Stability and performance of SISO control systems with First Order Reset Elements *

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Abstract

We consider set-point regulation and $\mathcal{L}_2$ robust stability properties of a class of reset control systems consisting of a minimum phase relative degree one linear SISO plant controlled by a novel first order reset element (FORE). In particular, we provide precise tuning rules for the parameters of the FORE that guarantee set-point regulation and $\mathcal{L}_2$ robust stability. These results rely on necessary and sufficient conditions for exponential and $\mathcal{L}_2$ finite gain stability of a class of planar reset systems consisting of a scalar linear plant controlled by the novel FORE. We construct a Lyapunov function for the planar reset system whenever our necessary and sufficient conditions for stability are satisfied. Moreover, we show that the $\mathcal{L}_2$ gain of the planar reset system decreases to zero as the pole and/or the gain of the FORE are increased to infinity. This result and a small gain theorem are then used to prove our main results for the class of SISO linear plants. A number of stability results, including Lyapunov conditions for $\mathcal{L}_p$ and exponential stability, for a larger class of reset and hybrid systems are presented and used to prove our main results.

1 Introduction

Reset controllers were proposed for the first time by Clegg in [17] with the aim of providing more flexibility in linear controller designs. In particular, Clegg proposed a particular reset element, the so called Clegg integrator, that acts like a linear integrator whenever its input and output have the same sign and it resets its output to zero otherwise. Clegg showed that the describing function of this device has the same magnitude plot as the linear integrator but it has 51.9° less phase lag than the linear integrator. He predicted that this added flexibility will be useful in overcoming some of the fundamental limitations of linear control. However, it was not until 1990's that this was demonstrated by way of an example in [5]; see also [19].

The first attempt to provide a systematic procedure for controller design exploiting Clegg integrators was presented in [30]. Subsequently, in [26], some limitations associated with the use of Clegg integrators were highlighted and a new

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reset device called the *first order reset element (FORE)* was introduced together with a corresponding FORE controller design procedure. These early results on reset control systems are summarized in a recent paper [12]. There has been a renewed interest in this class of systems in the late 1990's [8, 4, 6, 7, 12, 14, 15, 13, 23, 25, 24, 27, 46]. First attempts to rigorously analyze stability of reset systems with Clegg integrators can be found in [27, 25]. In particular, an integral quadratic constraint was proposed in [25] to analyze stability of a class of reset systems. BIBO stability analysis of reset systems consisting of a second order plant and a FORE was conducted in [13] (see also [15]). Stability analysis of general reset systems can be found in [7] (see also [24, 14]) where Lyapunov based conditions for asymptotic stability were presented and computable conditions for quadratic stability based on linear matrix inequalities (LMIs) were given. Moreover, in [7], BIBO stability of general reset systems was obtained as a consequence of quadratic stability and an internal model principle was proved for reference tracking and disturbance rejection. All above mentioned results use the same model of reset systems which in the case of FORE specializes to:

$$\begin{cases} \dot{x}_r = a_c x_r + b_c v, & \text{if } v \neq 0, \\ x_r^+ = 0, & \text{if } v = 0. \end{cases} \tag{1}$$

where a_c and b_c are respectively the pole and input gain of the FORE (the model of the Clegg integrator corresponds to $a_c = 0$ and $b_c = \frac{1}{RC}$); $x_r \in \mathbb{R}$ and $v \in \mathbb{R}$ are respectively the state and the input of the FORE. Note that in (1) the resetting rule of the FORE is characterized in terms of zero crossing of the input v. The dynamics (1) corresponds to the characterization of FOREs used in several recent papers (see, e.g., [24, 7, 2, 3] and references therein) which have translated into formulas the zero-crossing strategy commented in words in [26].

Recently, in [43] we proposed a different class of FORE models that were subsequently used in [34, 36, 35, 41, 44]. In particular, we allow resets to occur on more complicated sets and the model of the FORE within our modeling framework takes the following form:

$$\begin{cases} \dot{x}_r = a_c x_r + b_c v, & \text{if } x_r v \geq 0 \;, \\ x_r^+ = 0, & \text{if } x_r v \leq 0 \;, \end{cases} \tag{2}$$

Note that when using (1) in closed-loop (so that e will come from a plant output), extra conditions are enforced in [24, 7, 2, 3], to avoid the occurrence of Zeno solutions (see [29, 22] and [20, page 72]), namely solutions that jump infinitely many times in a compact time interval. On the other hand, it was proposed in [43] to augment the model (2) with a clock variable in order to achieve temporal regularization that avoids Zeno solutions (see also [36]).

A detailed motivation for considering the class of temporally regularized FORE models (2) is given in [36, 43] and we summarize it here: (i) it leads to a faithful (even though approximate) model of the Clegg integrator circuit originally proposed in [17] (see [43] for details); (ii) it avoids problems with Zeno solutions or undefined solutions; (iii) it allows us to state much less conservative Lyapunov conditions to estimate $\mathcal{L}_2$ gains than the class of models considered in [7]; (iv) models of the form (2) are fully consistent with the hybrid systems modeling framework developed in [21]. In this framework, the solutions of a hybrid system enjoy structural properties that guarantee robustness of asymptotic stability [21] and that enable results on converse Lyapunov theorems [10, 11] and invariance principles [37] for hybrid systems. The results in this paper rely heavily on the converse Lyapunov theorems developed in these references; (v) the models of the form (2) capture solutions that can be obtained by perturbing the nominal model with arbitrarily small noise (for more details see [21]).

Our new modeling framework has been already used in a range of recent papers to address various analysis and design questions for reset systems. We presented Lyapunov like conditions for $\mathcal{L}_2$ stability and exponential stability of reset systems in [36]. Our conditions involve locally Lipschitz Lyapunov functions as opposed to continuously differentiable ones considered in [7]. This allowed us to consider piecewise quadratic Lyapunov functions in verifying exponential or $\mathcal{L}_2$ stability of reset systems [42]. Some explicit Lyapunov functions have been computed in [44], while the properties of reset set point stabilizers and necessary and sufficient conditions for exponential and $\mathcal{L}_2$ stability have been reported, respectively, in [45] and [34]. LMI-based approaches for the $\mathcal{H}_2$ performance analysis and $\mathcal{L}_2$ performance analysis of reset control systems have been proposed, respectively, in [40] and [1], while [31] addressed the presence of input saturation in reset systems. Finally, [32] studied stability of reset systems in the presence of nonzero reference signals.

In this paper, we continue the investigation of exponential stability and $\mathcal{L}_2$ stability properties of reset systems. Some preliminary results presented here can be found in our conference papers [34, 43, 45]. First, we propose a novel class of FOREs with new resetting rules that lead to a subtly different model from the one considered in [36, 43]. Then, we consider set-point regulation, $\mathcal{L}_2$ stability and exponential stability of a special class of reset systems that consist of a relative degree one minimum phase linear SISO plant controlled with the proposed FOREs. The new reset rules in FORE simplify the stability analysis considerably and together with the reformulation of the main result from [36] become a key tool in establishing our main results. Another important technical contribution that we present are necessary and sufficient conditions for exponential and $\mathcal{L}_2$ stability of a class of planar reset systems that consist of a FORE controlling a scalar linear plant. Whenever the planar reset system is stable, we construct an appropriate Lyapunov function. Moreover, we use these Lyapunov results to show that the $\mathcal{L}_2$ gain of the planar system converges to zero if either the gain or the pole of the FORE (or both) are increased. This result is then used to show that the considered class of SISO linear plant controlled with the FORE is exponentially and $\mathcal{L}_2$ stable if the gain or pole of the FORE (or both) become sufficiently large. It should be remarked that the tools provided in this paper are a first step toward a more ambitious goal of developing design rules for FOREs or reset control systems that are able to systematically overcome intrinsic limitations of linear designs. As suggested in the conclusions of [26], this task will be possible "when stability criteria for feedback loops containing [...] FOREs" will be available. We also emphasize that one of the contributions of our paper are Lyapunov conditions for $\mathcal{L}_p$ and exponential stability of a class of nonlinear reset systems with new resetting rules, as well as equivalences among exponential stability, $\mathcal{L}_p$ stability and input-to-state stability of linear hybrid systems acting on cones. We believe that these results will be useful in many other situations.

The paper is organized as follows. Notation and mathematical preliminaries are given in Section 2. Section 3 contains the definition of the novel FORE that we propose in this paper, as well as its motivation and comparison to FOREs used in our earlier work. Results on set-point regulation of SISO plants with the novel FOREs are given in Section 4. Stabilization and $\mathcal{L}_2$ stabilization of relative degree one minimum phase linear SISO plants is given in Section 5. A complete characterization of stability and $\mathcal{L}_2$ stability of a class of planar FOREs is given in Section 6 as these results are crucial in proving the results in Sections 4 and 5. A range of useful stability results, including Lyapunov characterization of exponential and $\mathcal{L}_2$ stability, are given for a class of nonlinear reset systems in Section 7. While instrumental in proving our main results, these stability results are of interest in their own right and could be useful in a range of other situations not considered in this paper. Most proofs are included in Section 8 and a summary is given in the last section.

2 Definitions and notation

The sets of positive integers (including zero) and real numbers are respectively denoted as $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}$. The Euclidean norm is denoted $|\cdot|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t\to\infty} \beta(s,t) = 0$ for each $s \geq 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KLL}$ if, for each $r \geq 0$, $\beta(\cdot, r, \cdot)$ and $\beta(\cdot, \cdot, r)$ belong to class $\mathcal{KL}$. Given two vectors x and y, $(x, y) := [x^T \; y^T]^T$.

Given the following generic nonlinear hybrid system with temporal regularization,

$$\left.\begin{aligned} \dot{\tau} &= 1, \\ \dot{x} &= f(x,d) \end{aligned}\right\} \quad \text{if } x \in \mathcal{F} \text{ or } \tau \leq \rho, \qquad\qquad (3)$$

$$\left.\begin{aligned} \tau^+ &= 0, \\ x^+ &= g(x) \end{aligned}\right\} \quad \text{if } x \in \mathcal{J} \text{ and } \tau \geq \rho,$$

(where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $\tau \in \mathbb{R}_{\geq 0}$) we call its first equation the "flow" equation and its second equation the "jump" equation. The two conditions at the right hand side are the "flow" condition and the "jump" condition. The solution of the hybrid system will then flow or jump depending on whether its value at that time belongs to the so-called "jump set" (namely, the set of states and inputs for which the jump condition is true) or it belongs to the "flow set" or even both. Note that if an initial state belongs to both sets then both flows and jumps are possible which leads to non-unique solutions.

According to the hybrid systems framework of [20, 21], a hybrid time domain is defined as a subset of $[0, \infty) \times \mathbb{Z}_{\geq 0}$, given as a union of finitely or infinitely many intervals $[t_i, t_{i+1}] \times \{i\}$ where the numbers $0 = t_0, t_1, \ldots,$ form a finite or infinite nondecreasing sequence. The last interval is allowed to be of the form $[t_i, T)$ with T finite or $T = +\infty$. To guarantee existence of solutions (see [20, 21]), it is sufficient to assume that $\mathcal{F}$ and $\mathcal{J}$ in (3) are closed nonempty sets and that $f(\cdot, \cdot)$ is continuous in both arguments (in anticipation of the fact that d will be Lebesgue measurable). A hybrid signal is a function defined on a hybrid time domain. Specifically, a hybrid signal $d : \mathrm{dom}(d) \to \mathbb{R}^m$ is called a hybrid input in this paper. A hybrid signal $\xi : \mathrm{dom}(\xi) \to \mathbb{R}^{n+1}$ is called a hybrid arc if $\xi(\cdot, j)$ is locally absolutely continuous for each j. Denote $\xi := (x, \tau)$. A hybrid arc $\xi : \mathrm{dom}(\xi) \to \mathbb{R}^{n+1}$ and a hybrid input $d : \mathrm{dom}(d) \to \mathbb{R}^m$ is a solution pair (ξ, d) for the hybrid system (3) if:

(i) $\mathrm{dom}(\xi) = \mathrm{dom}(d)$;

(ii) for all j and almost all t such that $(t, j) \in \mathrm{dom}(\xi)$, we have

$$\xi(t,j) \in \mathcal{F} \text{ or } \tau(t,j) \leq \rho, \qquad \dot{x}(t,j) = f(x(t,j), d(t,j));\; \dot{\tau}(t,j) = 1\;;$$

(iii) for all $(t, j) \in \mathrm{dom}(\xi)$ such that $(t, j+1) \in \mathrm{dom}(\xi)$ we have

$$\xi(t,j) \in \mathcal{J} \text{ and } \tau(t,j) \geq \rho, \qquad x(t,j+1) = g(x(t,j));\; \tau(t,j+1) = 0\;.$$

Due to the special structure of system (3), and due to temporal regularization which enforces for all $(t,j) \in \text{dom}(\xi)$, $j \geq 2$ that $t_j - t_{j-1} \geq \rho$, Zeno solutions cannot occur and forward complete solutions are all forward complete in the t direction (namely, given any forward complete solution $\xi(\cdot,\cdot)$, for any $t \in \mathbb{R}_{\geq 0}$, there exists j such that $(t,j) \in \text{dom}(\xi)$). Then we can make use of the following signal norms in this paper because any forward complete solution will be well defined in the t direction (see also Lemma 1 in Section 7.1). Given any function $\eta(\cdot,\cdot)$ defined on the hybrid domain $\text{dom}(\eta)$, and any $(t,j) \in \text{dom}(\eta)$, denote: $\int_0^t \eta(s)ds := \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \eta(s,i)ds + \int_{t_j}^{t} \eta(s,j)ds$. Using this notation, given any hybrid signal $\zeta(\cdot,\cdot)$ and any $p \in [1,\infty)$, define its $\mathcal{L}_p$ norm of ζ as $\|\zeta\|_p := \lim_{t\to\infty} \left(\int_0^t |\zeta(s)|^p ds \right)^{1/p}$ and say that $\zeta \in \mathcal{L}_p$ whenever $\|\zeta\|_p < \infty$. Similarly, define its $\mathcal{L}_\infty$ norm of ζ as: [1] $\|\zeta\|_\infty := \operatorname*{ess.sup}_{(t,j)\in\text{dom}(\zeta)} |\zeta(t,j)|$. and say that $\zeta \in \mathcal{L}_\infty$ whenever $\|\zeta\|_\infty < \infty$.

Based on the signal norms above, we can introduce the following stability notions adopted in this paper. System (3) is *input-to-state stable (ISS) from d to x*, if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function γ such that for any initial condition $(x(0,0),\tau(0,0)) = (x_0,\tau_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ and any $d \in \mathcal{L}_\infty$, all solutions satisfy

$$|x(t,j)| \leq \max\{\beta(|x_0|,t), \gamma(\|d\|_\infty)\}, \quad \forall (t,j) \in \text{dom}(x). \tag{4}$$

The origin of the x dynamics of system (3) with $d = 0$ is *(locally) asymptotically stable* if there exists a ball $\mathcal{B} \subset \mathbb{R}^n$ centered at the origin such that for any initial condition $(x_0,\tau_0) \in \mathcal{B} \times \mathbb{R}_{\geq 0}$, the bound (4) with $d = 0$ holds for all solutions. System (3) is *finite gain exponentially ISS* from d to x, if there exist constants $\gamma_\infty,\ m,\ \ell > 0$ such that given any initial condition $(x_0,\tau_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, all solutions satisfy the bound (4) with $\beta(s,t) = mse^{-\ell t}$ and $\gamma(s) = \gamma_\infty\ s$. The origin of the x dynamics of system (3) with $d = 0$ is *globally exponentially stable* if there exist constants $m,\ \ell > 0$ such that given any initial condition $(x_0,\tau_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ the bound (4) with $d = 0$ and $\beta(s,t) = mse^{-\ell t}$ holds for all solutions. Given $p \in [1,+\infty)$, system (3) is *finite gain $\mathcal{L}_p$ stable from d to x* (respectively, *finite gain $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stable from d to x*), if there exist constants $\gamma_p,\ \gamma_0 > 0$ (respectively, $\gamma_{p,\infty},\ \gamma_0 > 0$) such that for any initial condition $(x_0,\tau_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ and any $d \in \mathcal{L}_p$,

$$\begin{aligned} \|x\|_p &\leq \gamma_0|x_0| + \gamma_p\|d(\cdot)\|_p, \\ \text{(respectively, } |x(t,j)| &\leq \gamma_0|x_0| + \gamma_{p,\infty}\|d(\cdot)\|_p, \quad \forall (t,j) \in \text{dom}(x)\ ). \end{aligned}$$

3 A new model of FORE and its Lyapunov implications

In this paper, we propose a novel model of FORE:

$$\begin{aligned} \dot{x}_r &= a_c x_r + b_c v, &&\text{if } \varepsilon v^2 + 2v x_r \geq 0 \\ x_r^+ &= 0, &&\text{if } \varepsilon v^2 + 2v x_r \leq 0, \end{aligned} \tag{5}$$

where $\varepsilon > 0$ is a small number associated with the tilting of the flow set boundary (see Figures 1 and 2). One way to understand the different FORE models (1), (2) and (5) is via the hybrid modeling framework of [21, 20]. Suppose that (x_r, v) is a solution pair for the system (5). Then, at any time $(t,j) \in \text{dom}(x_r)$, a solution of the hybrid system

[1] As compared to the definition in [9], we do not need to be careful about the ess. sup disregarding isolated values of ζ when multiple jumps occur at the same time t because this situation is ruled out by temporal regularization.

will flow or jump depending on whether its value at that time belongs to the jump set $\mathcal{J}$ or it belongs to the flow set $\mathcal{F}$ or even both.

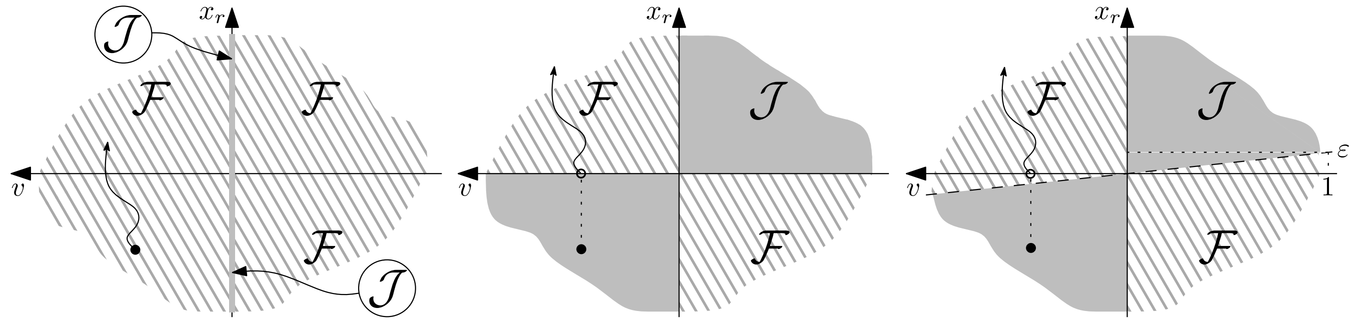

Figure 1: The jump (grey) and flow (striped) sets for the model (1) (left), the model (2) (middle) and the new model (5) (right).

Figure 1 shows the differences between the flow sets $\mathcal{F}$ and jump sets $\mathcal{J}$ for the three FORE models [2] (1), (2) and (5). For instance, in the FORE model (1) the jump set $\mathcal{J}$ is the vertical axis and the flow model $\mathcal{F}$ is its complement (left sub-figure). Note that in this case $\mathcal{J} \cap \mathcal{F} = \emptyset$. On the other hand, in the FORE model (2) the jump set $\mathcal{J}$ consists of the first and third quadrants including the axes and the flow set is the second and fourth quadrant including the axes (middle sub-figure). Note that in this case $\mathcal{J} \cap \mathcal{F} = \{(x_r, v) : x_r = 0 \text{ or } v = 0\}$. The final (right) sub-figure corresponds to the new FORE model and the only difference with the middle sub-figure is that one of the boundaries of the jump and flow sets is slightly tilted. A sample trajectory starting from the same initial condition and with the same input is given in all three sub-figures to illustrate the difference in dynamic behaviour that comes from the differences among the jump and flow sets.

In order to avoid Zeno solutions, we augment the model (5) with a clock variable in a similar manner as in [36, 43]:

$$\left\{ \begin{array}{ll} \left. \begin{array}{l} \dot{\tau} = 1, \\ \dot{x}_r = a_c x_r + b_c v \end{array} \right\} & \text{if } \varepsilon v^2 + 2 v x_r \geq 0 \text{ or } \tau \leq \rho, \\ \left. \begin{array}{l} \tau^+ = 0, \\ x_r^+ = 0 \end{array} \right\} & \text{if } \varepsilon v^2 + 2 v x_r \leq 0 \text{ and } \tau \geq \rho, \end{array} \right. \tag{6}$$

This type of rule has been used in [29] for instance where it was referred to as "temporal regularization". The same idea was also used in the context of reset systems in [13, §2]. The stability and performance results that we report in the sequel all refer to reset control systems modified as in (6) where ρ is a sufficiently small positive number. When dealing with the hybrid representation (5), temporal regularization is important. As a matter of fact, inspecting the right sub-figure of Figure 1 and equation (5), it appears that given any initial condition at the origin (namely anywhere in $x_r = 0, v = 0$), the FORE could jump and it would jump at the same point because x_r is already zero. For closed-loops with plants of order higher than one and with FOREs of the form (5), the origin of the (x_r, v) plane corresponds to a subspace of the closed-loop state-space where the above commented Zeno solutions may occur; note

[2] In Figure 1, the v axis direction is reversed so that in the case of negative feedback with one dimensional plants ($v = -x_p$), this is exactly the closed-loop phase plane, which has been commented in [36, 35, 43, 44, 34, 41].

that these Zeno solutions do not converge to the origin of the closed-loop system. Temporal regularization in FORE (6) overcomes this problem.

The motivation for using the model (2) as opposed to (1) was discussed in detail in [36, 43]. On the other hand, the motivation for using (5) as opposed to (2) stems from a Lyapunov analysis that is discussed next. When using the temporally regularized model (2) (see [36]), the temporal regularization causes the system's state to slightly overflow into the jump set $\mathcal{J}$ before the reset occurs and this makes the Lyapunov construction harder. In particular, all the Lyapunov-based results reported in [34, 35, 36, 43, 44, 45] are based on the existence of a Lyapunov function that satisfies standard regularity and growth conditions, in addition to the following flow and jump conditions:

1. it is a disturbance attenuation Lyapunov function in a *slightly inflated* version of the flow set $\mathcal{F}$;

2. it does not increase when jumping from the jump set $\mathcal{J}$.

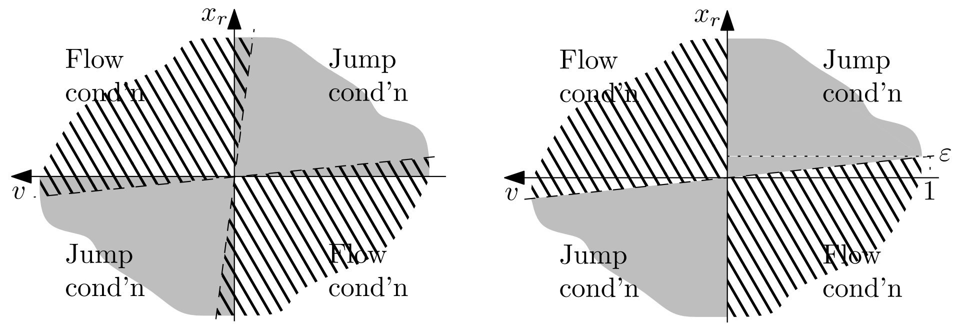

Figure 2: The sectors of the state space where the Lyapunov flow and jump conditions must hold for the temporally regularized FORE (2) without strict decrease at jumps (left) and the FORE (6) with strict decrease at jumps (right).

The two requirements above are graphically represented in the left sub-figure of Figure 2. The striped region represents the set on which the Lyapunov function is required satisfy the item 1 and the shaded region is where the item 2 above should hold. Note that there is an overlap of these two sets where both items 1 and 2 should hold. This stringent requirement makes the construction of appropriate Lyapunov functions hard in this case.

Conversely, the stability results for reset systems that use the model (5) and its temporally regularized version (6) that we propose in this paper require a Lyapunov function that:

1a. is a disturbance attenuation Lyapunov function in the flow set $\mathcal{F}$;

2a. *strictly decreases* when jumping from the jump set $\mathcal{J}$.

The right sub-figure of Figure 2 illustrates the two new conditions. The striped and shaded regions represent the sets on which the Lyapunov function needs to satisfy items 1a and 2a respectively. It turns out that it is much easier to construct Lyapunov functions satisfying conditions 1a and 2a rather than 1 and 2. Indeed, note that the model (2) as it stands will never satisfy the item 2a because there are cases when a state jumps onto itself (this is whenever $x_r = 0$). Therefore it is necessary to tilt the boundary between the flow and the jump set as shown in the right sub-figure of Figure 1. More specifically, the "tilting" corresponds to transforming the horizontal sector boundary from $x_r = 0$ (i.e., the x_p axis) into $x_r = \varepsilon x_p$. Then any state in the jump set will be mapped into the interior of the flow set (except for the origin) and it will be possible to construct Lyapunov functions guaranteeing the item 2a.

The advantage of this new technique is that the proof of the main results of [36, 35] becomes simpler (it is formalized in Theorem 6 in Section 7.1) and in some cases the Lyapunov construction might be simpler. Moreover, for situations where item 2a is a natural condition to impose, the new Lyapunov tools of Theorem 6 might be more effective at exploiting the underlying system features to effectively design Lyapunov functions that establish exponential and $\mathcal{L}_2$ stability of the system with temporal regularization. As a result, Theorem 6 in Section 7.1 is formulated for larger class of reset systems than those addressed in the rest of the paper.

4 Set point regulation of SISO linear plants with FOREs

Consider a strictly proper SISO linear plant whose dynamics is described by

$$\mathcal{P} \left\{ \begin{array}{rcl} \dot{x}_p &=& A_p x_p + B_{pu} u + B_{pd} d, \\ y &=& C_p x_p, \end{array} \right. \tag{7}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}^{n_d}$ is a disturbance input and $y \in \mathbb{R}$ is the measured plant output (A_p, B_{pu}, B_{pd} and C_p are matrices of appropriate dimensions). The goal of this section is to show how the set-point regulation of plant (7) using a FORE can be reduced to a stabilization problem in suitably transformed coordinates. A block diagram of the arising control scheme is represented in Figure 3.

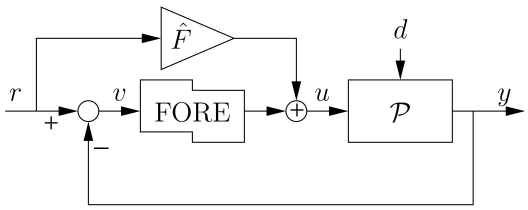

Figure 3: Set-point regulation of linear SISO plants using a FORE.

Suppose that the transfer function of the plant (7) from u to y does not have zeros at the origin. Then set point regulation of the output y can be achieved from u. To this aim, define the following scalar:

$$F = \left\{ \begin{array}{ll} -\dfrac{1}{C_p A_p^{-1} B_{pu}}, & \text{if } A_p \text{ is invertible,} \\ 0, & \text{otherwise,} \end{array} \right. \tag{8}$$

corresponding to the inverse of the DC gain of the plant, and define $\hat{F}$ as a nominal value for F. Then, if the closed-loop between the FORE (6) and the plant (7), with $d = 0$, $u = x_r$ and $v = -y$ is asymptotically stable, set point regulation can be achieved by using (6) with the following feedback interconnection:

$$u = x_r + \hat{F} r, \quad v = r - y. \tag{9}$$

The following statement, whose proof is reported in Section 8.1, establishes useful set-point regulation properties of the closed-loop under the assumption that the closed-loop with $r = 0$ is asymptotically stable. We use the notation $x := (x_p, x_r)$.

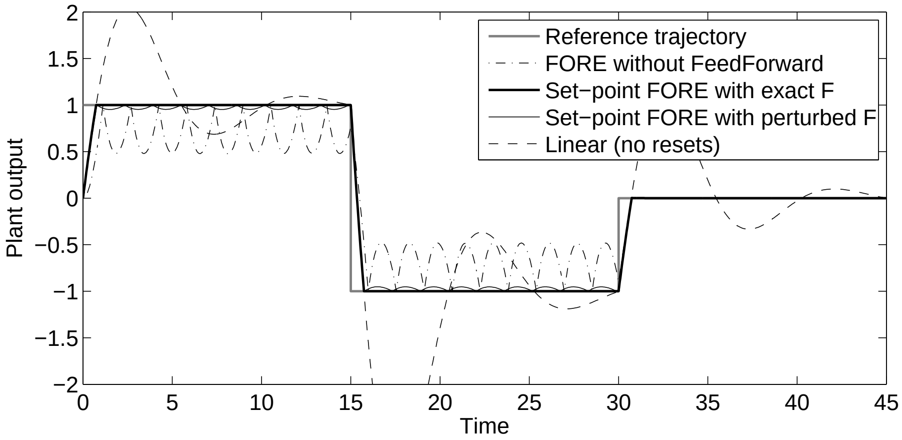

Figure 4: Example 1: linear response (dashed), FORE without feedforward (dash-dotted), FORE with feedforward and nominal F (bold) and FORE with feedforward and perturbed F (thin solid).

Theorem 1 *(FORE set point stabilizer) Suppose that the transfer function of the plant (7) from u to y does not have zeros at the origin and that the origin of the reset control system (7), (6), (9) with $r = 0$ and $d = 0$ is asymptotically stable. Then the closed-loop (7), (6), (9) is such that:*

1. *if $\hat{F} = F$, then for any constant reference $r^* \in \mathbb{R}$:*
   (a) *if $d = 0$, the equilibrium point $x^* = (x_p^*, 0)$, satisfying $y^* = C_p x_p^* = r^*$ is globally exponentially stable;*
   (b) *the system is finite gain $\mathcal{L}_p$ and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stable from d to $(x_p - x_p^*, x_r)$ for all $p \in [1, +\infty)$;*
2. *if $\hat{F} \neq F$, denoting $\Delta F = \hat{F} - F$, there exist positive constants k, ℓ and γ such that for any constant reference $r^* \in \mathbb{R}$:*

$$|y(t, j) - r^*| \leq \max\{k e^{\ell t} |\tilde{x}(0, 0)|,\ \gamma \|d\|_\infty,\ \gamma \Delta F r^*\}, \tag{10}$$

where $\tilde{x}(0, 0) = (x_p(0, 0) - x_p^*, x_r(0, 0))$.

The construction proposed in Theorem 1 and shown in Figure 3 generalizes the FORE control system construction to the set point regulation problem. This generalization is quite intuitive when the plant (7) has in it an integrator, or more generally an internal model of the reference, and is actually the case for all situations where FOREs have been used in the literature (see for example the discussion in [7, Section 4.1]). As a matter of fact in that case $F = 0$, the feedforward path in Figure 3 disappears and the scheme resembles the typical control scheme in feedback from the set-point tracking error. However, when the plant does not satisfy $\det(A_p) = 0$, this intuitive generalization is no longer effective and can lead to very undesirable closed-loop behavior, as shown in the next example.

Remark 1 Note that applying Theorem 7 to system (7), (6), (9) with $r = 0$ allows to conclude ISS and $\mathcal{L}_p$ stability from d to (x_p, x_r) for any $p \in [1, \infty)$. This property is less relevant here because we are dealing with set point regulation. Nevertheless, by virtue of Theorem 7, it follows directly from the asymptotic stability assumption. $\circ$

Example 1 Consider the plant (7) in feedback interconnection (9) with the FORE (6). Use the following parameters

$$A_p = -1.5, \quad B_{pu} = 1, \quad C_p = 1, \quad b_c = 2, \quad a_c = 1,$$

so that $F = 0.75$. In Figure 4, the grey curve represents the reference signal r and the dashed line represents the response of the system without resets and with the feedforward path (using $F = \hat{F}$), which is exponentially stable for these parameters. Note that this response is only illustrative of how the reset mechanism changes the underlying linear dynamics but can not be used to establish any superiority of reset control versus linear solutions. Indeed, many alternative linear control schemes could be considered for this set point regulation problem, depending on what the performance goal is. The solid line reports the response of the FORE control system implemented without the feedforward path (or, equivalently, by selecting $\hat{F} = 0$) and the bold line reports the response of the FORE control system implemented according to Figure 4 with $F = \hat{F}$. Finally, the thin solid curve shows the response when F is increased by 10% with respect to the correct value. The resulting response is a slight deterioration of the desirable bold response as anticipated by the result in item 2 of Theorem 1. ◦

5 Stabilization of minimum phase relative degree one linear SISO plants

In the previous section we addressed the set-point regulation problem under the assumption that the origin of the reset system with zero reference is asymptotically stable. In this section we provide tools for stabilizing a FORE closed-loop under the assumption that the plant is a linear SISO minimum phase relative degree one system. The underlying idea in the $\mathcal{L}_2$ stability proof is to use a small gain theorem with the trends proven later in Theorem 5. Then using the results of Theorem 7, we get exponential stability in the absence of disturbances.

Consider the plant (7) connected to the temporally regularized FORE (6) via the interconnection

$$u = x_r, \quad v = -y. \tag{11}$$

Under a minimum phase and relative degree one assumption on the plant dynamics, there exists a nonsingular change of coordinates so that we can write its dynamics as follows [28, Remark 4.3.1]:

$$\begin{aligned} \dot{z} &= A_z z + B_{zy} y + B_{zd} d && \text{(12a)} \\ \dot{y} &= a_p y + b_p u + \underbrace{C_z z + E_d d}_{\tilde{d}}, && \text{(12b)} \end{aligned}$$

where $y \in \mathbb{R}, z \in \mathbb{R}^{n_p - 1}$ and $u \in \mathbb{R}$ are, respectively, the plant output, a part of the state corresponding to the zero dynamics and the plant input. Since the plant is minimum phase, A_z is Hurwitz and we assume without loss of generality that $b_p > 0$. To state the next result we introduce the following definition:

Definition 1 Consider the FORE control system with temporal regularization (7), (6), (11). Assume that α is a suitable parameter of the closed-loop system. Then we say that the system is exponentially stable (or finite gain $\mathcal{L}_2$ stable) conditionally to large α and hierarchically small (ε, ρ) if there exists $\alpha^* > 0$ such that for each $\alpha \geq \alpha^*$ there exists ε^* such that for each $\varepsilon \in (0, \varepsilon^*]$ there exists ρ^* such that for all $\rho \in (0, \rho^*]$ we have that system (7), (6), (11) is exponentially stable (finite gain $\mathcal{L}_2$ stable). ◦

It is understood in the above definition that the only parameters that we can change are $\alpha, \varepsilon, \rho$, whereas all other constants in the model are fixed. Then we can state the following result, whose proof is reported in Section 8.1.

Theorem 2 *Consider the closed loop between the plant (12) and the FORE (6) via the interconnection (11), where the FORE (6) is parametrized by $(a_c, b_c, \varepsilon, \rho)$. Let A_z be Hurwitz and $b_p > 0$ in (12). Then, the following statements are true:*

1. *(unstable FORE feedback) the system is finite gain $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stable from d to x_p conditionally to large a_c and hierarchically small (ε, ρ). Moreover, when $d(t) \equiv 0$ the system is also exponentially stable conditionally to large a_c and hierarchically small (ε, ρ);*

2. *(high gain feedback) the system is finite gain $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stable from d to x_p conditionally to large b_c and hierarchically small (ε, ρ). Moreover, when $d(t) \equiv 0$ the system is also exponentially stable conditionally to large b_c and hierarchically small (ε, ρ);*

3. *(high gain+unstable FORE feedback) the system is finite gain $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stable from d to x_p conditionally to large (a_c, b_c) and hierarchically small (ε, ρ). Moreover, when $d(t) \equiv 0$ the system is also exponentially stable conditionally to large (a_c, b_c) and hierarchically small (ε, ρ).*

Remark 2 The three results in Theorem 2 can be interpreted in the context of the well known high gain feedback stabilization of linear systems. In particular, item 2 states that the same result holds for reset control systems. The novelty established here is in item 1 which states that high instability in the FORE is capable of stabilizing the reset control system. This result is new and significantly less trivial because the underlying linear dynamics become exponentially unstable for large values of a_c. Despite this fact, our novel proof technique allows to establish the exponential stability of the reset control system. Finally, item 3 simply states that if both the loop gain and the FORE pole go to infinity, the same stabilization result still works. ○

Remark 3 A result similar to Theorem 2 can be proved under appropriate conditions for a class of nonlinear SISO systems that are minimum phase (in an appropriate sense) and relative degree one. For instance, consider a nonlinear control affine system without disturbances:

$$\begin{aligned} \dot{x}_p &= f(x_p) + g(x_p)u_p \\ y &= h(x_p) \end{aligned}$$

and suppose that there exists a (global or local) nonsingular change of coordinates $(z, y) = T(x_p)$ and an input transformation $u_p = K(x_p) + L(x_p)u$ such that the system in new coordinates and with the new input u becomes: [3]

$$\begin{aligned} \dot{z} &= F(z, y) \\ \dot{y} &= u + G(z, y) \; . \end{aligned}$$

In other words, the nonlinear system is input-output linearized. Then, if we assume that the zero dynamics, which correspond to the z state are finite gain $\mathcal{L}_2$ stable from y to $G(z, y)$, we can apply FORE design to the linearized y

[3] Conditions under which this transformation is possible are given for instance in [28, Chapter 4].

state and use the same steps to conclude that increasing the gain or the pole (or both) of the FORE would stabilize the overall nonlinear system. ◦

Remark 4 Our results can be generalized to a class of nonlinear MIMO plants that have the same number χ of inputs and outputs if there exists a coordinate and input transformation that yields the system in the following form:

$$\begin{aligned} \dot{z} &= F(z,y) \\ \dot{y}_i &= u_i + G_i(z,y)\ ,\ i = 1,\ldots,\chi. \end{aligned}$$

For precise conditions under which such transformations are possible for control affine systems, see [28, Chapter 5]; typically, one would require vector relative degree $(1,1,\ldots,1)$ and zero dynamics that are stable in an appropriate sense. In this case, applying a FORE to each SISO pair (y_i, u_i) we will obtain the planar system considered in the next section and the overall system consists of χ such decentralized systems interconnected in feedback with the zero dynamics. With appropriate stability properties on the zero dynamics and by adjusting the parameters of all FOREs we can show that stability of the closed-loop holds. ◦

6 Lyapunov properties of planar FORE control loops

In this section we characterize the exponential and $\mathcal{L}_2$ stability properties of the planar reset systems that consist of the FORE (6), interconnection conditions (11) and the following scalar linear plant

$$\begin{aligned} \dot{x}_p &= a_p x_p + b_p u + d, \\ y &= x_p \end{aligned} \tag{13}$$

where $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ is a disturbance input and $x_p \in \mathbb{R}$ is the plant state.

Note that the above plant has the same form as (12b). Hence, the results of this section are instrumental in showing stability for the class of SISO linear systems considered in Section 5. Indeed, results of this section are essential in the proof of the results of Theorem 2 in Section 5. In particular, in Theorem 3 we characterize asymptotic and exponential stability of the closed-loop giving necessary and sufficient conditions in terms of the system parameters. Then in Theorems 4 and 5 we provide results about the $\mathcal{L}_2$ gain of the planar reset system from the input d to the input y. We note that these results are of interest in their own right as they completely characterize stability properties of a class of planar reset systems.

The closed loop (13), (6), (11) can be conveniently written in the following form

$$\begin{aligned} &\left.\begin{aligned} \dot{\tau} &= 1, \\ \dot{x} &= Ax + Bd \end{aligned}\right\} \quad \text{if } x^T M x \geq 0 \text{ or } \tau \leq \rho, \\ &\left.\begin{aligned} \tau^+ &= 0, \\ x^+ &= A_r x \end{aligned}\right\} \quad \text{if } x^T M x \leq 0 \text{ and } \tau \geq \rho, \end{aligned} \tag{14}$$

where $x := (x_p, x_r) \in \mathbb{R}^2$ and

$$\left[\begin{array}{c|c} A & B \\ \hline A_r & M \end{array}\right] = \left[\begin{array}{cc|cc} a_p & b_p & 1 & \\ -b_c & a_c & 0 & \\ \hline 1 & 0 & -\varepsilon & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]. \tag{15}$$

where the sign in the selection of M comes from the negative feedback interconnection in (11) (see also the right sub-figure of Figure 1). The next theorem establishes necessary and sufficient conditions for the exponential stability and finite $\mathcal{L}_2$ gain (from d to x) of the planar FORE control system (14), (15). The proof is based on the Lyapunov results of Theorems 6 and 7. It is reported in Section 8.2.

Theorem 3 *(Stability conditions) Consider the planar FORE control system (14), (15) (i.e., the closed-loop system (13), (6), (11)) and suppose that the loop gain $b_p b_c$ is positive. Then the following statements are equivalent.*

1. *The origin is (locally) asymptotically stable conditionally to hierarchically small (ε, ρ).*
2. *The origin is globally exponentially stable conditionally to hierarchically small (ε, ρ).*
3. *The system is finite gain $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stable from d to (x_p, x_r) conditionally to hierarchically small (ε, ρ).*
4. *At least one of the following two conditions holds:*
   - *(a) the matrix A in (15) is Hurwitz;*
   - *(b) the following condition is satisfied:*

$$2\sqrt{b_p b_c} + a_c - a_p > 0. \tag{16}$$

Theorem 3 establishes necessary and sufficient conditions for exponential stability and $\mathcal{L}_2$ stability of planar FORE systems. Another interesting aspect to study is to understand how the $\mathcal{L}_2$ gain compares to the $\mathcal{L}_2$ gain of the closed-loop without resets (whenever it exists) and also the trend of the gain as certain parameters get large. In particular, it is commonly acknowledged by practical experience that introducing resets improves the performance of a linear planar control system, even though a formal proof of this fact was not available. Such a proof is given in the next Theorem 4, whose proof is reported in Section 8.2. Moreover, it has been already noticed by studying certain gain estimates in [43] that the $\mathcal{L}_2$ gain seems to become smaller as the loop gain and/or the pole of the FORE (namely a_c) becomes larger. This intuition arises from the fact that the step response generated by the closed loop is faster, by corresponding to the patching of an exponentially diverging branch (having larger growth rate) followed by a flat-top at the desired steady state (see also the simulations in [36, Figure 3]). However, a formal proof of these $\mathcal{L}_2$ gain trends has not been established yet. It is now given in the following Theorem 5, whose proof is also reported in Section A. For the correct statement of these theorems we need to clarify a suitable concept of gain estimate and of gain convergence, introduced in the next definition.

Definition 2 Consider the FORE control system with temporal regularization (14), (15). Assume that γ is an input/output gain. Then we say that $\bar{\gamma}$ is an asymptotic estimate of the gain γ conditionally to hierarchically small

(ε, ρ) or alternatively, that

$$\gamma \overset{\varepsilon,\rho}{\leq} \bar{\gamma},$$

if for each (arbitrarily small) $\delta > 0$ there exists ε^* such that for each $\varepsilon \in (0, \varepsilon^*]$ there exists ρ^* such that for all $\rho \in (0, \rho^*]$, $\gamma \leq \bar{\gamma} + \delta$.

Assume that p is a suitable parameter of the closed-loop system and that $\gamma(p)$ is an input/output gain depending on p. Then we say that $\gamma(p)$ converges to zero conditionally to hierarchically small (ε, ρ) as p tends to $+\infty$, or alternatively, that

$$p \to \infty \ \Rightarrow \gamma(p) \xrightarrow{\varepsilon,\rho} 0,$$

if for each (arbitrarily small) $\bar{\gamma} > 0$ there exists[4] $p^* > 0$ such that for each $p \geq p^*$ there exists ε^* such that for each $\varepsilon \in (0, \varepsilon^*]$ there exists ρ^* such that for all $\rho \in (0, \rho^*]$, $\gamma(p) \leq \bar{\gamma}$. $\circ$

Remark 5 The goal of Definition 2 is to clarify what we mean by gain estimate and convergence to a value in terms of the small parameters of the system. In particular, the gain estimates and trends established in the next theorem require that first the parameter ε characterizing the FORE resetting rule in (6) is sufficiently small and then that the temporal regularization constant ρ is once again sufficiently small. This hierarchical selection is necessary because larger (possibly unstable) FORE poles will cause larger state evolution and smaller selections of ρ will be necessary. However, for fixed parameters, there always exists a small enough ρ for which the theorem statements hold. Similarly, with reference to the second part of Definition 2, we note that in Theorem 5 we consider various situations when $p = a_c$ or $p = k := b_c b_p$ or $p = (a_c, k)$. In a design context, one should first fix the desired gain $\bar{\gamma}$, then choose p sufficiently large and then impose first ε sufficiently small and subsequently ρ sufficiently small. $\circ$

Theorem 4 *($\mathcal{L}_2$ gain estimates) Consider the planar FORE control system (14), (15) (i.e., the closed-loop system (13), (6), (11)) with temporal regularization) where the loop gain $k := b_p b_c$ is positive. Whenever the closed-loop is exponentially, stable (so that, by Theorem 3 at least one of the two conditions at item 4 of Theorem 3 holds), the following asymptotic estimates hold conditionally to hierarchically small (ε, ρ) (in the sense of Definition 2) for the $\mathcal{L}_2$ gain γ of the closed-loop from d to y:*

1. *if item 4a of Theorem 3 holds, then*

$$\gamma \overset{\varepsilon,\rho}{\leq} \gamma_L, \tag{17}$$

where γ_L is the (finite, because A is Hurwitz) gain from d to y of the linear closed-loop without resets.

2. *if item 4b of Theorem 3 holds, then*

$$\gamma \overset{\varepsilon,\rho}{\leq} \frac{2(2+\kappa)\exp(\kappa\frac{\pi}{2})}{\kappa(2\sqrt{b_c b_p} - \max\{a_p - a_c, 0\}) - 4\max\{|a_c|, |a_p|\}}, \tag{18}$$

where κ is any constant satisfying $\kappa > \overline{\kappa} := \frac{4\max\{|a_c|,|a_p|\}}{2\sqrt{b_c b_p} - \max\{a_p - a_c, 0\}}$.

[4]The parameter p is allowed to be a vector and in this case $p > 0$ means that each entry of p is strictly larger than zero.

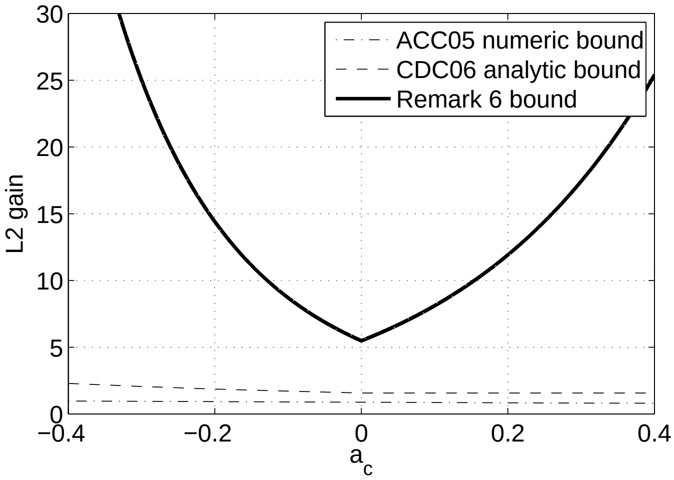

Figure 5: Comparison of the gain estimates obtained by using equation (19) from Remark 6 (bold), using the analytic tools from [44] (dashed) and the numerical tools from [43] (dash-dotted).

Remark 6 It is of interest to investigate whether for fixed values of the parameters there is an optimal selection of κ within (18) which gives the tightest estimate for the $\mathcal{L}_2$ gain. Indeed, by taking the derivative of the right-hand side of (18) with respect to κ and imposing that the derivative is zero, one gets two solutions (of a second order equation), one of them always being smaller than $\overline{\kappa}$ (thus not being usable) and one of them always being larger than $\overline{\kappa}$. In particular, the optimal κ is determined as

$$\kappa^* := \frac{\overline{\kappa}}{2} - 1 + \sqrt{\left(\frac{\overline{\kappa}}{2}+1\right)\left(\frac{\overline{\kappa}}{2}+1+\frac{4}{\pi}\right)},$$

and, when substituted into the gain bound equation (18) it gives the following value, which only depends on the system parameters:

$$\gamma^* = \frac{1+\kappa_0+\sqrt{\kappa_0(\kappa_0+2)}\exp\left(\kappa_1+\sqrt{\kappa_0(\kappa_0+2)}\right)}{2\sqrt{b_c b_p}-\max\{a_p-a_c,0\}} \tag{19}$$

where $\kappa_0 = \frac{\pi}{4}(\overline{\kappa}+2)$ and $\kappa_1 = \frac{\pi}{4}(\overline{\kappa}-2)$.

An example of the gain curve given by the function (19) is shown in Figure 5, when selecting $a_p = 0$ and $b_p b_c = 1$ and having a_c take values in $[-0.5, 0.5]$. This curve is compared to the gain estimates obtained when using the numerical and analytic tools given in [43] and [44], respectively. The latter estimates turn out to be tighter for this special case, but the advantage of this construction is that it provides an estimate of the gain for a larger class of systems (the constructions in [43] and [44] are limited to the case $a_p = 0$ and $b_p b_c = 1$). $\circ$

Theorem 5 *($\mathcal{L}_2$ gain trends) Consider the planar FORE control system (14), (15) (i.e., the closed-loop system (13), (6), (11)) where the loop gain $k := b_p b_c$ is positive. Let a_p be fixed. Denote by $\gamma(a_c, k)$ the $\mathcal{L}_2$ gain of the closed-loop from d to y as a function of the FORE pole a_c and of the loop gain $k := b_p b_c$. Then the following trends hierarchically conditioned by (ε, ρ) in the sense of Definition 2 hold for the closed-loop system:*

1. $k \to +\infty \;\Rightarrow \gamma(a_c, k) \xrightarrow{\varepsilon,\rho} 0,$

2. $a_c \to +\infty \;\Rightarrow \gamma(a_c, k) \xrightarrow{\varepsilon,\rho} 0$,

3. $k \to +\infty$ *and* $a_c \to +\infty \;\Rightarrow \gamma(a_c, k) \xrightarrow{\varepsilon,\rho} 0$,

namely, the $\mathcal{L}_2$ *gain of the closed-loop decreases to zero (conditionally to hierarchical selections of* (ε, ρ)*) as the loop gain and/or the FORE pole are increased.*

7 Properties of temporally regularized systems

In this section we provide a range of stability results for a class of nonlinear and a class of linear reset systems. These results are essential in proving our main results presented in earlier sections but they are of interest in their own right and, hence, they are stated here in more generality than what is needed in this paper.

7.1 Lyapunov conditions for exponential and $\mathcal{L}_2$ stability

Even though the rest of the paper is concerned with homogeneous dynamics, we switch to a nonlinear formulation here because the proof of the Lyapunov theorem reported next does not require linearity nor homogeneity of the reset system dynamics. The theorem reported here is an interesting result in its own right and it deals with a class of nonlinear reset systems. In this paper it will be instrumental for the proof of Theorems 3, 4 and 5.

We first establish formally the fact that the temporally regularized nonlinear system (3), does not have Zeno solutions.

Lemma 1 *All the solutions of (3) are uniformly non-Zeno.*

Proof. Given any solution $\xi(\cdot,\cdot) := (x(\cdot,\cdot), \tau(\cdot,\cdot))$ of (3) it is evident that $t_j - t_{j-1} \geq \rho$ for all $(t, j) \in \mathrm{dom}(x)$, $j \geq 2$. This implies that the uniformly non-Zeno definition in [21] (see also [18]) is satisfied with $T = \rho$ and $J = 2$. •

Then we formalize the requirement that there is a Lyapunov function for the x dynamics of system (3), which decreases both during flows and along jumps.

Assumption 1 *Given system (3), a suitable output* y *satisfying* $|y|^2 \leq \lambda_0 |x|^2$ *and an integer* $p \in [1, +\infty)$*, the locally Lipschitz Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is such that there exist positive real numbers* λ_i*,* $i = 1, \dots, 7$ *and* $\eta \in (0, 1)$ *such that for all* d*:*

$$\lambda_1 |x|^p \leq V(x) \leq \lambda_2 |x|^p \tag{20a}$$

$$\frac{\partial V(x)}{\partial x} f(x, d) \leq \lambda_3 V(x) + \lambda_4 |x|^{p-1} |d|, \qquad \textit{for almost all } x \in \mathbb{R}^n, \tag{20b}$$

$$\max_{v \in \partial V(x)} v^T f(x, d) \leq -\lambda_5 V(x) - \lambda_6 |y|^p + \lambda_7 |d|^p, \quad \forall x \in \mathcal{F}, \tag{20c}$$

$$V(g(x)) \leq \eta V(x), \quad \forall x \in \mathcal{J}. \tag{20d}$$

where $\partial V(x)$ *is the Clarke generalized gradient of* $V(\cdot)$ *at* x *(see [16] for its definition).*

Remark 7 Note that global asymptotic stability of the origin of

$$\begin{cases} \dot{x} &= f(x,d), \quad \text{if } x \in \mathcal{F}, \\ x^+ &= g(x), \quad \text{if } x \in \mathcal{J}, \end{cases} \tag{21}$$

(which has no temporal regularization) with $d = 0$ is equivalent to (20c) and (20d) with $d = 0$ and $V(\cdot)$ positive definite and radially unbounded. These conditions, a subset of the conditions in Assumption 1, are sufficient to establish semiglobal practical asymptotic stability for the system (21) augmented with temporal regularization, i.e., the system (3). For details, see [21] or [11]. Here we also impose the stronger conditions (20a) and (20b) because we are interested in establishing global asymptotic stability results, in addition to properties of the system with disturbance inputs. This is why the extra conditions (20a) and (20b) are required. $\circ$

The following theorem establishes the sufficiency of the Lyapunov conditions of Assumption 1 to establish the exponential and $\mathcal{L}_p$ stability properties of the reset system with temporal regularization (3). This theorem should be thought of as a valuable alternative to the approaches in [36, 35, 43, 44, 45] to analyze reset systems with temporal regularization. In particular, as compared to [36, Theorem 1], the next theorem exploits the strict decrease at jumps (which was not required in [36]) to compensate for a possible growth of V outside the flow set $\mathcal{F}$. This was not possible in [36] because the Lyapunov functions were only required to not increase at jumps, thereby extra flow conditions had to be imposed on a slightly inflated version of the flow set. This new technique leads to a simpler proof than that of [36]. See Section 3 for a qualitative discussion of this fact. Note also that the results in [36] were stated for linear dynamics while we allow the nonlinear case and, in addition, we don't enforce here the assumption required in [36] that after a jump the state belongs to the flow set.

Theorem 6 *Given an integer $p \in [1, +\infty)$, assume that there exists a function $V(\cdot)$ satisfying Assumption 1. Then if $f(\cdot, \cdot)$ is continuous in its first argument, the reset system with temporal regularization (3) satisfies the following:*

1. *there exists $\rho^* > 0$ such that for any $\rho \in (0, \rho^*]$ the origin of the x dynamics with $d = 0$ is exponentially stable;*
2. *the system is finite gain $\mathcal{L}_p$ stable from d to y and for any $\epsilon > 0$, there exists ρ^* such that for all $\rho \leq \rho^*$ the $\mathcal{L}_p$ gain from d to y is upper bounded by $\left(\frac{\lambda_7}{\lambda_6}\right)^{1/p} + \epsilon$.*

Proof. For clarity of exposition we carry out the proof for the case $p = 2$. In the general case, the proof should be extended by using Young's inequality before (24) as follows:

$$\lambda_4 |x|^{p-1}|d| \leq (p-1) \sqrt[p-1]{\frac{1}{\lambda_7}\left(\frac{\lambda_4}{p}\right)^p} |x|^p + \lambda_7 |d|^p \tag{22}$$

while the rest of the proof remains substantially unchanged. [5]

Since equation (20b) holds almost everywhere in $\mathbb{R}^n$, and around each point where the condition does not hold there's a full measure set of points where it holds, then following the reasoning in [38, page 100], it follows that

$$\max_{v \in \partial V(x)} v^T f(x,d) \leq \lambda_3 V(x) + \lambda_4 |x||d|, \quad \forall x \in \mathbb{R}^n. \tag{23}$$

[5] For $p = 1$, (22) reduces to $\lambda_4|d|$ and λ_6 in (26) should be replaced by $\frac{\lambda_4 \lambda_6}{\lambda_7}$.

Adding to the right hand side the square $\lambda_7(|d| - \frac{\lambda_4}{2\lambda_7}|x|)^2$ we get

$$\max_{v \in \partial V(x)} v^T f(x,d) \leq \lambda_8 V(x) + \lambda_7 |d|^2, \quad \forall x, \tag{24}$$

where $\lambda_8 = \lambda_3 + \frac{\lambda_4^2}{4\lambda_7\lambda_1}$.

Given any $\alpha > 1$, define the function $W : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as follows:

$$W(\tau, x) := \exp(-L \min\{\tau, \alpha\rho\}) V(x) \tag{25}$$

where $L > 0$ is to be determined. Denote $\xi := (\tau, x)$. By [16, p. 48], $\partial W(\tau, x) \subseteq \partial_\xi e(\tau) V(x) + e(\tau)\partial_\xi V(x)$, where $e(\tau) = \exp(-L \min\{\tau, \alpha\rho\})$ and ∂_ξ denotes the generalized gradient with respect to ξ, which is zero in the x direction for $e(\tau)$ and is zero in the τ direction for $V(x)$. Then, we get for all $x \in \mathbb{R}^n$ and all $\tau \in [0, \rho]$

$$\begin{aligned}
\max_{v \in \partial W(\tau,x)} v^T \begin{bmatrix} 1 \\ f(x,d) \end{bmatrix} &\leq \max_{v \in \partial_\xi e(\tau)} V(x) v^T \begin{bmatrix} 1 \\ f(x,d) \end{bmatrix} + \max_{v \in \partial_\xi V(x)} e(\tau) v^T \begin{bmatrix} 1 \\ f(x,d) \end{bmatrix} \\
&= -L\exp(-L\tau)V(x) + \exp(-L\tau) \max_{v \in \partial V(x)} v^T f(x,d) \\
&\leq -L\exp(-L\tau)V(x) + \exp(-L\tau)\left(\lambda_8 V(x) + \lambda_7 |d|^2\right) \\
&\leq W(\tau,x)\left(-L + \lambda_8 + \tfrac{\lambda_6\lambda_0}{\lambda_1}\right) - \lambda_6 \exp(-L\tau)|y|^2 + \lambda_7|d|^2 \ .
\end{aligned} \tag{26}$$

where we used (24) in the next to last step and added the term $\exp(-L\tau)\left(-\lambda_6|y|^2 + \frac{\lambda_6\lambda_0}{\lambda_1}V(x)\right)$, which is positive by (20a), in the last step.

Similarly, noting that for all τ, $\max_{v \in \partial_\xi e(\tau)} V(x) v^T \left[\begin{smallmatrix} 1 \\ \star \end{smallmatrix}\right] \leq 0$ (where $\star$ denotes "don't care") and also using (20c), we get for all $x \in \mathcal{F}$:

$$\begin{aligned}
\max_{v \in \partial W(\tau,x)} v^T \begin{bmatrix} 1 \\ f(x,d) \end{bmatrix} &\leq \exp(-L\min\{\tau,\alpha\rho\}) \max_{v \in \partial V(x)} v^T f(x,d) \\
&\leq \exp(-L\min\{\tau,\alpha\rho\})(-\lambda_5 V(x) - \lambda_6|y|^2 + \lambda_7|d|^2) \\
&\leq -\lambda_5 W(\tau,x) - \lambda_6 \exp(-L\alpha\rho)|y|^2 + \lambda_7|d|^2 \ .
\end{aligned} \tag{27}$$

Combining the two bounds (26) and (27) and selecting $L \geq \lambda_5 + \lambda_8 + \frac{\lambda_6\lambda_0}{\lambda_1}$ we have

$$\max_{v \in \partial W(\tau,x)} v^T \begin{bmatrix} 1 \\ f(x,d) \end{bmatrix} \leq -\lambda_5 W(\tau,x) - \exp(-L\alpha\rho)\lambda_6|y|^2 + \lambda_7|d|^2, \quad \forall x \in \mathcal{F} \text{ or } \tau \leq \rho, \tag{28}$$

which corresponds to the flow set condition in (3).

Consider the change in W due to jumps. We have

$$\begin{aligned}
W(0, g(x)) &= V(g(x)) \\
&\leq \eta V(x) \\
&\leq \eta \exp(L\alpha\rho) W(\tau, x) \ .
\end{aligned} \tag{29}$$

Therefore, selecting $\rho \leq \rho^* = \dfrac{\log(1/\eta)}{\alpha L}$, we have

$$W(0, g(x)) \leq W(\tau, x), \quad \forall x \in \mathcal{J} \text{ and } \tau \geq \rho, \tag{30}$$

which corresponds to the jump set condition in (3).

The proof is completed integrating equations (28) and (30) along the trajectories of the system to derive an exponential bound on $|x|$ and the $\mathcal{L}_2$ bound from $\|d\|_2$ to $\|y\|_2$. In particular, by the results in [38, page 99], the good bounds on the generalized gradients of W imply that the time derivative of W evaluated along trajectories satisfies the same bound. $\bullet$

Remark 8 The use of generalized gradients in Theorem 6 is motivated by the fact that the result is used in Theorems 3–5 with locally Lipschitz Lyapunov functions. Note that it is not sufficient to impose the flow conditions (20c) stated for the gradient of V for almost all x. More specifically, for (20b) it is sufficient to restrict the attention to almost everywhere because by continuity of f, for each disregarded point there is a full measure set of points where the condition holds, and (23) holds. However, this reasoning doesn't hold for (20c) where the condition is restricted to the set $\mathcal{F}$ for which no extra assumptions hold. In particular, one can construct defective cases with *thin* selections of $\mathcal{F}$, namely sets of measure zero, so that imposing a flow condition almost everywhere in $\mathcal{F}$ corresponds to not imposing it at all.

It should be also emphasized that, different from here, in [36, Theorems 1 & 2] the flow condition has been imposed on the gradient of V almost everywhere in the flow set. This is sufficient there because in [36] the conditions on the gradient are required almost everywhere in an inflated version $\mathcal{F}_\epsilon$ of the flow set $\mathcal{F}$, such that no points of $\mathcal{F}$ except for the origin belong to the boundary of $\mathcal{F}_\epsilon$. Then except for the origin, any point in $\mathcal{F}$ has a neighborhood contained in $\mathcal{F}_\epsilon$, which allows to establish a flow condition on the generalized gradient of V everywhere except for the origin. This is enough to apply the results of [38] and obtain a global flow condition on the generalized gradient. $\circ$

7.2 Stability of linear reset systems acting on cones

In this section we will further restrict the class of systems analyzed in the previous section to the case of linear dynamics acting on jump and flow sets that are cones. Under this homogeneity property, based on the results that recently appeared in [39, 22], parallel properties to the well-known properties of linear (discrete-time or continuous-time) systems are proved:

- (local) asymptotic stability of the origin $\Leftrightarrow$ global exponential stability,
- global exponential stability $\Leftrightarrow$ finite gain $\mathcal{L}_p$ and finite gain $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stability from d to x with $p \in [1, +\infty)$,
- global exponential stability $\Leftrightarrow$ finite gain exponential ISS from d to x.

The results reported here are instrumental for the proof of Theorem 1 in Section 4 and of Theorem 2 in Section 5.

More specifically, we focus on the following class of temporally regularized linear reset systems, which generalizes the systems characterized by (14):

$$\left.\begin{aligned} \dot{\tau} &= 1, \\ \dot{x} &= Ax + Bd \end{aligned}\right\} \quad \text{if } x \in \mathcal{F} \text{ or } \tau \in [0,\ \rho], \qquad\qquad (31)$$
$$\left.\begin{aligned} \tau^+ &= 0, \\ x^+ &= A_r x \end{aligned}\right\} \quad \text{if } x \in \mathcal{J} \text{ and } \tau \geq \rho,$$

where $x \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. We assume that (31) satisfies the following assumption.

Assumption 2 *The sets $\mathcal{F}$ and $\mathcal{J}$ are closed nonempty subsets of $\mathbb{R}^n$. Moreover, they both are cones namely, for each $\lambda > 0$ and each $x \in \mathbb{R}^n$, $x \in \mathcal{F} \Rightarrow \lambda x \in \mathcal{F}$ and $x \in \mathcal{J} \Rightarrow \lambda x \in \mathcal{J}$.*

Theorem 7 *If Assumption 2 holds, then the following statements are equivalent:*

1. *the origin of the x dynamics of (31) with $d = 0$ is (locally) asymptotically stable;*
2. *the origin of the x dynamics of (31) with $d = 0$ is globally exponentially stable;*
3. *given $p \in [1, +\infty)$, system (31) is finite gain $\mathcal{L}_p$ stable and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stable from d to x;*
4. *system (31) is finite gain exponentially ISS from d to x.*
5. *given $p \in [1, \infty)$, there exists a function $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is continuously differentiable in $\mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ and positive constants λ_i, $i = 1, \ldots, 8$ and $\nu \in (0, 1)$ satisfying for all d the following bounds:*

$$\lambda_1 |x|^p \ \leq \ W(\tau, x) \ \leq \ \lambda_2 |x|^p, \quad \forall (\tau, x) \in \mathbb{R} \times \mathbb{R}^n \tag{32a}$$

$$\nabla_\tau W(\tau, x) + < \nabla_x W(\tau, x), Ax + Bd > \leq \ -\lambda_5 W(\tau, x) - \lambda_6 |x|^p + \lambda_7 |x|^{p-1} |d|, \quad \forall (\tau, x) \in \bar{\mathcal{F}} \setminus \{0\} \tag{32b}$$

$$W(0, A_r x) \leq \nu W(\tau, x), \quad \forall (\tau, x) \in \bar{\mathcal{J}} \tag{32c}$$

$$|\nabla_x W(\tau, x)| \ \leq \ \lambda_8 |x|^{p-1}, \quad \forall (\tau, x) \in \bar{\mathcal{F}} \setminus \{0\}, \tag{32d}$$

where $\bar{\mathcal{F}} := \{(\tau, x) :\ (x \in \mathcal{F}$ and $\tau \in [\rho, 1+\rho])$ or $\tau \in [0, \rho]\}$ and $\bar{\mathcal{J}} := \{(\tau, x) :\ (x \in \mathcal{J}$ and $\tau \in [\rho, 1+\rho])$ or $\tau \in (-\infty, 0] \bigcup [1 + \rho, +\infty)\}$.

The proof of Theorem 7 is carried out by introducing the following system

$$\left\{ \begin{array}{ll} \left. \begin{array}{l} \dot{\tau} = \min\{1, 1 + \rho - \tau\}, \\ \dot{x} = Ax + Bd \end{array} \right\} & \text{if } (x \in \mathcal{F} \text{ and } \tau \in [\rho, 1 + \rho]) \text{ or } \tau \in [0, \rho], \\ \left. \begin{array}{l} \tau^+ = 0, \\ x^+ = A_r x \end{array} \right\} & \text{if } x \in \mathcal{J} \text{ and } \tau \in [\rho, 1 + \rho], \\ \left. \begin{array}{l} \tau^+ = 0, \\ x^+ = 0 \end{array} \right\} & \text{if } \tau \in (-\infty, 0] \bigcup [\rho, +\infty), \end{array} \right. \tag{33}$$

which is more convenient than (31) because for any initial condition the clock variable τ converges to the compact set $[0, 1 + \rho]$. Then, Theorem 7 is proven by relying on the following lemma.

Lemma 2 *The bounds in items 1 and 2 hold for the solutions of (31) with $d = 0$ if and only if they hold for the solutions of (33) with $d = 0$.*

Proof. Consider the invertible transformation $\Sigma : \mathbb{R}_{\geq 0} \to [0, 1 + \rho)$:

$$\Sigma(\tau) = \begin{cases} \tau, & \text{if } \tau \in [0, \rho], \\ \rho + 1 - \exp(\rho - \tau) & \text{if } \tau \geq \rho, \end{cases}$$

whose inverse is given by

$$\Sigma^{-1}(\tau_a) = \begin{cases} \tau_a, & \text{if } \tau_a \in [0, \rho], \\ \rho - \log(1 + \rho - \tau_a) & \text{if } \tau_a \in [\rho, 1 + \rho). \end{cases}$$

It can be verified by direct calculation that given any solution $(\tau(\cdot,\cdot), x(\cdot,\cdot))$ of (31) with $d = 0$, the function $(\Sigma(\tau(\cdot,\cdot)), x(\cdot,\cdot))$ is a solution of the auxiliary system (33) with $d = 0$. Therefore any bound on the x response of (33), also holds for the x response of (31). Conversely, given any solution $(\tau_a(\cdot,\cdot), x_a(\cdot,\cdot))$ of the auxiliary system (33) with $d = 0$, denote by (t^*, j^*) the smallest hybrid time when the solution obeys the bottom jump rule in (33), where possibly $(t^*, j^*) = (0, 0)$ or $(t^*, j^*) = \infty$. Then for all $(t, j) \geq (t^*, j^*)$ (this should be disregarded if $(t^*, j^*) = \infty$), and $(t, j) \in \operatorname{dom}((\tau_a, x_a))$, $x_a(t, j) = 0$ and for all $(t, j) < (t^*, j^*)$ (this means for all (t, j) if $(t^*, j^*) = \infty$), and $(t, j) \in \operatorname{dom}((\tau_a, x_a))$, the function $(\Sigma^{-1}(\tau_a(\cdot,\cdot)), x_a(\cdot,\cdot))$ is a solution of (31). Therefore any bound on the x response of (31), also holds for the x response of (33). •

Proof of Theorem 7.

1 ⇒ 2. Consider a sufficiently small scalar $s > 0$ so that the asymptotic stability set $\mathcal{B}$ contains the set $\{x : |x| \leq s\}$. Consider now any initial condition $x(0, 0), \tau(0, 0)$ and denote by $z(t, j)$ the response of the x dynamics from the initial condition $z(0, 0), \tau(0, 0)$ with $z(0, 0) = \min\left\{\frac{s}{|x(0,0)|}, 1\right\} x(0, 0)$. From Assumption 2, given any solution $(x(\cdot,\cdot), \tau(\cdot,\cdot))$ of (31) and any $\lambda > 0$, $(\lambda x(\cdot,\cdot), \tau(\cdot,\cdot))$ is a solution and, since $z(0, 0) \in \mathcal{B}$ it follows that

$$|x(t, j)| = \max\left\{\frac{|x(0, 0)|}{s}, 1\right\} |z(t, j)| \leq \max\left\{\frac{|x(0, 0)|}{s}, 1\right\} \beta(\min\{|x(0, 0)|, s\}, t) =: \beta_G(|x(0, 0)|, t),$$

which establishes a global $\mathcal{KL}$ bound on the x response.

To obtain a $\mathcal{KLL}$ bound from the $\mathcal{KL}$ bound above, note that because of temporal regularization, for any $t, j \in \operatorname{dom}(x)$, $j \geq 1$, $\rho \leq t_{j+1} - t_j$. Taking a sum on both sides, for any $(t, k) \in \operatorname{dom}(x)$ we get $k\rho \leq \sum_{j=1}^{k} t_{j+1} - t_j = t_{k+1} - t_1$, which implies for all $t \in [t_{k+1}, t_{k+2}]$, $k\rho \leq t$. Therefore, for all $t, k \in \operatorname{dom}(x)$, we have $\max\{k - 1, 0\}\rho \leq t$, which implies $\frac{t}{2} + \frac{\max\{k-1,0\}\rho}{2} \leq t$ (see also the similar discussion in [20, pp. 34-35]). Define now the class $\mathcal{KLL}$ function $\bar{\beta}(s, t, j) := \beta_G\left(s, \frac{t}{2} + \frac{\max\{j-1,0\}\rho}{2}\right)$. Then for all $t, j \in \operatorname{dom}(x)$, we have $\bar{\beta}(|x(0, 0)|, t, j) \geq \beta_G(|x(0, 0)|, t)$ and, consequently,

$$|x(t, j)| \leq \beta_G(|x(0, 0)|, t) \leq \bar{\beta}(|x(0, 0)|, t, j), \tag{34}$$

which provides a class $\mathcal{KLL}$ bound on $|x(t, j)|$.

Based on Lemma 2, the bound (34) holds also for the solutions of (33) with $d = 0$ and moreover all solutions to (33) in the τ variable converge to the set $[0, 1 + \rho]$. Then, applying [11, Theorem 7.9], we get that the class $\mathcal{KLL}$ bound is robust because the asymptotically stable set $\mathcal{A} = \{(x, \tau) : x = 0, \tau \in [0, 1 + \rho]\}$ is compact. In light of the robust bound (34), consider now system (33) and note that by Assumption 2, it satisfies the Standing Assumption 1 in [39] and [39, Definition 5] with, $\mathcal{O} = \mathbb{R}^{n+1}$, $\Gamma(s, (\tau, x)) = sx$ and $\delta = 0$. Moreover, the bound (34) establishes a $\mathcal{KLL}$ bound in the sense of [39, Definition 1] with $\omega(\tau, x) = |x|$. Then, all the assumptions of [39, Theorem 2] with $\theta = \kappa = 1$ and $\vartheta \equiv \omega$ are satisfied and by item 2 of [39, Proposition 3] there exist $m, \ell > 0$ such that $|x(t, j)| \leq m|x(0, 0)|e^{-\ell t}$, $\forall (t, j) \in \operatorname{dom}(x)$, for all solutions to (33) with $d = 0$. Applying once again Lemma 2, the exponential bound applies to all the solutions of (31) as to be proven.

2 ⇒ 5. Similar to the previous step, by Lemma 2, if the exponential bound holds for all the solutions of (31), then it also holds for all the solutions of (33).

Using the same function Γ as in the previous step, all the assumptions of [39, Theorem 2] with $\theta = \kappa = 1$ and $\vartheta \equiv \omega$ are satisfied again. Therefore there exist positive constants $\mu > 0$ and $\nu \in (0,1)$ and a continuous function $W : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is smooth on $\mathbb{R}\times\mathbb{R}^n \setminus \{0\}$ and homogeneous with degree p with respect to Γ (namely satisfying $W(\tau, \lambda x) = \lambda^p W(\tau, x)$ for all τ, x and $\lambda > 0$) such that for some $\alpha_1(\cdot)$ and $\alpha_2(\cdot) \in \mathcal{K}_\infty$:

$$
\begin{aligned}
\alpha_1(|x|) \leq W(\tau, x) &\leq \alpha_2(|x|), && \forall(\tau, x) && (35a)\\
\nabla_\tau W(\tau, x) + < \nabla_x W(\tau, x), Ax > &\leq -\mu W(\tau, x), && \forall(\tau, x) \in \bar{\mathcal{F}} \setminus \{0\}, && (35b)\\
W(0, A_r x) &\leq \nu W(\tau, x), && \forall(\tau, x) \in \bar{\mathcal{J}}. && (35c)
\end{aligned}
$$

[(32c), (32a)] Equation (35c) coincides with (32c). Define $\lambda_1 := \alpha_1(1)$ and $\lambda_2 := \alpha_2(1)$. Then, since $W(\cdot)$ is homogeneous of degree p with respect to Γ (that is, $W(\tau, sx) = s^p W(\tau, x)$ for any $s > 0$, and any τ, x), given any τ and $x \in \mathbb{R}^n \setminus \{0\}$, $W(\tau, x) = W\left(\tau, |x|\frac{x}{|x|}\right) = |x|^p W\left(\tau, \frac{x}{|x|}\right)$, which from (35a) leads to (32a).

[(32d)] Consider now the directional derivative of $W(\tau, \cdot)$ at any $x \neq 0$ in the direction w given by

$$< \nabla_x W(\tau, x), w >= \lim_{h\to 0} \frac{W(\tau, x + hw) - W(\tau, x)}{h},$$

and define the unit vector $z := \frac{x}{|x|}$ and $\bar{h} := \frac{h}{|x|}$. Then, since $W(\tau, \cdot)$ is homogeneous of degree p, we get

$$
\begin{aligned}
\frac{W(\tau, x + hw) - W(\tau, x)}{h} &= \frac{W\left(\tau, |x|z + |x|\bar{h}w\right) - W(\tau, |x|z)}{|x|\bar{h}}\\
&= |x|^{p-1}\frac{W\left(\tau, z + \bar{h}w\right) - W(\tau, z)}{\bar{h}},
\end{aligned}
$$

therefore, since w is arbitrary, for any $x \neq 0$, $\nabla_x W(\tau, x) = |x|^{p-1}\nabla_x W\left(\tau, \frac{x}{|x|}\right)$. Since $W(\cdot, \cdot)$ is smooth, there exists $\lambda_8 \in \mathbb{R}_{\geq 0}$ such that $\max\limits_{\tau\in[0,1+\rho],|z|=1} |\nabla_x W(\tau, z)| \leq \lambda_8$, which combined with the previous bound gives (32d).

[(32b)] Using equations (35b) and trading some of the μ to get (from (32a)) a good term in $|x|^p$, also using (32d), we get (32b) with $\lambda_5 = \frac{\mu+1}{2}$, $\lambda_6 = \frac{1-\mu}{2\lambda_1}$ and $\lambda_7 = \lambda_8|B|$.

5 ⇒ 3. Consider the function $U(t, j) = W(\tau(t, j), x(t, j))$, where $(\tau(t, j), x(t, j))$ is a solution of the hybrid system having hybrid time domain $\mathrm{dom}(x) := [t_0, t_1] \times 0 \bigcup [t_1, t_2] \times 1 \bigcup \cdots$. Then, using Young's inequality in (32b) we get:

$$\dot{U}(t, j) \quad\leq\quad -k_x|x(t, j)|^p + k_d|d(t, j)|^p, \qquad \text{for almost all } t \in [t_j, t_{j+1}], \tag{36}$$

where $k_x = \frac{\lambda_6}{2}$ and $k_d = \frac{\lambda_6}{2(p-1)}\left(\frac{2(p-1)\lambda_7}{p\lambda_6}\right)^p$ and using the same approach as in [9] the hybrid domain of the disturbance d is selected as $\mathrm{dom}(d) = \mathrm{dom}(x)$. Consider now any $(t, j) \in \mathrm{dom}(x)$ and denote for simplicity $t_{j+1} = t$. Then, integrating equation (36) and rearranging (32c) gives:

$$
\begin{aligned}
&0 \leq -U(t_{k+1}, k) + U(t_k, k) - k_x\int_{t_k}^{t_{k+1}} |x(s, k)|^p ds + k_d\int_{t_k}^{t_{k+1}} |d(s, k)|^p ds, \quad k = 0,\ 1,\ \ldots,\ j, && (37a)\\
&0 \leq -U(t_k, k) + U(t_k, k-1), \quad k = 1,\ \ldots,\ j. && (37b)
\end{aligned}
$$

Summing up all the $2j+1$ equations in (37), many terms are evidently simplified and the following bound is derived (recall that $t = t_{j+1}$):

$$\begin{aligned} U(t,j) &\leq U(t_0,0) - k_x \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} |x(s,i)|^p ds + k_d \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} |d(s,i)|^p ds, \\ &\leq U(t_0,0) - k_x \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} |x(s,i)|^p ds + k_d \|d\|_p^p, \quad \forall (t,j) \in \mathrm{dom}(x). \end{aligned} \tag{38}$$

By the left inequality in (32a), the bound (38) guarantees the desired finite gain $\mathcal{L}_p$ to $\mathcal{L}_\infty$ bound with finite gain $\gamma_{p,\infty} = \left(\frac{k_d}{c_1}\right)^{1/p}$. Moreover, (38) guarantees the desired finite gain $\mathcal{L}_p$ property with $\gamma_p = \left(\frac{2k_d}{k_x}\right)^{1/p}$.

3 ⇒ 1. Based on the finite gain $\mathcal{L}_p$ and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ bounds, we use [6] [33, Theorem 3] to conclude exponential stability when $d = 0$. Indeed, we have that all conditions of [33, Proposition 2] hold in our case and, hence, the closed loop system is uniformly globally fixed time interval stable (UGFTIS) with linear gain (see [33, Definition 6]). This implies that all the conditions of [33, Theorem 3] hold and, hence, we can conclude that the x dynamics of the system is UGES.

5 ⇒ 4. Since the function $W(\cdot,\cdot)$ satisfies the assumptions of [9, Proposition 2.6] with $\omega(x,\tau) = |x|$, then by [9, Proposition 2.7] (see also [9, Definition 2.1] the system is ISS from d to x. Finite gain exponential ISS can be derived by slightly generalizing the proof in [9] using the Lyapunov function. Alternatively, the bound can be directly proven by nesting the bounds given in the proof (5 ⇒ 3) above. The details are omitted because they are straightforward.

4 ⇒ 1. Local asymptotic stability trivially follows from the first term of the right hand side of the ISS bound (4). •

8 Proof of Theorems 1–5

8.1 Proof of Theorems 1 and 2

Proof of Theorem 1 Since $-C_pA_p^{-1}B_{pu}$ is the static plant gain, then by the assumption on the zeros of the plant it follows that $-C_pA_p^{-1}B_{pu} \neq 0$. Therefore, F in (8) is well defined and corresponds to the inverse of the static plant gain. Consider now the dynamics (7), (6), (9) and for any $r \in \mathbb{R}$ perform the change of coordinates $x_p \to \tilde{x}_p := x_p - x_p^*$, where x_p^* is a vector satisfying the following (always solvable) set of equations: $A_p x_p^* = -B_{pu} x_r^*$, $C_p x_p^* = r$ (note that if A_p is invertible, then $x_p^* = -A_p^{-1} B_{pu} x_r^*$, otherwise $x_p^* \in \ker(A_p)$ such that $C_p x_p^* = r$). Then the arising dynamics

[6] The results in [33] were stated for a special class of hybrid systems that does not include reset systems. However, all the results of [33] that we use can be restated for general hybrid models that include the class of reset systems considered in this paper.

corresponds to

$$\left.\begin{aligned}&\dot{\tau} = 1,\\ &\dot{\tilde{x}}_p = A_p\tilde{x}_p + B_{pu}x_r + B_{pu}\Delta F r^* + B_{pd}d\\ &\dot{x}_r = a_c x_r + b_c\tilde{y}\end{aligned}\right\} \quad \text{if } \varepsilon\tilde{y}^2 - 2\tilde{y}x_r \geq 0 \text{ or } \tau \leq \rho,$$

$$\left.\begin{aligned}&\tau^+ = 0,\\ &\tilde{x}_p^+ = \tilde{x}_p,\\ &x_r^+ = 0\end{aligned}\right\} \quad \text{if } \varepsilon\tilde{y}^2 - 2\tilde{y}x_r \leq 0 \text{ and } \tau \geq \rho,$$

(where $\tilde{y} = C_p\tilde{x}_p$) which coincide with the dynamics (7), (6), (9), with $r = 0$, an extra constant disturbance proportional to $\Delta F r^*$ acting on the plant flow equation, and x_p replaced by $\tilde{x}_p$. Therefore, by assumption, the origin of the reset closed-loop with $\Delta F = 0$ and $d = 0$ in the $(\tilde{x}_p, x_r)$ coordinates is asymptotically stable. Applying Theorem 7 we conclude that the origin of $(\tilde{x}_p, x_r)$ is globally exponentially stable (this proves item 1a) and that (39) is finite gain $\mathcal{L}_p$ and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stable for all $p \in [1,\infty)$ and finite gain exponentially ISS from the disturbance $B_{pu}\Delta F r^* + B_{pd}d$ to the state $(\tilde{x}_p, x_r)$. Item 1b then follows from the $\mathcal{L}_p$ and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stability properties. Finally, item 2 follows from the finite gain exponential ISS property. •

Proof of Theorem 2 We only prove the first case as the proofs of the two other cases follow almost identical steps by using the results of Theorem 5. Consider the overall system as a feedback interconnection between the linear system (12a) having inputs (y, d) and output z and the second order reset system consisting of (12b), (6), (11) that has inputs (z, d) and output y. From item 2 of Theorem 5 we have that the gain of the reset system from $\tilde{d} := C_z z + E_d d$ to y can be reduced arbitrarily by adjusting a_c, ε and ρ. Hence, the gain from (z, d) to y can be reduced arbitrarily for the reset system. Since A_z in the linear system (12a) is Hurwitz, then the system is finite gain $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stable from (d, y) to z with some $\mathcal{L}_2$ gain γ_z. Hence, there exist sufficiently large a_c and sufficiently small ε and ρ such that the small gain condition (see also [33])

$$\gamma(a_c) \cdot \gamma_z < 1$$

holds, which implies that the closed loop system (12a), (12b), (6), (11) is finite gain $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stable from d to (z, y). This completes the $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stability proof. Global exponential stability then follows from the equivalence between items 3 and 2 in Theorem 7. •

8.2 Proof of Theorems 3 and 4

We prove Theorems 3 and 4 together, by applying Theorem 6 and Theorem 7. The proof is carried out based on a preliminary step and based on the following steps (in different order):

Step S1 Proves that item 4a of Theorem 3 implies the existence of a Lyapunov function satisfying the conditions of Theorem 6 and guaranteeing (via Theorem 6) item 2 of Theorem 3 and equation (17) of Theorem 4.

Step S2 Proves that item 4b of Theorem 3 implies the existence of a Lyapunov function satisfying the conditions of Theorem 6 and guaranteeing (via Theorem 6) item 2 of Theorem 3 and equation (18) of Theorem 4.

Step N Proves that if both the conditions at items 4a and 4b of Theorem 3 don't hold, then the reset system generates responses not converging to zero (therefore, items 1 and 2 of Theorem 3 cannot hold).

Theorem 4 is then proven from Steps S1 and S2. As for Theorem 3, we have ($1 \Leftrightarrow 2$) and ($1 \Leftrightarrow 3$) from Theorem 7. Moreover, ($4 \Rightarrow 2$) from Steps S1 and S2 and ($\neg 4 \Rightarrow \neg 2$, namely $2 \Rightarrow 4$) from Step N, which completes the proof.

Preliminary step. Consider system (14), (15) without temporal regularization and perform the change of coordinates $\tilde{x} = Tx$, where $T := \left[\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{b_p/b_c} \end{smallmatrix}\right]$ to get

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}d, & \text{if } \tilde{x} \in \tilde{\mathcal{F}}, \\ \tilde{x}^+ = \tilde{A}_r\tilde{x}, & \text{if } \tilde{x} \in \tilde{\mathcal{J}}, \end{cases} \tag{39}$$

where $\tilde{\mathcal{F}} := \{\tilde{x} :\ \tilde{x}\tilde{M}\tilde{x} \geq 0\}$ and $\tilde{\mathcal{J}} := \{\tilde{x} :\ \tilde{x}\tilde{M}\tilde{x} \leq 0\}$ with the following parameters selection:

$$\begin{aligned} \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{A}_r & \tilde{M} \end{array}\right] &= \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline TA_rT^{-1} & T^{-1}\sqrt{\frac{b_p}{b_c}}MT^{-1} \end{array}\right] \\ &= \left[\begin{array}{cc|cc} a_p & \sqrt{b_pb_c} & 1 & \\ -\sqrt{b_pb_c} & a_c & 0 & \\ \hline 1 & 0 & -\tilde{\varepsilon} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{aligned} \tag{40}$$

with $\tilde{\varepsilon} = \varepsilon\sqrt{b_p/b_c}$. The rest of the proof will be carried out in the coordinates $\tilde{x}$.

Step S2. Define $\theta(\cdot) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ and $r(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ implicitly through the relationship:

$$\tilde{x} = \begin{bmatrix} r(\tilde{x})\sin(\theta(\tilde{x})) \\ r(\tilde{x})\cos(\theta(\tilde{x})) \end{bmatrix}. \tag{41}$$

Differentiating (41) with respect to $\tilde{x}$ and using $|\tilde{x}|^2 = r(\tilde{x})^2$, we get

$$\begin{aligned} I &= \nabla r(\tilde{x}) \begin{bmatrix} \sin(\theta(\tilde{x})) \\ \cos(\theta(\tilde{x})) \end{bmatrix}^T + \nabla\theta(\tilde{x}) \left(J \begin{bmatrix} r(\tilde{x})\sin(\theta(\tilde{x})) \\ r(\tilde{x})\cos(\theta(\tilde{x})) \end{bmatrix} \right)^T \\ &= \tfrac{\nabla r(\tilde{x})}{r(\tilde{x})}\tilde{x}^T + \nabla\theta(\tilde{x})(J\tilde{x})^T, \end{aligned} \tag{42}$$

where $J := \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Then, right-multiplying (42) by $J\tilde{x}$ and $\tilde{x}$, respectively, and using $J^TJ = I$, $J + J^T = 0$ and $|\tilde{x}|^2 = r(\tilde{x})^2$ the following two equations are respectively obtained:

$$\begin{aligned} J\tilde{x} &= r(\tilde{x})^2\nabla\theta(\tilde{x}) \\ \tilde{x} &= r(\tilde{x})\nabla r(\tilde{x}). \end{aligned} \tag{43}$$

Define $\theta_{\tilde{\epsilon}} := \arctan(\tilde{\epsilon}/2)$ and $\Theta_j := \bigcup_{i=0}^{1} [i\pi + 2\theta_{\tilde{\epsilon}}, i\pi + \pi/2 - 2\theta_{\tilde{\epsilon}}]$, $\Theta_f := \bigcup_{i=0}^{1} [i\pi + \pi/2 - 2\theta_{\tilde{\epsilon}}, i\pi + \pi + 2\theta_{\tilde{\epsilon}})$, so that $\Theta_f \bigcup \Theta_j = [2\theta_{\tilde{\epsilon}}, 2\pi + 2\theta_{\tilde{\epsilon}})$. Consider now the following candidate Lyapunov function:

$$V(\tilde{x}) := r(\tilde{x})^2 \exp(f(\theta(\tilde{x}))), \tag{44}$$

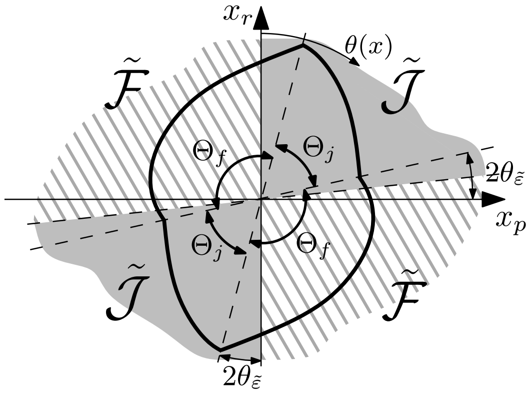

Figure 6: A level set of the function $V(\cdot)$ in (44) (bold) and its relation to the sets Θ_f and Θ_j (spanned by the double arrows) and $\tilde{\mathcal{F}}$ (striped) and $\tilde{\mathcal{J}}$ (grey).

where for any constant $\kappa > 0$, $f(\cdot) : [2\theta_{\tilde{\epsilon}}, 2\pi + 2\theta_{\tilde{\epsilon}}) \to \mathbb{R}$ is defined as:

$$f(\theta) := \begin{cases} -\kappa \operatorname{mod}_\pi(\theta - \pi/4), & \text{if } \theta \in \Theta_f, \\ \log\left(\frac{\sin^2(\theta)}{a_1} + \frac{\cos^2(\theta)}{a_2}\right) & \text{if } \theta \in \Theta_j, \end{cases} \tag{45}$$

with a_1 and a_2 ensuring continuity (see below) and $\operatorname{mod}_\pi(\theta) := \begin{cases} \theta, & \text{if } \theta \in [2\theta_{\tilde{\epsilon}}, \pi + 2\theta_{\tilde{\epsilon}}) \\ \theta - \pi, & \text{if } \theta \in [\pi + 2\theta_{\tilde{\epsilon}}, 2\pi + 2\theta_{\tilde{\epsilon}}). \end{cases}$ For illustration purposes, Figure 6 represents a level set of $V(\cdot)$ on the $\tilde{x} = (\tilde{x}_p, \tilde{x}_r)$ plane for some value of κ and $\tilde{\epsilon}$. To ensure continuity at the patching surfaces between Θ_f and Θ_j (namely, $\theta = \theta_{\tilde{\epsilon}}$ and $\theta = \pi/2 - 2\theta_{\tilde{\epsilon}}$), the constants a_1 and a_2 are selected as $a_1 := \dfrac{\cos(4\theta_{\tilde{\epsilon}})}{\exp(f_{max})\cos^2(2\theta_{\tilde{\epsilon}}) - \exp(f_{min})\sin^2(2\theta_{\tilde{\epsilon}})}$ and $a_2 := \dfrac{\cos(4\theta_{\tilde{\epsilon}})}{\exp(f_{min})\cos^2(2\theta_{\tilde{\epsilon}}) - \exp(f_{max})\sin^2(2\theta_{\tilde{\epsilon}})}$, where f_{max} and f_{min} are defined below in (46). These selections are guaranteed to be positive for a small enough $\theta_{\tilde{\epsilon}}$ (see also Figure 6).

Based on (45), since $f(\cdot)$ is decreasing in the upper equation and it is increasing in the lower equation, it follows that:

$$\begin{aligned} f_{min} \ &:= \ -\kappa\left(\frac{3\pi}{4} + 2\theta_{\tilde{\epsilon}}\right) = f(\pi + 2\theta_{\tilde{\epsilon}}) \\ &\leq \ f(\theta) \ \leq \ f(\pi/2 - 2\theta_{\tilde{\epsilon}}) \\ &= \ -\kappa\left(\frac{\pi}{4} - 2\theta_{\tilde{\varepsilon}}\right) := f_{max} \end{aligned} \tag{46}$$

for all $\theta \in [0, 2\pi]$, so that the candidate Lyapunov function (44) satisfies

$$\exp(f_{min})|\tilde{x}|^2 \leq V(\tilde{x}) \leq \exp(f_{max})|\tilde{x}|^2. \tag{47}$$

To show that the function (44), (45) decreases at jumps, we use the following claim whose proof is reported at the end of the section to avoid breaking the flow of the current proof.

Claim 1 *Consider the continuous function expressed in polar coordinates as $V(x) = r^2(x)\varphi(\theta(x))$, where*

$$\varphi(\theta) = \begin{cases} \varphi_f(\theta), & \textit{if } \theta \in \Theta_f, \\ \frac{\sin^2(\theta)}{a_1} + \frac{\cos^2(\theta)}{a_2} & \textit{if } \theta \in \Theta_j, \end{cases}$$

where $\varphi_f(\cdot)$ is continuous and satisfies $\varphi_f(\theta+\pi)=\varphi_f(\theta)$ for all $\theta\in[2\theta_{\tilde\varepsilon},\pi+2\theta_{\tilde\varepsilon}]$, and a_1 and a_2 are selected to guarantee continuity of $\varphi(\cdot)$. If $\nabla\varphi(\theta)<0$ for almost all $\theta\in\Theta_f$, then $V(A_r x)\le\eta V(x)$, for all x in $\tilde{\mathcal{J}}$, with $\eta=\cos^2(\theta_{\tilde\varepsilon})$.

It follows from (45) that $\exp(-\kappa\frac{\pi}{4})-\exp(-\kappa(\frac{\pi}{4}-\theta_{\tilde\varepsilon}))=\exp(-\kappa\frac{\pi}{4})(1-\exp(\kappa\theta_{\tilde\varepsilon}))>0$, then from Claim 1, the following strict decrease condition at jumps

$$V(\tilde A_r\tilde x)\le\eta V(\tilde x),\quad \text{if } \tilde x\in\tilde{\mathcal{J}}. \tag{48}$$

is guaranteed for some $\eta\in(0,1)$.

Consider now the derivative of $V(\tilde x)$ along the system's dynamics and observe that by virtue of (43) the following (conservative) bound holds for almost all $\tilde x\in\mathbb{R}^2$ and all d:

$$\begin{aligned}<\nabla V(\tilde x),\tilde A\tilde x+\tilde Bd> &= \exp(f(\theta(\tilde x)))\Big(2r(\tilde x)<\nabla r(\tilde x),\tilde A\tilde x+\tilde Bd>+\nabla f(\theta)r(\tilde x)^2<\nabla\theta(\tilde x),\tilde A\tilde x+\tilde Bd>\Big)\\ &= \exp(f(\theta(\tilde x)))\Big(\tilde x^T(\tilde A^T+\tilde A)\tilde x+\tfrac{\nabla f(\theta)}{2}\tilde x^T(\tilde A^TJ+J^T\tilde A)\tilde x+2\tilde x^T\tilde Bd+\nabla f(\theta)\tilde x^TJ^T\tilde Bd\Big),\\ &\le \lambda_3V(\tilde x)+\lambda_4|\tilde x||d|\end{aligned} \tag{49}$$

where λ_3 and λ_4 are large enough constants which always exist because $|\nabla f(\theta)|$ is bounded for almost all θ and by the bounds given in (47).

Regarding the directional derivative of $V(\cdot)$ in the flow set, first note that for all $x\in\tilde{\mathcal{F}}$, $x\neq0$, $\theta(x)$ is in the interior $\overset{\circ}{\Theta}_f$ of Θ_f, so that, by (45), $V(\cdot)$ is differentiable in $\tilde{\mathcal{F}}\setminus\{0\}$. In $\tilde{\mathcal{F}}$, the bound (49) can be improved noting that $\nabla f(\theta)=-\kappa$ for all $\theta\in\overset{\circ}{\Theta}_f$, so that the second line in (49) gives

$$<\nabla V(\tilde x),\tilde A\tilde x+\tilde Bd> = \exp(f(\theta(\tilde x)))\Big(\tilde x^T(\tilde A^T+\tilde A)\tilde x-\frac{\kappa}{2}\tilde x^T(\tilde A^TJ+J^T\tilde A)\tilde x+2\tilde x^T\tilde Bd-\kappa\tilde x^TJ^T\tilde Bd\Big), \tag{50}$$

for all $\tilde x\in\tilde{\mathcal{F}}\setminus\{0\}$ and all d. To suitably bound the right hand side of (50), define

$$\eta_1=2\sqrt{b_cb_p}-\max\{a_p-a_c,0\}-\sigma_\eta, \tag{51}$$

where $\sigma_\eta>0$ is a sufficiently small positive number, to guarantee that $\eta_1>0$. Such a σ_η always exists because, by assumption, $2\sqrt{b_cb_p}+a_c-a_p>0$. Also note that η_1 defined in (51) satisfies

$$\tilde x^T(J^T\tilde A+\tilde A^TJ)\tilde x\ge\eta_1|\tilde x|^2,\quad\forall\tilde x\in\tilde{\mathcal{F}}, \tag{52}$$

As a matter of fact, by applying the S-procedure, and by the definition of $\tilde{\mathcal{F}}$ the above is true if $J^T\tilde A+\tilde A^TJ-\eta_1I+\mu\tilde M\ge0$ for some $\mu>0$. This last inequality is always satisfied using $\mu=\max\{a_c-a_p,\epsilon_\mu\}$, because based on the explicit expressions in (40), it can be written as

$$\begin{bmatrix}2\sqrt{b_cb_p}-\eta_1-\mu\tilde\varepsilon & a_p-a_c+\mu\\ a_p-a_c+\mu & 2\sqrt{b_cb_p}-\eta_1\end{bmatrix}=\begin{bmatrix}\max\{a_p-a_c,0\}+\sigma_\eta-\mu\tilde\varepsilon & \max\{a_p-a_c+\epsilon_\mu,0\}\\ \max\{a_p-a_c+\epsilon_\mu,0\} & \max\{a_p-a_c,0\}+\sigma_\eta\end{bmatrix}\ge0,$$

as long as $\epsilon_\mu>0$ and $\tilde\varepsilon>0$ are sufficiently small.

Then, using once again (40), it is straightforward to check that $\tilde x^T(\tilde A^T+\tilde A)\tilde x\le2\max\{|a_c|,|a_p|\}|\tilde x|^2$, so that the flow set bound (50) becomes

$$\begin{aligned}<\nabla V(\tilde x),\tilde A\tilde x+\tilde Bd> &= \exp(f(\theta(\tilde x)))\left(2\max\{|a_c|,|a_p|\}|\tilde x|^2-\frac{\kappa\eta_1}{2}|\tilde x|^2+(2+\kappa)|\tilde x||\tilde Bd|\right)\\ &= \exp(f(\theta(\tilde x)))\left(-\frac{\eta_2}{2}|\tilde x|^2+(2+\kappa)|\tilde x||d|\right)\\ &\le -\exp(f_{min})\frac{\eta_2}{2}|\tilde x|^2+(2+\kappa)\exp(f_{max})|\tilde x||d|,\end{aligned}$$

for all $\tilde{x} \in \tilde{\mathcal{F}}$, where $\eta_2 := \kappa\eta_1 - 4\max\{|a_c|, |a_p|\}$ is a positive constant as long as $\kappa > \frac{4\max\{|a_c|,|a_p|\}}{\eta_1}$ and where in the last line we have used the uniform upper and lower bounds on $f(\theta)$ in (46).

By completing squares in the above bound we finally obtain:

$$< \nabla V(\tilde{x}), \tilde{A}\tilde{x} + \tilde{B}d > \quad \leq \quad -\exp(f_{min})\frac{\eta_2}{4}|\tilde{x}|^2 + \frac{((2+\kappa)\exp(f_{max}))^2}{\exp(f_{min})\eta_2}|d|^2, \quad \forall \tilde{x} \in \tilde{\mathcal{F}} \setminus \{0\},$$

which can be combined with (47) (48) and (49) to apply Theorem 6 from which it follows that the following estimate for the $\mathcal{L}_2$ gain from d to $\tilde{x}$ holds:

$$\gamma = \exp(f_{max} - f_{min})\frac{2(2+\kappa)}{\eta_2}, \tag{53}$$

for any $\kappa > \frac{4\max\{|a_c|,|a_p|\}}{\eta_1}$ (so that $\eta_2 > 0$), whose right hand side converges to the right hand side of (18) as ε (therefore $\tilde{\varepsilon}$, $\theta_{\tilde{\epsilon}}$ and σ_η) converges to zero. Finally, since $|x_p| = |\tilde{x}_p| \leq |\tilde{x}|$, the estimate (53) also applies to the $\mathcal{L}_2$ gain from d to x_p, as to be proven.

Step N. If item 4b does not hold, then when $2\sqrt{b_p b_c} + a_c - a_p \leq 0$ and note that the eigenvalues of $\tilde{A}$, corresponding to $\lambda_{1,2} = \frac{a_c + a_p \pm \sqrt{\Delta}}{2}$, where $\Delta := (a_c - a_p)^2 - 4b_p b_c$, are both real because $\Delta \geq 0$ by assumption. Moreover, a possible choice of the corresponding eigenvectors (whenever $\Delta \geq 0$) is given by

$$v_{1,2} = \begin{bmatrix} a_p - a_c \pm \sqrt{\Delta} & -2\sqrt{b_p b_c} \end{bmatrix}^T, \tag{54}$$

from which it appears that, under the stated conditions, both eigenvectors belong to the flow set. Indeed, their second component is always negative and their first component is always positive because $a_p - a_c \geq \sqrt{b_p b_c} > 0$ and $\sqrt{\Delta} < a_p - a_c$.

If also item 4a does not hold, then $\tilde{A}$ is non-Hurwitz. Then at least one of the two eigenvalues is real non-negative and by picking an initial condition equal to the corresponding eigenvector, the response of the reset system will remain in the eigenspace which is completely contained in the flow set. Therefore the response of the reset system will never jump and its solution, coinciding with the solution of the linear system will not be exponentially converging.

Step S1. We carry out the analysis in the transformed coordinates (40) after observing that $\tilde{x}_p = x_p$ by definition. Since $\tilde{A}$ is Hurwitz, denote by γ_L the $\mathcal{L}_2$ gain of the linear system without resets, then there exists $P = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right]$ such that $\tilde{x}^T P(\tilde{A}\tilde{x} + \tilde{B}d) + |\tilde{x}_p|^2 - \gamma_L^2|d|^2 < 0$, for all $\tilde{x}$ and d. We break the analysis in two cases: $p_{12} \geq 0$ and $p_{12} < 0$.

Case 1: $p_{12} \geq 0$. Since $p_{12} \geq 0$, then the Lyapunov function $V_L(\tilde{x}) := \tilde{x}^T P \tilde{x}$ satisfies Assumption 1 for the reset system and exponential stability and finite $\mathcal{L}_2$ gain with bound γ_L follows from Theorem 6. Indeed, the first two conditions in (20) hold because $V_L(\cdot)$ is quadratic, the flow condition follows trivially from the fact that the reset system coincides with the linear one in the flow set and the jump condition follows from noting that applying the S-procedure, condition $\tilde{x}^T(\tilde{A}_r P \tilde{A}_r - P)\tilde{x} \leq 0$ for all $\tilde{x} \in \tilde{\mathcal{J}}$ is equivalent to the existence of $\mu \geq 0$ such that

$$\begin{bmatrix} \mu\tilde{\epsilon} & p_{12} - \mu \\ p_{12} - \mu & p_{22} \end{bmatrix} \geq 0, \tag{55}$$

which is evidently satisfied selecting $\mu = p_{12} \geq 0$.

Case 2: $p_{12} < 0$. In this case consider the following candidate Lyapunov function:

$$V_2(\tilde{x}) := \begin{cases} \tilde{x}^T \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right] \tilde{x} & \text{if } \tilde{x}_r\,(\tilde{x}_p - \sigma\tilde{x}_r) \geq 0, \\ \tilde{x}^T \left[\begin{smallmatrix} p_{11} & 0 \\ 0 & p_{22}+2\sigma p_{12} \end{smallmatrix}\right] \tilde{x} & \text{if } \tilde{x}_r\,(\tilde{x}_p - \sigma\tilde{x}_r) \leq 0, \end{cases} \tag{56}$$

where σ is a small enough positive constant such that $\sigma < \frac{p_{22}}{2|p_{12}|}$, which ensures $p_{22} + 2\sigma p_{12} > 0$. Note that $V_2(\cdot)$ is Lipschitz because the two definitions coincide at the patching surfaces $\tilde{x}_r = 0$ and $\tilde{x}_p = \sigma\tilde{x}_r$.

Consider now the jump condition (20d) and note that for all $\tilde{x}$ satisfying the bottom condition in (56), it is satisfied for some $\eta = \bar{\eta}$ decreasing with $\theta_{\tilde{\varepsilon}}$. In the remaining set, namely for all $\tilde{x}$ such that $\tilde{x}_p\,(\tilde{x}_p - \sigma\tilde{x}_r) \leq 0$, the jump condition (20d) holds for the same $\eta = \bar{\eta}$ as long as σ is small enough. Indeed, applying the S-procedure we get the following inequality

$$V_2(A_r x) - \eta V(x) - \mu\tilde{x}_p\left(\tilde{x}_p + \frac{p_{22}}{4p_{12}}\tilde{x}_r\right) =$$

$$\tilde{x}^T \begin{bmatrix} p_{11} - \eta p_{11} - \mu & -\eta p_{12} - \mu\frac{p_{22}}{8p_{12}} \\ -\eta p_{12} - \mu\frac{p_{22}}{8p_{12}} & -\eta p_{22} \end{bmatrix} \tilde{x} \leq 0$$

which, by positive definiteness of $\left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right]$, is satisfied with $\mu = p_{11}$ and σ small enough.

By the definition of $V_L(\cdot)$ in case 1 above, the function $V_2(\cdot)$ in (56) also satisfies the flow condition (20c) in all the flow set but the small sector corresponding to the $\tilde{\varepsilon}$ inflation, namely the set

$$\mathcal{E} := \{\tilde{x} :\ \tilde{x}^T \begin{bmatrix} 0 & \tilde{\varepsilon} \\ \tilde{\varepsilon} & -2 \end{bmatrix} \tilde{x} \geq 0\}.$$

All the other properties of (20) trivially hold because $V_2(\cdot)$ is the patching of two quadratic functions. Therefore, if $\left[\begin{smallmatrix} \tilde{A}^T P_d + P_d\tilde{A} + C^T C & P_d\tilde{B} \\ B^T P_d & -\gamma^2 \end{smallmatrix}\right] < 0$ for all $x \in \mathcal{E}$ and all d (where $P_d := \left[\begin{smallmatrix} p_{11} & 0 \\ 0 & p_{22} \end{smallmatrix}\right]$), the result follows from applying Theorem 6 using $V_2(\cdot)$. Indeed $V_2(\cdot)$ is differentiable everywhere in $\tilde{\mathcal{F}}$ except for the horizontal axis, where the generalized gradient is given by the maximum of the directional derivatives and where we establish good bounds for them on both sides. To show this final step we use the S-procedure and seek for a positive scalar μ such that

$$\begin{bmatrix} 2a_p p_{11} + 1 & \tilde{k}(p_{11} - p_{22}) + \tilde{\varepsilon}\mu & p_{11} \\ \star & -2\mu + 2a_c p_{22} & 0 \\ \star & \star & -\gamma_L^2 \end{bmatrix} < 0, \tag{57}$$

where $\star$ denote symmetric entries and $\tilde{k} = \sqrt{b_p b_c} > 0$.

To find this value of μ, consider the flow property of $V_L(\cdot)$, $\tilde{x}^T P(\tilde{A}\tilde{x} + \tilde{B}d) + |\tilde{x}_p|^2 - \gamma_L^2|d|^2 < 0$, which can be written in matrix form as $\left[\begin{smallmatrix} \tilde{A}^T P + P\tilde{A} + C^T C & P\tilde{B} \\ B^T P & -\gamma_L^2 \end{smallmatrix}\right] < 0$, where $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. After suitable calculations, the condition becomes

$$\begin{bmatrix} 2a_p p_{11} - 2\tilde{k}p_{12} + 1 & \tilde{k}(p_{11} - p_{22}) + p_{12}(a_p + a_c) & p_{11} \\ \star & 2\tilde{k}p_{12} + 2a_c p_{22} & p_{12} \\ \star & \star & -\gamma_L^2 \end{bmatrix} < 0, \tag{58}$$

which holds for all $\tilde{x}$ and d. Then, to guarantee (57), it is sufficient to select a $\mu = \mu^*$ large enough so that

$$\begin{bmatrix} 2a_p p_{11} + 1 & \tilde{k}(p_{11} - p_{22}) + p_{12}(a_p + a_c) & p_{11} \\ \star & -2\mu^* + 2a_c p_{22} & 0 \\ \star & \star & -\gamma_L^2 \end{bmatrix} < 0 \tag{59}$$

(where we have used $\tilde{k} > 0$ and $p_{12} < 0$). Such a μ^* always exists because of (58) and because μ^* allows to make the central diagonal term negative enough to compensate for replacing the off-diagonal term p_{12} in (58) by a 0. Then, selecting $\tilde{\varepsilon}^* = \frac{p_{12}(a_p+a_c)}{\mu^*}$, inequality (57) coincides with (59) and is guaranteed to hold. Note that $\tilde{\varepsilon}^* > 0$ because $p_{12} < 0$ and $a_p + a_c < 0$ from the assumption that $\tilde{A}$ is Hurwitz.

Since (57) guarantees that the flow condition holds for $\tilde{\varepsilon} = \tilde{\varepsilon}^*$, it also guarantees that it holds for all $\tilde{\varepsilon} < \tilde{\varepsilon}^*$ because they lead to smaller flow sets.

Proof of Claim 1 Define $\varphi_j(\theta) := \frac{\sin^2(\theta)}{a_1} + \frac{\cos^2(\theta)}{a_2}$. Then observe that $r^2(x) = x_p^2 + x_r^2 = r^2(x)\sin^2(\theta(x)) + r^2(x)\cos^2(\theta(x))$ and since $A_r \left[\begin{smallmatrix} x_p \\ x_r \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_p \\ 0 \end{smallmatrix}\right]$ for all x, then

$$r^2(A_r x) = r^2(x)\sin^2(\theta(x)). \tag{60}$$

Since by assumption φ is continuous and $\nabla\varphi(\theta) < 0$ for almost all $\theta \in \Theta_f$, then

$$\begin{aligned} \varphi_f(\pi/2) &\leq \varphi(\theta), \quad \forall \theta \in \left[\frac{\pi}{2} - 2\theta_{\tilde{\varepsilon}}, \frac{\pi}{2}\right] && \text{(61a)} \\ \varphi_f(\pi + 2\theta_{\tilde{\varepsilon}}) &\leq \varphi(\theta), \quad \forall \theta \in [\pi, \pi + 2\theta_{\tilde{\varepsilon}}]. && \text{(61b)} \end{aligned}$$

Note also the following simple relations:

$$\frac{\sin(\theta)}{\cos(2\theta_{\tilde{\varepsilon}})} \leq 1; \quad \frac{\sin(2\theta_{\tilde{\varepsilon}})}{\cos(\theta)} \leq 1, \quad \forall \theta \in \left[0, \frac{\pi}{2} - 2\theta_{\tilde{\varepsilon}}\right]. \tag{62}$$

Now we establish three useful properties.

1. If $\theta \in \left[2\theta_{\tilde{\varepsilon}}, \frac{\pi}{2} - 2\theta_{\tilde{\varepsilon}}\right]$, then applying (61a), using (62) and also using $\cos^2(2\theta_{\tilde{\varepsilon}}) < \cos^2(\theta_{\tilde{\varepsilon}})$, we get:

$$\begin{aligned} \sin^2(\theta)\varphi\left(\frac{\pi}{2}\right) &\leq \sin^2(\theta)\varphi\left(\frac{\pi}{2} - 2\theta_{\tilde{\varepsilon}}\right) = \sin^2(\theta)\varphi_j\left(\frac{\pi}{2} - 2\theta_{\tilde{\varepsilon}}\right) \\ &= \sin^2(\theta)\left(\tfrac{\cos^2(2\theta_{\tilde{\varepsilon}})}{a_1} + \tfrac{\sin^2(2\theta_{\tilde{\varepsilon}})}{a_2}\right) \\ &= \cos^2(2\theta_{\tilde{\varepsilon}})\left(\tfrac{\sin^2(\theta)}{a_1} + \tfrac{\cos^2(\theta)}{a_2}\left(\tfrac{\sin^2(\theta)}{\cos^2(2\theta_{\tilde{\varepsilon}})}\tfrac{\sin^2(2\theta_{\tilde{\varepsilon}})}{\cos^2(\theta)}\right)\right) \\ &\leq \cos^2(2\theta_{\tilde{\varepsilon}})\varphi_j(\theta) < \eta\varphi(\theta). \end{aligned} \tag{63a}$$

2. If $\theta \in [\pi, \pi + 2\theta_{\tilde{\varepsilon}}]$, applying the previous inequality (63a) with $\theta = \pi + 2\theta_{\tilde{\varepsilon}}$ and using (61b), we get:

$$\begin{aligned} \sin^2(\theta)\varphi\left(\frac{\pi}{2}\right) &\leq \sin^2(\pi + 2\theta_{\tilde{\varepsilon}})\varphi\left(\frac{\pi}{2}\right) \\ &\leq \eta\varphi(\pi + 2\theta_{\tilde{\varepsilon}}) \leq \eta\varphi(\theta). \end{aligned} \tag{63b}$$

3. If $\theta \in \left[\frac{\pi}{2} - 2\theta_{\tilde{\varepsilon}}, \frac{\pi}{2} - \theta_{\tilde{\varepsilon}}\right]$, since $\sin(\theta) \leq \cos(\theta_{\tilde{\varepsilon}})$ and from (61a), we get:

$$\sin^2(\theta)\varphi\left(\frac{\pi}{2}\right) \leq \cos^2(\theta_{\tilde{\varepsilon}})\varphi(\theta) = \eta\varphi(\theta). \tag{63c}$$

Finally, since $x \in \tilde{\mathcal{J}}$ implies $\text{mod}_\pi(\theta(x)) \in \left[2\theta_{\tilde{\varepsilon}}, \frac{\pi}{2} - \theta_{\tilde{\varepsilon}}\right] \bigcup [\pi, \pi + 2\theta_{\tilde{\varepsilon}}]$ and since $\varphi(\theta + \pi) = \varphi(\theta)$ by assumption, combining equation (60) with the three bounds (63), we get

$$\begin{aligned} V(A_r x) &= r^2(A_r x)\varphi(\pi/2) \\ &= r^2(x)\sin^2(\theta(x))\varphi(\pi/2) \\ &\leq r^2(x)\eta\varphi(\theta(x)) = \eta V(x). \end{aligned}$$

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9 Conclusions

In this paper we introduced a new type of First Order Reset Element and discussed how it can be used for set point regulation of a class of SISO linear plants. Then we discussed how this new FORE can be used for exponential and $\mathcal{L}_2$ stabilization of SISO minimum phase relative degree one linear plants and stated necessary and sufficient conditions for the planar reset closed-loop betwen a first order linear plant and this new FORE. Some results that are instrumental to our proof but also of interest on their own have been also introduced: Lyapunov conditions for exponential and $\mathcal{L}_2$ stability of nonlinear reset systems with temporal regularization and equivalences among exponential stability, asymptotic stability, $\mathcal{L}_p$ and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stability and input-to-state (ISS) stability of temporally regularized linear reset systems acting on cones.

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A Proof of Theorem 5

Proof of item 1. The proof follows in a straightforward way from the bounds established in item 2 of Theorem 4. Indeed, pick $b_c b_p$ sufficiently large (which can be done because $b_c b_p$ approaches $+\infty$) so that condition (16) holds and fix κ in (18) so that $\kappa > \overline{\kappa} := \frac{4\max\{|a_c|,|a_p|\}}{2\sqrt{b_c b_p} - \max\{a_p - a_c, 0\}}$. Then (18) will hold with this κ for any larger value of $b_c b_p$ (because $\overline{\kappa}$ is strictly decreasing, as $b_c b_p$ grows). Finally, since κ is fixed, (18) shows that the gain tends asymptotically to zero as $b_c b_p$ approaches infinity.

Proof of items 2 and 3. The proof of these two items is carried out by constructing a Lyapunov function satsifying the conditions in Assumption 1 with $y = x_p$ and where λ_6 is an increasing function of a_c while λ_7 is independent of a_c, so that the $\mathcal{L}_2$ gain bound of Theorem 6 will prove the $\mathcal{L}_2$ gain trends. Since the proof is quite involved and extensive, we summarize it here. First, in (64), (65) the Lyapunov function is proposed. Then all the properties in Assumption 1 are proven for this function: equation (66) implies (20a), equation (69) implies (20d), equation (70) implies (20b) and the remainder of the proof is used to show that (73) holds with ϵ_d and ν^* independent of a_c and k. This last fact implies (20c) with λ_6 and λ_7 having the properties summarized above. It is emphasized that when $x \in \mathcal{F}$ (equivalentely $\theta \in \Theta_f$), the Lyapunov function that we select is the patching of three pieces, so the proof of (73) is carried out analyzing separately the three pieces, one by one. More specifically, for each piece first the useful inequalities (75), (77) and (79) are derived, respectively, and then these three inequalities are used to show (73) in the three remaining items of the proof.

The Lyapunov function used here is defined along similar steps as those carried out in the proof of Theorems 3 and 4. In particular, consider the same common step used at the beginning of Section 8.2 and the polar coordinates proposed in (41). Then, relations (43) will hold but instead of (44) we will use the following candidate Lyapunov function:

$$V(\tilde{x}) := r(\tilde{x})^2 \varphi(\theta(\tilde{x})), \tag{64}$$

where $\varphi(\cdot)$ is defined as

$$\begin{aligned}
\varphi(\theta) &:= \begin{cases} \min\{\sqrt{a_c}t^3(\theta_e(\theta)), \sqrt{t^5(\theta_e(\theta))}, \sqrt{\sigma^3}t(\theta_e(\theta))\}, & \text{if } \theta \in \Theta_f, \\ \frac{\sin^2(\theta)}{a_1} + \frac{\cos^2(\theta)}{a_2}, & \text{if } \theta \in \Theta_j, \end{cases} && \text{(65a)} \\
\theta_e(\theta) &:= \mathrm{mod}_\pi(\theta) + \nu(3/4\pi - \mathrm{mod}_\pi(\theta)), \qquad \forall \theta \in \Theta_f && \text{(65b)}
\end{aligned}$$

where $t(\theta_e) := -\tan(\theta_e)$, $\text{mod}_\pi(\theta) := \begin{cases} \theta, & \text{if } \theta \in [2\theta_{\tilde{\varepsilon}}, \pi + 2\theta_{\tilde{\varepsilon}}) \\ \theta - \pi, & \text{if } \theta \in [\pi + 2\theta_{\tilde{\epsilon}}, 2\pi + 2\theta_{\tilde{\epsilon}}). \end{cases}$ and θ_e corresponds to a slightly inflated version of θ in the set Θ_f. Choose now $\theta_{\tilde{\varepsilon}} = \frac{\nu\pi}{16(1-\nu)}$ and after some computations, it can be verified that for all $\theta \in \Theta_f$, $t(\theta_e(\theta)) \geq t\left(\theta_e\left(\frac{\pi}{2} - 2\theta_{\tilde{\varepsilon}}\right)\right) = \frac{\cos\left(\frac{\nu\pi}{8}\right)}{\sin\left(\frac{\nu\pi}{8}\right)}$ and $t(\theta_e(\theta)) \leq t(\theta_e(\pi + 2\theta_{\tilde{\varepsilon}})) = \frac{\sin\left(\frac{\nu\pi}{8}\right)}{\cos\left(\frac{\nu\pi}{8}\right)}$, so that for any arbitrarily small selection of $\nu > 0$ and given the selection above for $\theta_{\tilde{\varepsilon}}$, there exist suitable constants φ_m, $\varphi_M > 0$ such that the function (65) satisfies the following bounds:

$$\varphi_m|\tilde{x}|^2 \leq V(\tilde{x}) \leq \varphi_M|\tilde{x}|^2 \quad \forall \tilde{x} \in \mathbb{R}^2, \tag{66}$$

Finally, a_1 and a_2 are selected in the same exact way as in Step S2 of the proof of Theorems 3 and 4 to ensure continuity of $\varphi(\cdot)$.

Since the selection in (65b) is continuous and all trigonometric functions are continuous too, then there exist $\mathcal{K}$ functions (namely continuous functions that are zero at zero and strictly increasing) $\ell(\cdot)$, and $\delta(\cdot)$ satisfying the following bounds for all $\theta \in \Theta_f$:

$$\begin{aligned} &|\sin(\theta)| \leq (1 + \ell(\nu))|\sin(\theta_e(\theta))|, \quad |\cos(\theta)| \leq (1 + \ell(\nu))|\cos(\theta_e(\theta))|, \\ &\delta(\nu) \geq |\sin(\theta)\cos(\theta) - \sin(\theta_e(\theta))\cos(\theta_e(\theta))|. \end{aligned} \tag{67}$$

Since $\delta(\cdot)$ is a class $\mathcal{K}$ function, then it is possible to choose ν small enough so that the following bound holds:

$$\delta(\nu) \leq \tfrac{(1-0.1)\sqrt{b_p b_c}}{2a_c}. \tag{68}$$

In particular, without loss of generality, we assume in the following that $\nu \leq \nu^* := \min\{0.1, \nu_1\}$, uniformly over all selections of a_c and $k = b_p b_c$, where ν_1 is any positive number satisfying $\ell(\nu_1) \leq 1$.

Note that the bound (68) imposes that ν should become smaller as a_c grows larger. Therefore also $\theta_{\tilde{\varepsilon}}$ (namely, $\tilde{\varepsilon}$) should become smaller for (66) to hold and consequently, the strict decrease at jumps given by η will also become smaller. By Theorem 6 this also implies that the time regularization constant ρ will become smaller. Nevertheless, according to the statement of this theorem, the $\mathcal{L}_2$ gain will converge to zero in the sense of Definition 2.

Regarding the jump properties of $V(\cdot)$, since $t(\theta_e(\theta)) = -\tan(\theta_e(\theta))$ is a strictly decreasing function of θ, then the three terms in (65a) are all decreasing in Θ_f and $\nabla\varphi(\theta) < 0$ for almost all $\theta \in \Theta_f$. Therefore, applying Claim 1, there exists $\eta < 1$ such that the following strict decrease at jumps holds:

$$V(A_r\tilde{x}) \leq \eta V(\tilde{x}), \quad \forall \tilde{x} \in \tilde{\mathcal{J}}. \tag{69}$$

Consider now the derivative of V along the systems dynamics and note that, using (43), similar to (49), the following bounds hold for almost all $x \in \mathbb{R}^2$ and all d:

$$\begin{aligned} < \nabla V(\tilde{x}), \tilde{A}\tilde{x} + \tilde{B}d > &= 2r(\tilde{x})\varphi(\theta(\tilde{x})) < \nabla r(\tilde{x}), \tilde{A}\tilde{x} + \tilde{B}d > + r(\tilde{x})^2 \nabla\varphi(\theta(\tilde{x})) < \nabla\theta(\tilde{x}), \tilde{A}\tilde{x} + \tilde{B}d > \\ &= 2\varphi\tilde{x}^T(\tilde{A}\tilde{x} + \tilde{B}d) + \nabla\varphi\tilde{x}^T J^T(\tilde{A}\tilde{x} + \tilde{B}d) \\ &= \varphi(2a_p\tilde{x}_p^2 + 2a_c\tilde{x}_r^2 + 2\tilde{x}_p d) + \nabla\varphi(\sqrt{b_p b_c}(\tilde{x}_p^2 + \tilde{x}_r^2) - (a_c - a_p)\tilde{x}_p\tilde{x}_r + \tilde{x}_r d), \\ &\leq \lambda_3 V(\tilde{x}) + \lambda_4|\tilde{x}||d| \end{aligned} \tag{70}$$

where we have omitted the dependence on $\tilde{x}$, and where λ_3 and λ_4 are large enough constants which always exists because $|\varphi(\theta)|$ and $|\nabla\varphi(\theta)|$ are bounded for all θ.

When only focusing on the directional derivative of $V(\cdot)$ in the flow set, the bound in (70) can be improved as follows. First note that combining terms and by completion of squares, the third line in (70) leads to

$$\begin{aligned}
&< \nabla V(\tilde{x}), \tilde{A}\tilde{x} + \tilde{B}d > \leq r^2\Big(2\varphi(a_p s_\theta^2 + a_c c_\theta^2) + \nabla\varphi(\sqrt{b_p b_c} - (a_c - a_p)s_\theta c_\theta) + \epsilon_d \varphi^2 s_\theta^2 + \frac{\epsilon_d}{2}(\nabla\varphi)^2 c_\theta^2\Big) + \frac{d^2}{\epsilon_d} \\
&\leq r^2\Big(2\varphi(|a_p| + a_c c_\theta^2) + \nabla\varphi(\sqrt{b_p b_c} - a_c sc) + |\nabla\varphi|((1+\ell(\nu))^2|a_p||sc| + \delta(\nu)|a_c|) + \epsilon_d\varphi^2 + \frac{\epsilon_d}{2}(\nabla\varphi)^2(1+\ell(\nu))^2c^2\Big) + \frac{d^2}{\epsilon_d} \\
&=: r^2 z + \frac{d^2}{\epsilon_d},
\end{aligned} \tag{71}$$

where we have used (67) and the shortcuts s_θ, s, c_θ and c, respectively, for $\sin(\theta(\tilde{x}))$ $\sin(\theta_e(\theta(\tilde{x})))$, $\cos(\theta(\tilde{x}))$ and $\cos(\theta_e(\theta(\tilde{x})))$, respectively. Moreover, the dependence of r, φ and $\nabla\varphi$ on $\tilde{x}$ has been omitted for the sake of conciseness. Finally, the positive number ϵ_d is used in the completion of squares. Its selection will be clarified next.

The rest of the proof amounts to showing that for sufficiently large a_c, given a suitable selection for ϵ_d (independent of a_c and uniform over arbitrarily large values of $k = b_p b_c$), the term z in (71) satisfies

$$z < -k_x\sqrt{a_c}s^2, \tag{72}$$

where $k_x = \min\{\frac{1}{8}, \frac{3}{4}\sqrt{b_p b_c}\}$. Using (67), and since (71) holds for all of the three pieces of V in (65a), then for all d and for all $\tilde{x}$ in the flow set

$$\begin{aligned}
\max_{v\in\partial V(\tilde{x})} v^T(\tilde{A}\tilde{x} + \tilde{B}d) &\leq -(1+\ell(\nu))^{-1}r^2\sin^2(\theta)k_x\sqrt{a_c} + \frac{d^2}{\epsilon_d} \\
&\leq -(1+\ell(\nu^*))^{-1}k_x\sqrt{a_c}|x_p|^2 + \frac{d^2}{\epsilon_d},
\end{aligned} \tag{73}$$

where ν^* is the uniform upper bound on ν introduced above.

Combined with (66), (69) and (70), equation (73) can be used in Theorem 6 to show that as a_c (and possibly also $k = b_p b_c$) approaches infinity, an estimate of the $\mathcal{L}_2$ gain is given by $\sqrt{\dfrac{1+\ell(\nu^*)}{\epsilon_d k_x\sqrt{a_c}}}$, which approaches zero, as to be proved.

To show (72), note that the upper formula for $\varphi(\theta)$ in (65) (which applies to the flow set), leads to the following selections in three different and disjoint subsets of Θ_f:

1. Θ_{f1}, where $\min\{\sqrt{a_c}t^3(\theta_e(\theta)), \sqrt{t^5(\theta_e(\theta))}, \sqrt{\sigma^3}t(\theta_e(\theta))\} = \sqrt{a_c}t^3(\theta_e(\theta))$. In this set we have

$$\varphi = \sqrt{a_c}t^3; \quad \nabla\varphi = -3(1-\nu)\sqrt{a_c}t^2c^{-2}, \tag{74}$$

   and as long as $a_c \geq 1$, then the following bound holds:

$$\sqrt{a_c}t^2 \leq 1, \tag{75}$$

   indeed, the fact that we are in Θ_{f1} implies that $\sqrt{a_c}\sqrt{t} \leq 1$ (otherwise we would be in Θ_{f2}). Therefore $\sqrt{t} \leq 1$ and $t^2 \leq \sqrt{t} \leq 1$, which implies (75).

2. Θ_{f2}, where $min\{\sqrt{a_c}t^3(\theta_e(\theta)), \sqrt{t^5(\theta_e(\theta))}, \sqrt{\sigma^3}t(\theta_e(\theta))\} = \sqrt{t^5(\theta_e(\theta))}$. In this set we have

$$\varphi = t^{2.5}; \quad \nabla\varphi = -2.5(1-\nu)t^{1.5}c^{-2}, \tag{76}$$

and the following bounds hold (otherwise we would be in the sets Θ_{f3} and Θ_{f1}, respectively):

$$t^{1.5} \leq \sigma^{1.5}, \quad \sqrt{a_c}\sqrt{t} \geq 1. \tag{77}$$

3. Θ_{f3}, where $\min\{\sqrt{a_c}t^3(\theta_e(\theta)), \sqrt{t^5(\theta_e(\theta))}, \sqrt{\sigma^3}t(\theta_e(\theta))\} = \sqrt{\sigma^3}t(\theta_e(\theta))$. In this set we have

$$\varphi = \sigma^{1.5}t; \quad \nabla\varphi = -(1-\nu)\sigma^{1.5}c^{-2}, \tag{78}$$

and since we are in Θ_{f3}, then $\sigma^{1.5} \leq t^{1.5}$ (otherwise we would be in Θ_{f2}), therefore the following bounds hold:

$$\sigma \leq t, \quad c^2 \leq \sigma^{-2}s^2 \leq \sigma^{-2} \tag{79}$$

(where we also used $\sigma > 0$, $t > 0$).

To show that (72) holds for large enough a_c, we will use equations (74)–(79) and derive bounds for z in (71) in the three cases, as shown next.

1. Substituting (74) in (71), re-arranging and using (75) and $t = -s/c$, we get

$$\begin{aligned} z \ &\leq \ \sqrt{a_c}t^3\left(-3(1-\nu)a_c + 2|a_c| + (2+3(1+\ell(\nu))^2)|a_p|\right) \\ &\quad -3\sqrt{a_c}t^2c^{-2}\left((1-\nu)\sqrt{b_pb_c} - \delta(\nu)|a_c| - \frac{\epsilon_d}{3}s^2 - \frac{3(1+\ell(\nu))^2\epsilon_d}{2}\right). \end{aligned}$$

   Since $s^2 \leq 1$, $\nu \leq \nu^* \leq 0.1$ and based on the bound (68), the equation above becomes:

$$z \leq \sqrt{a_c}t^3\left(-3(0.9)a_c + 2|a_c| + (2+3(1+\ell(0.1))^2)|a_p|\right) - 3\sqrt{a_c}t^2c^{-2}\left(\frac{0.9\sqrt{b_pb_c}}{2} - \frac{\epsilon_d}{3} - \frac{3(1+\ell(0.1))^2\epsilon_d}{2}\right).$$

   The first term in brackets is then negative as long as $a_c > \frac{2+3(1+\ell(0.1))^2}{3(0.9)-2}|a_p|$. The remaining term will provide the decrease condition, as long as $\epsilon_d \leq \sqrt{b_cb_p}\frac{1}{5}\left(\frac{2+9(1+\ell(0.1))^2}{6}\right)^{-1}$ so that $\frac{0.9\sqrt{b_pb_c}}{2} - \left(\frac{1}{3} - \frac{3(1+\ell(0.1))^2}{2}\right)\epsilon_d \geq \frac{\sqrt{b_cb_p}}{4}$. Indeed, it will imply $z \leq -3\sqrt{a_c}t^2c^{-2}\frac{\sqrt{b_cb_p}}{4}$ which, using $c^{-2} \geq 1$, leads to (72) with $k_x = \frac{3}{4}\sqrt{b_pb_c}$. Note that the bound on ϵ_d is independent of a_c and remains satisfied for a fixed ϵ_d if the loop gain b_cb_p tends to infinity.

2. Substituting (76) in (71), re-arranging and using $t = -s/c$, we get

$$\begin{aligned} z \ &\leq \ t^{2.5}\left(-2.5(1-\nu)a_c + 2|a_c| + (2+2.5(1+\ell(\nu))^2)|a_p|\right) \\ &\quad -2.5t^{1.5}c^{-2}\left((1-\nu)\sqrt{b_pb_c} - \delta(\nu)|a_c| - \frac{\epsilon_d}{2.5}t^{1.5}s^2 - \frac{2.5(1+\ell(\nu))^2\epsilon_d}{2}t^{1.5}\right). \end{aligned}$$

   Since $s^2 \leq 1$, $\nu \leq \nu^* \leq 0.1$ and based on the first bound in (77) and the bound (68), the equation above becomes:

$$z \leq t^{2.5}\left(-2.5(0.9)a_c + 2|a_c| + (2+2.5(1+\ell(0.1))^2)|a_p|\right) - 2.5t^{1.5}c^{-2}\left(\frac{0.9\sqrt{b_pb_c}}{2} - \frac{\epsilon_d}{2.5}\sigma^{1.5} - \frac{2.5(1+\ell(\nu))^2\epsilon_d}{2}\sigma^{1.5}\right).$$

   Picking $\epsilon_d \leq \frac{\sqrt{b_cb_p}}{4}\left(\frac{2+7(1+\ell(0.1))^2}{5}\sigma^{1.5}\right)^{-1}$, the right term in brackets is greater than zero and the arising negative term can be disregarded. The remaining term will provide the decrease condition. This follows from picking $a_c \geq 8(2+2.5(1+\ell(0.1))^2)|a_p|$ and noting that, similar to the previous item, this is sufficient to bound the first term in brackets by $-\frac{1}{8}a_c$. Then using the second bound in (77) we get

$$\begin{aligned} z \ &\leq \ -\frac{1}{8}a_ct^{2.5} = -\frac{1}{8}\sqrt{a_c}\sqrt{a_c}\sqrt{t}t^2 \\ &\leq \ -\frac{1}{8}\sqrt{a_c}t^2 \leq -\frac{1}{8}\sqrt{a_c}s^2, \end{aligned}$$

where we used $c^{-2} \geq 1$. This proves (72) with $k_x = \frac{1}{8}$. Note that once again the bound on ϵ_d is independent of a_c and remains satisfied if the loop gain $b_c b_p$ tends to infinity.

3. Substituting (78) in (71), re-arranging and using $t = -s/c$, we get

$$\begin{aligned} z &\leq \sigma^{1.5} t \left(-(1-\nu)a_c + 2|a_c|(1+\ell(\nu))^2 c^2 + (2+(1+\ell(\nu))^2)|a_p|\right) \\ &\quad -\sigma^{1.5} c^{-2} \left((1-\nu)\sqrt{b_p b_c} - \delta(\nu)|a_c| - \epsilon_d \sigma^{1.5} s^2 - \frac{(1+\ell(\nu))^2 \epsilon_d}{2} \sigma^{1.5}\right). \end{aligned}$$

Since $s^2 \leq 1$, $\nu \leq \nu^* \leq 0.1$ and based on the bound (68), the equation above becomes:

$$z \leq \sigma^{1.5} t \left(-0.9 a_c + 2|a_c|(1+\ell(\nu))^2 c^2 + (2+(1+\ell(0.1))^2)|a_p|\right) - \sigma^{1.5} c^{-2} \left(\frac{0.9\sqrt{b_p b_c}}{2} - \left(1 + \frac{(1+\ell(\nu))^2}{2}\right) \sigma^{1.5} \epsilon_d\right).$$

Picking $\epsilon_d \leq \frac{\sqrt{b_c b_p}}{4} \left(\left(1 + \frac{(1+\ell(\nu))^2}{2}\right) \sigma^{1.5}\right)^{-1}$, the right term in brackets is greater than zero and the arising negative term can be disregarded. The remaining term will provide the decrease condition. To show this, first note that based on the second bound in (79) and since $\nu \leq \nu^* \leq \nu_1$, where $\ell(\nu_1) = 1$, then $2|a_c|(1+\ell(\nu))^2 c^2 \leq 2|a_c| 2^2 \sigma^{-2}$, therefore selecting $\sigma = 5$, the following bound holds:

$$\begin{aligned} z &\leq \sigma^{1.5} t \left(-\frac{2a_c}{5} - \frac{a_c}{2} + 8|a_c|\frac{1}{25} + (2+(1+\ell(0.1))^2)|a_p|\right) \\ &\leq \sigma^{1.5} t \left(-\frac{a_c}{2} + (2+(1+\ell(0.1))^2)|a_p|\right) \\ &\leq -\sigma^{1.5} t \frac{a_c}{4}, \end{aligned}$$

as long as $a_c \geq 4(2+(1+\ell(0.1))^2)|a_p|$. Finally, using the first bound in (79), z can be bounded as $z \leq -\frac{1}{4} a_c \sigma^{1.5} t \leq -0.25\sigma^{2.5} a_c s^2 \leq -0.25\sqrt{a_c} s^2$ whenever $a_c \geq 1$. This proves (72) with $k_x = \frac{1}{4} > \frac{1}{8}$. Note that once again the bound on ϵ_d is independent of a_c and remains satisfied if the loop gain $k = b_c b_p$ tends to infinity.

Based on the above items, the bound (72) is proved and the proof is completed.