > 0$ such that for all $T \in (0, T^*)$, $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ we have:*

$$\frac{\Delta V}{T} \leq y^T u - \frac{1}{2}\psi_1(x) - \frac{1}{2}\psi_2(u) \tag{75}$$

■

Sketch of proof of Corollary 5.3: Using Proposition 3.4, we see that given any (Δ_x, Δ_u) there exists $T_1^* > 0$ such that $\forall T \in (0, T_1^*)$, $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ we have:

$$\frac{\Delta V}{T} \leq y^T u - \psi_1(x) - \psi_2(u) + TK_1|x|^2 + TK_2|u|^2 \ ,$$

and from properties of ψ_1 and ψ_2, it follows that there exists $T^* \leq T_1^*$ such that $\forall T \in (0, T^*)$, $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ we have that (75) holds. ■

We emphasize that the above approach can be used for more general properties than passivity to cancel ν in the dissipation inequality for the discrete-time system.

6 Conclusions

We have presented general results on preservation of general dissipation inequalities under sampling in the emulation controller design. We have covered the closed-loop and open-loop cases. These results generalize all available results on emulation design in the sampled-data literature that we are aware of (see [6, 21, 26, 28, 29, 33, 35]) and provide a unified framework for digital controller design using the emulation method for general nonlinear systems.

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A Appendix

Proof of Lemma 2.1: Let $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$ be given. Using $(\Delta_x, R_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$, where $R_z = \Delta_z + 1$, let the second condition of the lemma generate $T^* > 0$, $M > 0$, $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{K}_\infty$. Let $T_1^* := \min\{T^*, 1/M\}$. It follows from condition 2a of the lemma that, for each $|x| \le \Delta_x$, $|z| \le \Delta_z$, $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$ and all $t \in [0, T]$, where $T \in (0, T_1^*)$, the solution $z(t)$ of

$$\dot{z}(t) = g(x, z(t), d_c(t), d_s) \ , \qquad\qquad z(0) = z \tag{76}$$

satisfies $|z(t)| \le R_z$ and $|z(t) - z| \le Mt$. Condition 3 guarantees that $|d_c(t)| \le \Delta_{d_c}$ and $|d_c(t) - d_c(0)| \le \Delta_{\dot{d}_c} t$, for all $t \in [0, T_1^*]$. It then follows from condition 2b of the lemma that, for all $|z| \le \Delta_z$, $|x| \le \Delta_x$, $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \le \Delta_{\dot{d}_c}$, $|d_s| \le \Delta_{d_s}$ and all $T \in (0, T_1^*)$,

$$\begin{aligned} \left| \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)] d\tau \right| &\le \int_0^T \rho_1(|z(\tau) - z|) d\tau + \int_0^T \rho_2(|d_c(\tau) - d_c|) d\tau \\ &\le T\rho_1(MT) + T\rho_2(\Delta_{\dot{d}_c} T) \ . \end{aligned} \tag{77}$$

Hence, we can write

$$\left| \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)] d\tau \right| \le T\tilde{\rho}(T) \ , \tag{78}$$

where $\tilde{\rho}(s) := \rho_1(Ms) + \rho_2(\Delta_{\dot{d}_c} s)$ is a $\mathcal{K}_\infty$ function since ρ_1 and ρ_2 are $\mathcal{K}_\infty$. Since

$$G_T^e(x, z, d_c[0], d_s) = z + Tg(x, z, d_c, d_s) + \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)] d\tau \ , \tag{79}$$

the result follows from (78) and the first condition of the lemma, which implies the existence of $\tilde{\rho}_1 \in \mathcal{K}_\infty$, such that

$$\left| G_T^a - G_T^{Euler} \right| \le T\tilde{\rho}_1(T) \ .$$

Finally, by letting $\rho = \tilde{\rho} + \tilde{\rho}_1$ we prove that G_T^a is one-step weakly consistent with G_T^e. ■

Proof of Lemma 2.2: Let $(\Delta_x, \Delta_z, \Delta_{d_s})$ be given. Using the triple $(\Delta_x, R_z, \Delta_{d_s})$, where $R_z = \Delta_z + 1$, let the second condition of the lemma generate $T^* > 0$, $M > 0$, $\rho_1 \in \mathcal{K}_\infty$. Let $T_1^* := \min\{T^*, 1/M\}$. It follows from condition 2a of the lemma that, for each $|x| \le \Delta_x$, $|z| \le \Delta_z$, $|d_s| \le \Delta_{d_s}$ and all $t \in [0, T]$, where $T \in (0, T_1^*)$, the solution $z(t)$ of

$$\dot{z}(t) = g(x, z(t), d_s) \ , \qquad\qquad z(0) = z \tag{80}$$

satisfies $|z(t)| \le R_z$ and $|z(t) - z| \le Mt$. It then follows from condition 2b of the lemma that, for all $|z| \le \Delta_z$, $|x| \le \Delta_x$, $|d_s| \le \Delta_{d_s}$ and all $T \in (0, T_1^*)$,

$$\left| \int_0^T [g(x, z(\tau), d_s) - g(x, z, d_s)] d\tau \right| \le \int_0^T \rho_1(|z(\tau) - z|) d\tau \le T\rho_1(MT) \ , \tag{81}$$

and we can write that

$$\left| \int_0^T [g(x, z(\tau), d_s) - g(x, z, d_s)] d\tau \right| \le T\tilde{\rho}(T) \ , \tag{82}$$

where $\tilde{\rho}(s) := \rho_1(Ms)$. Since

$$G_T^e(x, z, d_s) = z + Tg(x, z, d_s) + \int_0^T [g(x, z(\tau), d_s) - g(x, z, d_s)] d\tau \ , \tag{83}$$

the result follows from (82) and the first condition of the lemma, which implies the existence of $\tilde{\rho}_1 \in \mathcal{K}_\infty$, such that

$$\left| G_T^a - G_T^{Euler} \right| \le T\tilde{\rho}_1(T) \ .$$

Finally, by letting $\rho = \tilde{\rho} + \tilde{\rho}_1$ we prove that G_T^a is one-step strongly consistent with G_T^e. ■