Further results on stability of networked control systems: a Lyapunov approach

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***Abstract*— Simple Lyapunov proofs are given for an improved (relative to previous results that have appeared in the literature) bound on the maximum allowable transfer interval to guarantee global asymptotic or exponential stability in networked control systems and also for semiglobal practical asymptotic stability with respect to the length of the maximum allowable transfer interval. We apply our results to emulation of nonlinear controllers in sampled-data systems.**

I. Introduction

A networked control system (NCS) is composed of multiple feedback control loops that share a serial communication channel. This architecture promotes ease of maintenance, greater flexibility, and low cost, weight and volume. On the other hand, if the communication is substantially delayed or infrequent, the architecture can degrade the overall system performance significantly. Results on the analysis of an NCS include [12], [13], [14], [7], [8]. In an NCS, the delay and frequency of communication between sensors and actuators in a given loop is determined by a combination of the channel's limitations and the transmission protocol used. Various protocols have been proposed in the literature, including the "round robin" (RR) and "try-once-discard" (TOD) protocols discussed in [12] and [13]. We note that sampled-data systems are a special case of NCS since in this case all sensor and control signals are transmitted at each transmission instant.

When the individual loops in an NCS are designed assuming perfect communication, the stability of the NCS is largely determined by the transmission protocol used and by the so-called "maximum allowable transfer interval" (MATI), i.e., the maximum allowable time between any two transmissions in the network. When specialized to sampled-data systems, this controller design approach is called "emulation" (see [5], [2]). Following pioneering work of Walsh and co-authors [13], [12], we consider the problem of characterizing the length of the MATI for a given protocol to ensure uniform global asymptotic or exponential stability. We also demonstrate that our results apply in a straightforward manner to emulation of nonlinear controllers for digital implementation. This appears to be the first result in the literature that provides an explicit formula for the computation of MATI in the context of emulation of nonlinear controllers.

In [7] the authors were able to improve on the initial MATI bounds given in [13], [12] by efficiently summarizing the properties of protocols through Lyapunov functions and characterizing the effect of transmission errors through $\mathcal{L}_p$ gains. They established uniform asymptotic or exponential stability and input-output stability when the MATI$\in [0,\tau_{MATI}]$ with

$$\tau_{MATI} \le \frac{1}{L}\ln\left(1+\frac{1-\lambda}{\frac{\gamma}{L}+\lambda}\right) \tag{1}$$

where $\lambda\in[0,1)$ characterized the contraction of the protocol's Lyapunov function at transmission times while $L>0$ described its expansion between transmission times, and $\gamma>0$ captured the effect of the error signals on the behavior of the ideal system through an $\mathcal{L}_p$ gain.[1]

In this paper, we will give a simple Lyapunov proof of an improved (larger) MATI bound, expressed in terms λ, L and γ corresponding to the case of $\mathcal{L}_2$ gains. (A similar approach can be taken for the general $\mathcal{L}_p$ case.) In particular, we establish uniform asymptotic or exponential stability when

$$\tau_{MATI} \le \begin{cases} \dfrac{1}{Lr}\arctan\left(\dfrac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma>L \\ \dfrac{1}{L}\dfrac{1-\lambda}{1+\lambda} & \gamma=L \\ \dfrac{1}{Lr}\text{arctanh}\left(\dfrac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma<L\ , \end{cases} \tag{2}$$

and note that in the first and last expressions we use respectively the trigonometric and hyperbolic functions, where

$$r := \sqrt{\left|\left(\frac{\gamma}{L}\right)^2-1\right|}\ . \tag{3}$$

It is not obvious, except for the case $\gamma=L$, that (2) provides a larger bound than (1). We will establish that it is an improvement by first noting that the bound in (1) is the value τ_1 satisfying

$$\dot\phi_1 = -L\phi_1-\gamma\ ,\quad \phi_1(0)=1\ ,\quad \phi_1(\tau_1)=\lambda \tag{4}$$

whereas the bound in (2) is the value τ_2 satisfying

$$\dot\phi_2 = -2L\phi_2-\gamma(\phi_2^2+1)\ ,\quad \phi_2(0)=\lambda^{-1}\ ,\quad \phi_2(\tau_2)=\lambda \tag{5}$$

and that necessarily $\tau_2>\tau_1$ for all $L>0$, $\gamma>0$ and $\lambda\in(0,1)$. The above equations can be obtained from appropriate Lyapunov arguments that are presented later in our proofs. The difference in the bounds for the batch reactor system considered in [7] is reported in Table I. (For more discussion, see Remark 1.) The improvement is on the order of 10%. (The values $L=15.73$, $\lambda=\sqrt{1/2}$, $\gamma=15.9222$ for the

[1]For convenience, in a minor departure from the description in [7], we use an inequality rather than a strict inequality in (1) but take γ to be any number that is *strictly greater than* the $\mathcal{L}_p$ gain used in [7].

TABLE I

BOUNDS COMPARISON FOR TOD/RR PROTOCOLS: BATCH REACTOR IN [7]

Definition	TOD	RR
theoretical bound on MATI τ_{MATI} computed in [7]	0.01	0.0082
theoretical bound on MATI τ_{MATI} via Theorem 1	0.0108	0.009
percentage of increment using the new τ_{MATI}	8.4 %	9.76%

TOD protocol and $\gamma = 21.5275$ for the RR protocol are reported in [7].) For some systems, the improvement could be over 50%. See the figures below, which address separately the case $\gamma < L$ and $\gamma \geq L$.

We emphasize that the contribution of this paper is not only a (modest) improvement in the MATI bound relative to [7] but also a simple Lyapunov proof. At the same time, we give a direct Lyapunov proof of a result in [8] which states that if an NCS is asymptotically stable with perfect communication then it is semiglobally practically asymptotically stable with respect to τ_{MATI}. This proof also generalizes easily to the case, addressed in [8], where there are exogenous inputs to which the system with perfect communication is input-to-state stable.

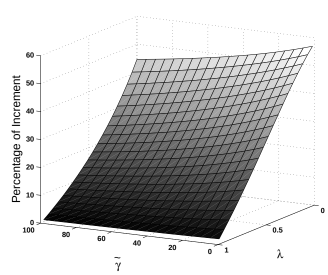

Fig. 1. Percentage improvement in the MATI bound using Theorem 1 compared to using [7, Theorem 4], $\tilde{\gamma} = \gamma/L \geq 1$.

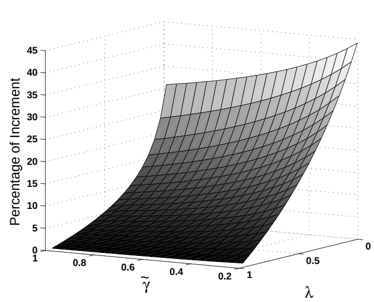

Fig. 2. Percentage improvement in the MATI bound using Theorem 1, compared to using [7, Theorem 4], $\tilde{\gamma} = \gamma/L < 1$.

II. NOTATION AND DEFINITIONS

We denote by $\mathbb{R}$ and $\mathbb{Z}$ the sets of real and integer numbers, respectively. Also $\mathbb{R}_{\geq 0} = [0, +\infty)$, and $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. The Euclidean norm is denoted $|\cdot|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t\to\infty} \beta(s,t) = 0$ for each $s \geq 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KLL}$ if, for each $r \geq 0$, $\beta(\cdot, r, \cdot)$ and $\beta(\cdot, \cdot, r)$ belong to class $\mathcal{KL}$.

We recall definitions given in [3] that we will use to develop a hybrid model of a NCS. The reader should refer to [3] for the motivation and more details on these definitions.

Definition 1: A *compact hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ given by :

$$\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j\ )$$

where $J \in \mathbb{Z}_{\geq 0}$ and $0 = t_0 \leq t_1 \cdots \leq t_J$. A *hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that, for each $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, \dots, J\})$ is a compact hybrid time domain. ■

Definition 2: A *hybrid trajectory* is a pair (dom ξ, ξ) consisting of hybrid time domain dom ξ and a function ξ defined on dom ξ that is continuously differentiable in t on (dom ξ) $\cap$ $(\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{Z}_{\geq 0}$. ■

Definition 3: For the hybrid system $\mathcal{H}$ given by the open state space $O \subset \mathbb{R}^n$ and the data (F, G, C, D) where $F : O \to \mathbb{R}^n$ is continuous, $G : O \to O$ is locally bounded, and C and D are subsets of O, a hybrid trajectory ξ : dom $\xi \to O$ is a *solution to* $\mathcal{H}$ if

1) For all $j \in \mathbb{Z}_{\geq 0}$ and for almost all $t \in I_j :=$ dom $\xi \cap (\mathbb{R}_{\geq 0} \times \{j\})$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j))$.
2) For all $(t, j) \in$ dom ξ such that $(t, j+1) \in$ dom ξ, we have $\xi(t, j) \in D$ and $\xi(t, j+1) = G(\xi(t, j))$.

Hence, the hybrid system models that we consider are of the form:

$$\begin{array}{rcll} \dot{\xi}(t,j) & = & F(\xi(t,j)) & \xi(t,j) \in C \\ \xi(t_{j+1}, j+1) & = & G(\xi(t_{j+1}, j)) & \xi(t_{j+1}, j) \in D\ . \end{array}$$

We sometimes omit the time arguments and write:

$$\begin{array}{rcll} \dot{\xi} & = & F(\xi) & \xi \in C \\ \xi^+ & = & G(\xi) & \xi \in D\ , \end{array} \tag{6}$$

where we denoted $\xi(t_{j+1}, j+1)$ as ξ^+ in the last equation. We also note that typically $C \cap D \neq \emptyset$ and, in this case, if $\xi(0,0) \in C \cap D$ we have that either a jump or flow is possible, the latter only if flowing keeps the state in C. Hence, the hybrid model we consider may have non-unique solutions.

III. Problem statement

In this section, we formally state the problem that we consider and summarize the model of NCS from [7]. In the next section, we will embed this model within the hybrid framework of [3] by representing it in the form (6) that is useful in our proofs.

We pursue the controller design technique proposed in [12], [13] and further developed in [7], [8]. The plant model is given by equations:

$$\begin{aligned} \dot{x}_P &= f_P(x_P, u) \\ y &= g_P(x_P) \ . \end{aligned} \tag{7}$$

The first step in controller design is to ignore the network and design a stabilizing controller for the plant:

$$\begin{aligned} \dot{x}_C &= f_C(x_C, y) \\ u &= g_C(x_C) \ . \end{aligned} \tag{8}$$

The second step in the design is to implement the above controller over the network and determine the value of a network parameter (MATI) that guarantees that the same controller implemented over the network will yield stability. Note that this approach is very similar to the emulation approach to controller design of sampled-data systems.

Now we describe the model of NCS. Let the sequence $t_j, j \in \mathbb{Z}_{\geq 0}$ of monotonically increasing transmission times satisfy $\epsilon \leq t_{j+1} - t_j \leq \tau$ for all $j \in \mathbb{Z}_{\geq 0}$ and some fixed $\epsilon, \tau > 0$. Note that ϵ is arbitrary and it is used to prevent Zeno solutions in the model given below. At each t_j, the protocol gives access to the network to one of the nodes $i \in \{1, 2, \ldots, \ell\}$. We refer to τ as the *maximum allowable transmission interval* (MATI). Using the plant (7) and controller (8), we introduce the nonlinear NCS of the following form

$$\begin{aligned} \dot{x}_P &= f_P(x_P, \hat{u}) & t \in [t_{j-1}, t_j] \\ y &= g_P(x_P) & \\ \dot{x}_C &= f_C(x_C, \hat{y}) & t \in [t_{j-1}, t_j] \\ u &= g_C(x_C) & \\ \dot{\hat{y}} &= \hat{f}_P(x_P, x_C, \hat{y}, \hat{u}) & t \in [t_{j-1}, t_j] \\ \dot{\hat{u}} &= \hat{f}_C(x_P, x_C, \hat{y}, \hat{u}) & t \in [t_{j-1}, t_j] \\ \hat{y}(t_j^+) &= y(t_j) + h_y(i, e(t_j)) & \\ \hat{u}(t_j^+) &= u(t_j) + h_u(i, e(t_j)) & \end{aligned} \tag{9}$$

where x_P and x_C are respectively states of the plant and the controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of most recently transmitted plant and controller output values via the network; e is the network induced error defined as

$$e(t) := \begin{pmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{pmatrix} = \begin{pmatrix} e_y \\ e_u \end{pmatrix} \ .$$

We often use the choice $\hat{f}_P = 0$ and $\hat{f}_C = 0$ which means that the networked version of the output $\hat{y}$ and control $\hat{u}$ are kept constant during the transmission intervals (i.e., the network nodes operate in a similar manner to a zero order hold).

Note that if NCS has ℓ links, then the error vector can be partitioned as follows $e = [e_1^T \ e_2^T \ \ldots \ e_\ell^T]^T$. The functions h_u and h_y are typically such that, if the ith link gets access to the network at some transmission time t_j we have that the corresponding part of the error vector has a jump. For several protocols, such as the round robin and Try-Once-Discard protocols (see [7]), we typically assume that e_i is reset to zero at time t_j^+, that is $e_i(t_j^+) = 0$. However, we emphasize that this assumption is not needed in general. This allows us to write the models h_u, h_y for protocols commonly found in the literature (see [7], [8] for more details).

We combine the controller and plant states into a vector $x := (x_P, x_C)$ and using the error vector defined earlier $e = (e_y, e_u)$, we can rewrite (9) as a system with jumps that is more amenable for analysis:

$$\dot{x} = f(x, e) \qquad \forall t \in [t_{j-1}, t_j] \tag{10}$$

$$\dot{e} = g(x, e) \qquad \forall t \in [t_{j-1}, t_j] \tag{11}$$

$$e(t_j^+) = h(j, e(t_j)) \ , \tag{12}$$

where $\epsilon \leq t_{j+1} - t_j \leq \tau$ for all $j \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$ and f, g and h are obtained using straightforward calculations from (9), see [7]. In order to write (11), we assumed that functions g_P and g_C in (9) are continuously differentiable (this assumption can be relaxed). We refer to (12) as a protocol. The protocol determines the algorithm by which access to the network is assigned to different nodes in the system. For more details on protocol modelling in this manner, see [7], [8]. Note that

$$\dot{x} = f(x, 0) \tag{13}$$

represents the closed loop system (7), (8) without the network. We consider the following problem:

> *Problem: Suppose that the controller (8) was designed for the plant (7) so that the closed loop system (7), (8) without network (equivalently, the system (13)) is globally asymptotically stable. Determine the value of τ_{MATI} so that for any $\epsilon \in (0, \tau_{MATI}]$ and all $\tau \in [\epsilon, \tau_{MATI}]$, we have that the NCS described by (10), (11), (12) is stable in an appropriate sense.*

Moreover, we show that the value of τ_{MATI} computed in [7] and given by (1) is always smaller than the value of τ_{MATI} given by (2). Hence, our new result provides a less conservative analytical bound for τ_{MATI} that is very important in implementing the controller (8) via the network in the manner described by (9). Indeed, this bound shows that stabilization is possible with lower bandwidth of the communication channel (since τ_{MATI} is inversely proportional to the channel bandwidth).

IV. Main results

In order to streamline the proofs, we map the model (10), (11), (12) of an NCS that was introduced in the previous section into a hybrid system of the type (6) discussed in the preliminaries section. In particular, we consider systems of the form

$$\left.\begin{array}{rcl}\dot{x} &=& f(x,e)\\ \dot{e} &=& g(x,e)\\ \dot{\tau} &=& 1\\ \dot{\kappa} &=& 0\end{array}\right\} \qquad \tau\in[0,\tau_{MATI}]$$
$$\left.\begin{array}{rcl}x^+ &=& x\\ e^+ &=& h(\kappa,e)\\ \tau^+ &=& 0\\ \kappa^+ &=& \kappa+1\end{array}\right\} \qquad \tau\in[\varepsilon,\infty) \tag{14}$$

where $\varepsilon > 0$ can be arbitrarily small, $\tau_{MATI} \geq \varepsilon$ and $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $\tau \in \mathbb{R}_{\geq 0}$ and $\kappa \in \mathbb{Z}_{\geq 0}$.

In what follows we will consider the behavior of all possible solutions to the hybrid system (14) subject to $\tau(0,0) \geq 0$. Since the derivative of τ is positive (equal to one) and when τ jumps it is reset to zero, it follows that τ will never take on negative values. According to the definition of solution for a hybrid system, the error vector e can jump, following the rules of the protocol, after ϵ seconds have elapsed from the previous jump. This is because at the previous jump τ was reset to zero, when the system is not jumping we have $\dot{\tau} = 1$, and the D set, which enables jumps, is the set $\{(x,e,\tau,\kappa) : \tau \in [\varepsilon,\infty)\}$. On the other hand, if τ_{MATI} seconds have elapsed from the previous jump then the error vector e must jump. This is because the C set is $\{(x,e,\tau,\kappa) : \tau \in [0,\tau_{MATI}]\}$, and thus flows are not allowed after τ reaches τ_{MATI}. In this way, the time-invariant hybrid system (14) covers all of the possible behaviors described by (10), (11), (12).

Standing Assumption 1: f and g are continuous and h is locally bounded. ■

We will give an upper bound on τ_{MATI} to guarantee asymptotic or exponential stability.

Definition 4: For the hybrid system (14), the set $\{(x,e,\tau,\kappa) : x = 0\ ,\ e = 0\}$ is *uniformly globally asymptotically stable* if there exists $\beta \in \mathcal{KLL}$ such that, for each initial condition $\tau(0,0) \in \mathbb{R}_{\geq 0}$, $\kappa(0,0) \in \mathbb{Z}_{\geq 0}$, $x(0,0) \in \mathbb{R}^{n_x}$, $e(0,0) \in \mathbb{R}^{n_e}$, and each corresponding solution,

$$\left|\left[\begin{array}{c} x(t,j)\\ e(t,j)\end{array}\right]\right| \leq \beta\left(\left|\left[\begin{array}{c} x(0,0)\\ e(0,0)\end{array}\right]\right|, t, \varepsilon j\right) \tag{15}$$

for all (t,j) in the solution's domain with $\varepsilon > 0$ (it avoids Zeno solutions). The set is *uniformly globally exponentially stable* if β can be taken to have the form $\beta(s,t,k) = Ms\exp(-\lambda(t+k))$ for some $M > 0$ and $\lambda > 0$. ■

Definition 5: For the hybrid system (14) the set $\{(x,e,\tau,\kappa) : x = 0\ ,\ e = 0\}$ is *uniformly semiglobally practically asymptotically stable* (USPAS) with respect to τ_{MATI} if there exists $\beta \in \mathcal{KLL}$ and for any pair of positive real numbers (δ,Δ) there exists $\tau_{MATI} > 0$ such that for each $0 < \varepsilon \leq \tau_{MATI}$, each initial condition $\tau(0,0) \in \mathbb{R}_{\geq 0}$, $\kappa(0,0) \in \mathbb{Z}_{\geq 0}$, $|x(0,0)| \leq \Delta$, $|e(0,0)| \leq \Delta$ and each corresponding solution we have

$$\left|\left[\begin{array}{c} x(t,j)\\ e(t,j)\end{array}\right]\right| \leq \max\left\{\beta\left(\left|\left[\begin{array}{c} x(0,0)\\ e(0,0)\end{array}\right]\right|, t, \varepsilon j\right)\ ,\ \delta\right\}, \tag{16}$$

for all (t,j) in the solution's domain. ■

In order to guarantee asymptotic or exponential stability, we make the following assumption:

Assumption 1: There exist a function $W : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz, positive definite, radially unbounded function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, real numbers $\lambda \in (0,1)$, $L \geq 0$, $\gamma > 0$, $\underline{\alpha}_W, \overline{\alpha}_W \in \mathcal{K}_\infty$ and a continuous, positive definite function ϱ such that, $\forall \kappa \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$

$$\underline{\alpha}_W(|e|) \leq W(\kappa,e) \leq \overline{\alpha}_W(|e|) \tag{17}$$

$$W(\kappa+1,h(\kappa,e)) \leq \lambda W(\kappa,e) \tag{18}$$

and for all $\kappa \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$,

$$\left\langle \frac{\partial W(\kappa,e)}{\partial e}, g(x,e)\right\rangle \leq LW(\kappa,e) + H(x)\ ; \tag{19}$$

moreover, for all $e \in \mathbb{R}^{n_e}$, all $\kappa \in \mathbb{Z}_{\geq 0}$, and almost all $x \in \mathbb{R}^{n_x}$,

$$\begin{array}{rcl}\langle \nabla V(x), f(x,e)\rangle &\leq& -\varrho(|x|) - \varrho(W(\kappa,e)) - H^2(x)\\ && +\gamma^2 W^2(\kappa,e)\ .\end{array} \tag{20}$$

■

Remark 1: This assumption is essentially the same as the main assumption of [7, Theorem 4] when considering $\mathcal{L}_2$ gains. The condition on $\dot{x} = f(x,e)$ is expressed here in terms of a Lyapunov function that establishes an $\mathcal{L}_2$ gain γ from W to H whereas in [7, Theorem 4] it is directly in terms of the $\mathcal{L}_2$ gain γ. However, in practice the $\mathcal{L}_2$ gain is often verified with a Lyapunov function V that satisfies (20). For example, the results in the first row of Table I, which come from [7], use values (λ, L, γ) that admit functions W, H and a positive definite, quadratic function V that satisfy (17)-(20) with $\varrho(s) = \varepsilon s^2$ for some $\varepsilon > 0$ sufficiently small. ■

Theorem 1: Under Assumption 1, if τ_{MATI} in (14) satisfies the bound (2) and $0 < \varepsilon \leq \tau_{MATI}$ then, for the system (14), the set $\{(x,e,\tau,\kappa) : x = 0, e = 0\}$ is uniformly globally asymptotically stable. If, in addition, there exist strictly positive real numbers $\underline{\alpha}_W, \overline{\alpha}_W$, a_1, a_2, and a_3 such that $\underline{\alpha}_W|e| \leq W(\kappa,e) \leq \overline{\alpha}_W|e|$, $a_1|x|^2 \leq V(x) \leq a_2|x|^2$, and $\varrho(s) \geq a_3 s^2$ then this set is uniformly globally exponentially stable.

In the proof of Theorem 1, Sec. VI-A it is shown that $V(x) + \gamma W^2(\kappa,e)$ is a strict Lyapunov function for the discrete-time system that is generated as the composition of flows and jumps in the system (14).

Theorem 2: Consider the hybrid NCS (14). Suppose that the following conditions hold.

1) There exist a function $W : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a continuous, positive definite function ϱ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_W, \overline{\alpha}_W, \alpha$ such that, $\forall \kappa \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$,

$$\underline{\alpha}_W(|e|) \leq W(\kappa,e) \leq \overline{\alpha}_W(|e|) \tag{21}$$

$$W(\kappa+1,h(\kappa,e)) \leq W(\kappa,e) - \varrho(e) \tag{22}$$

and for all $\kappa \in \mathbb{Z}_{\geq 0}$ and almost all $e \in \mathbb{R}^{n_e}$,

$$\left|\frac{\partial W(\kappa, e)}{\partial e}\right| \leq \alpha(|e|) \ . \qquad (23)$$

2) The origin of $\dot{x} = f(x, 0)$ is globally asymptotically stable.

Then, for (14), the set $\{(x, e, \tau, \kappa) : x = 0, e = 0\}$ is US-PAS with respect to τ_{MATI}. ■

V. Emulation in sampled-data systems

In this section, we specialize Theorem 1 to the case of emulation of continuous-time controllers. We believe that the explicit formula which we provide is the first one reported in the literature in this context. For space reasons, we do not re-state the result of Theorem 2.

First, we note that sampled-data systems are a special case of NCS (see (14)) and they can be described by the following model:

$$\left.\begin{array}{rcl}\dot{x} & = & f(x,e) \\ \dot{e} & = & g(x,e) \\ \dot{\tau} & = & 1\end{array}\right\} \quad \tau \in [0, \tau_{MATI}] \qquad \left.\begin{array}{rcl} x^+ & = & x \\ e^+ & = & 0 \\ \tau^+ & = & 0\end{array}\right\} \quad \tau \in [\varepsilon, \infty) \qquad (24)$$

where the main difference with (14) is the simplified model of the protocol ($e^+ = 0$) and the absence of the κ equations. In other words, u and y are transmitted at each transmission instant, or equivalently, there is only one link. A straightforward consequence of this special structure is that for any function W that satisfies (17), (19) and (20), we have that it also satisfies (18) for any $\lambda \in [0, 1)$ (in particular, we can let $\lambda = 0$). Using this, a direct consequence of Theorem 1 is the following result on emulation of controllers in sampled-data systems:

Corollary 1: Suppose that (17), (19) and (20) in Assumption 1 hold. If MATI in (24) satisfies the bound (2) with $\lambda = 0$ and $0 < \varepsilon \leq \tau_{MATI}$ then, for the system (24), the set $\{(x, e, \tau) : x = 0, e = 0\}$ is uniformly globally asymptotically stable. If, in addition, there exist strictly positive real numbers $\underline{\alpha}_W, \overline{\alpha}_W$, a_1, a_2, and a_3 such that $\underline{\alpha}_W|e| \leq W(e) \leq \overline{\alpha}_W|e|$, $a_1|x|^2 \leq V(x) \leq a_2|x|^2$, and $\varrho(s) \geq a_3 s^2$ then this set is uniformly globally exponentially stable.

The proof of Corollary 1 follows directly from the proof of Theorem 1 by letting $\lambda \to 0^+$ in the formula (2).

VI. Proofs of main results

A. Proof of Theorem 1

Let $\phi : [0, \tau_{MATI}] \to \mathbb{R}$ be the solution to

$$\dot{\phi} = -2L\phi - \gamma(\phi^2 + 1) \qquad \phi(0) = \lambda^{-1} \ . \qquad (25)$$

We will establish the following claim in the next section:

Claim 1: $\phi(\tau) \in [\lambda, \lambda^{-1}]$ for all $\tau \in [0, \tau_{MATI}]$.

We will use the definitions $\xi := \left[x^T, e^T, \tau, \kappa\right]^T$ and $F(\xi) := \left[f(x,e)^T, g(x,e)^T, 1, 0\right]^T$. Define

$$U(\xi) := V(x) + \gamma\phi(\tau)W^2(\kappa, e) \ . \qquad (26)$$

Below, by abuse of notation, we consider the quantity $\langle \nabla U(\xi), F(\xi)\rangle$ even though W is not differentiable with respect to κ. This is justified since the component in $F(\xi)$ corresponding to κ is zero. It is easy to check that for all (τ, κ) and that for almost all (x, e)

$$\begin{aligned} U(\xi^+) &= V(x^+) + \gamma\phi(\tau^+)W^2(\kappa^+, e^+) \\ &\leq V(x) + \gamma\lambda W^2(\kappa, e) \leq U(\xi). \qquad (27) \\ \langle \nabla U(\xi), F(\xi)\ \rangle &\leq -\varrho(|x|) - \varrho(W(\kappa, e)) \ . \end{aligned}$$

Since ϱ is positive definite, V is positive definite and radially unbounded, and Claim 1 holds, it follows that there exists a continuous, positive definite function $\widetilde{\varrho}$ such that

$$\langle \nabla U(\xi), F(\xi)\rangle \leq -\widetilde{\varrho}(U(\xi)) \ . \qquad (28)$$

Then, by standard results for continuous-time systems (see, for example, [10]) and using (27), we have the existence of $\beta \in \mathcal{KL}$ satisfying

$$\beta(s, t_1 + t_2) = \beta(\beta(s, t_1), t_2) \quad \forall (s, t_1, t_2) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0},$$
$$U(\xi(t,j)) \leq \beta(U(\xi(0,0)), 0.5t + 0.5\varepsilon j) \quad \forall (t,j) \in \operatorname{dom}\xi \ .$$

Then, using that V is positive definite and proper, using (17), Claim 1, and the definition of U in (26), uniform global asymptotic stability of the set $\{(x, e, \tau, \kappa) : x = 0, e = 0\}$ follows.

Under the assumptions made in the theorem to guarantee uniform global exponential stability, it follows that $\widetilde{\varrho}$ can be taken to be linear and β can be taken to be of the form $\beta(s,t) = Ms\exp(-\lambda t)$. Then uniform exponential stability follows from the quadratic upper and lower bounds on $V(x)$ and $W^2(\kappa, e)$. The proof will be complete after we prove Claim 1, which we will do in Section VII-B.

B. Proof of Theorem 2

Using (21) and (22), one can combine the ideas in [6] and [9, p. 22-23] to get a continuously differentiable function $\rho \in \mathcal{K}_\infty$ and $\sigma > 0$ such that with $\widetilde{W}(\kappa, e) := \rho(W(\kappa, e))$ we have

$$\widetilde{W}(\kappa + 1, h(k, e)) \leq e^{-\sigma}\widetilde{W}(\kappa, e) \ . \qquad (29)$$

Using the last assumption of the Theorem, let the smooth function V be the one obtained from Kurzweil's converse Lyapunov theorem [4], satisfying

$$\langle \nabla V(x), f(x, 0)\rangle \leq -\alpha_V(|x|) \qquad (30)$$

for some $\alpha_V \in \mathcal{K}_\infty$. Using the definition of ξ and $F(\xi)$ from the proof of Theorem 1, define

$$U(\xi) := V(x) + e^{-\sigma\tau/\tau_{MATI}}\widetilde{W}(\kappa, e) \ . \qquad (31)$$

Then, using (29), (14), and (31), we get

$$U(\xi^+) \leq V(x) + e^{-\sigma}\widetilde{W}(\kappa, e) \leq U(\xi) \ . \qquad (32)$$

Using the continuity of f, (30), (21) and (23), we also have the existence of a continuous function φ satisfying $\varphi(x, 0) = 0$ for all x and such that

$$\begin{aligned}\langle \nabla U(\xi), F(\xi)\rangle \leq{}& -\alpha_V(|x|) + \varphi(x, e) \\ & - \tau_{MATI}^{-1}\sigma e^{-\sigma\tau/\tau_{MATI}}\rho \circ \underline{\alpha}_W(|e|) \ . \end{aligned} \qquad (33)$$

Now the continuous-time arguments given in [1] or [11, Lemma 2.1] can be used to assure that, for each pair of strictly positive real numbers $\widetilde{\delta} < \widetilde{\Delta}$ there exists $\tau_{MATI} > 0$ such that, for almost all ξ in the set

$$\left\{ (x, e, \tau, \kappa) : \widetilde{\delta} \leq \left| \left[\begin{array}{c} x \\ e \end{array} \right] \right| \leq \widetilde{\Delta}, \tau \in [0, \tau_{MATI}] \ , \ \kappa \in \mathbb{Z}_{\geq 0} \right\}$$

we have

$$\langle \nabla U(\xi), F(\xi) \rangle \leq -0.5\alpha_V(|x|) - 0.5\underline{\alpha}_W(|e|) \ . \tag{34}$$

The result follows using standard continuous-time arguments like in the proof of Theorem 1.

VII. Proof of Claim 1 and that the bound is better

A. A race between differential equations

In this section we establish the following fact:

Lemma 1: For each $\lambda \in (0, 1)$, the value τ_1 in (4) is less than the value τ_2 in (5).

This lemma shows that Claim 1 in the proof of Theorem 1 holds when τ_{MATI} satisfies the bound given by the right-hand side of (1). Thus, the proof of Theorem 1 is complete in this case. In the next subsection, we establish that the bound (2) is equal to the value τ_2 in (5). This will establish Claim 1 and finish the proof of Theorem 1 as it is stated. It will also confirm that the bound on τ_{MATI} reported here is larger than the bound reported in [7].

Proof of Lemma 1. Note that $\tau_2 = \tau_{2+} + \tau_{2-}$ where τ_{2+} and τ_{2-} satisfy

$$\begin{aligned} \dot{\phi}_2 &= -2L\phi_2 - \gamma(\phi_2^2 + 1), \\ & \phi_2(0) = 1, \ \phi_2(\tau_{2+}) = \lambda, \ \phi_2(-\tau_{2-}) = \lambda^{-1} \ . \end{aligned} \tag{35}$$

Define $\tau_{1+} := \tau_1$. Let $\phi_1(\cdot)$, respectively $\phi_2(\cdot)$, denote the solution of (4), respectively (35). Since $\phi_1(\tau_{1+}) = \phi_2(\tau_{2+}) = \lambda$ and $\phi_2(-\tau_{2-}) = \lambda^{-1}$, we have $1 = \frac{d\phi_i(\tau_{i+})}{d\lambda} = \frac{d\phi(\tau_{i+})}{d\tau_{i+}} \frac{d\tau_{i+}}{d\lambda}$ and $-\lambda^{-2} = \frac{d\phi_2(\tau_{2-})}{d\lambda} = -\frac{d\phi(\tau_{2-})}{d\tau_{2-}} \frac{d\tau_{2-}}{d\lambda}$. These equations yield

$$\frac{d\tau_{1+}}{d\lambda} = \frac{-1}{L\lambda + \gamma}, \quad \frac{d\tau_{2+}}{d\lambda} = \frac{-1}{2L\lambda + \gamma(\lambda^2 + 1)},$$

$$\frac{d\tau_{2-}}{d\lambda} = \frac{-1}{\lambda^2(2L\lambda^{-1} + \gamma(\lambda^{-2} + 1))} = \frac{-1}{2L\lambda + \gamma(\lambda^2 + 1)} \ .$$

Using $\lambda^2 + 1 < 2$, $\tau_{1+} = \tau_1$, and $\tau_2 = \tau_{2+} + \tau_{2-}$ gives

$$\frac{d\tau_2}{d\lambda} < \frac{d\tau_1}{d\lambda} \ . \tag{36}$$

Since $\tau_1 = \tau_2 = 0$ when $\lambda = 1$, the condition (36) establishes the lemma.

B. Proof of Claim 1

Claim 1 follows immediately from the following lemma.

Lemma 2: The right-hand side of (2) is equal to the value τ_2 in (5) (cf. (25)).

Proof. By definition we can write $\tau_2 = -\int_{\lambda^{-1}}^{\lambda} \frac{d\phi}{\gamma\phi^2 + 2L\phi + \gamma} = -\frac{1}{\gamma} \int_{\lambda^{-1} + \frac{L}{\gamma}}^{\lambda + \frac{L}{\gamma}} \frac{ds}{s^2 - \text{sgn}(L-\gamma)\left(\frac{Lr}{\gamma}\right)^2}$, where $s := \phi + \frac{L}{\gamma}$, r is defined in (3) and $\text{sgn}(\cdot)$ is the sign function with $\text{sgn}(0) = 0$. The first formula in (2), when $\gamma > L$, comes from using the fact that $-\frac{1}{\gamma} \int_a^b \frac{ds}{(Lr/\gamma)^2 + s^2} = -\frac{1}{Lr} \left[\arctan\left(\frac{b\gamma}{Lr}\right) - \arctan\left(\frac{a\gamma}{Lr}\right) \right]$ and that for all $c_2 \geq c_1 \geq 0$ we have $\arctan(c_2) - \arctan(c_1) = \arctan((c_2 - c_1)/(1 + c_1 c_2))$. The second formula in (2), when $L = \gamma$, follows from the fact that $-\frac{1}{\gamma} \int_a^b \frac{ds}{s^2} = \frac{1}{\gamma} \left(\frac{1}{b} - \frac{1}{a} \right)$. The third formula in (2), when $\gamma < L$ follows from $\frac{1}{\gamma} \int_a^b \frac{ds}{(Lr/\gamma)^2 - s^2} = \frac{1}{Lr} \left[\text{arctanh}\left(\frac{b\gamma}{Lr}\right) - \text{arctanh}\left(\frac{a\gamma}{Lr}\right) \right]$. Then, the last formula in (2) is obtained by using the identity $\text{arctanh}(c_2) - \text{arctanh}(c_1) = \text{arctanh}\left(\frac{c_2 - c_1}{1 - c_2 c_1}\right)$.

VIII. Conclusions

We have provided a simple Lyapunov proof for certain results that have appeared previously in the literature on the stability of networked control systems. Along the way, we have provided some modest improvements to the previous results. We hope that the Lyapunov approach to proving stability for networked control systems will lead to better insight into the design of protocols for these systems and will also inspire even sharper analysis tools.

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