Path-Following for Nonlinear Systems with Unstable Zero Dynamics: an Averaging Solution

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August 26, 2008

Abstract

We consider a path-following problem in which the goal is to ensure that the error between the system output and the geometric path be asymptotically less than a prespecified constant, while guaranteeing a forward motion along the path and boundedness of all states. A solution to this problem was given in [12] for a class of nonlinear systems and for paths satisfying a certain geometric condition. In this paper, we exploit averaging techniques to develop an alternative simpler solution to the above problem for the same class of systems but under stronger conditions on the path geometry.

1 Introduction

Path-following has recently been introduced as an interesting alternative to the more classical problem of reference tracking [1]-[8]. The primary task in path-following is to ensure that the system output converges to a geometric path, while the properties of the output's motion along the geometric path are of secondary importance. In particular, instead of requiring a specific motion along the geometric path, which is typical in reference tracking, in path-following problems the designer is allowed to select a motion along the geometric path from a large class of possible motions. This additional flexibility is often a major advantage of path-following over reference tracking [7]-[8]. For instance, it was shown in [9]-[12] that this extra degree of design freedom can be used to avoid the fundamental limitations on achievable tracking accuracy imposed by unstable zero dynamics in reference tracking problems.

Our results are most closely related to [11] and [12] which we briefly summarize. In [11] the goal is to ensure asymptotic convergence of the system output to the geometric path, while guaranteeing a forward motion along the path and boundedness of all states. This problem was solved in [11] for linear systems and paths satisfying a certain geometric condition, the so called repeatability. A more general problem was considered in [12] where practical (instead of asymptotic) convergence of the system output to the geometric path was required. While in [12] the same class of repeatable geometric paths was considered, the results were proved for a more general class of nonlinear systems with input-to-state stabilizable zero dynamics.

Here, we consider the same path-following problem and class of systems as in [12] but under more stringent conditions on the path geometry. In particular, we assume that the geometric path can be parameterized by a scalar positive parameter θ and that the parameterized path is periodic in the parameter. Periodic paths are repeatable in the sense of [12] but the opposite is not true. For periodic paths, we provide a novel and significantly simpler controller design than the one developed in [12]. We note that the proof of our main result differs from the proof in [12] and it is of independent interest as it exploits the averaging techniques (see e.g. [13]) for the first time in this context.

*Supported by the Australian Research Council under the Discovery Project and Australian Professorial Fellow program.

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In Section 2 we formulate the path-following problem of interest and give sufficient conditions for its solvability for a class of nonlinear systems with potentially unstable zero dynamics. Our novel design and the main result are presented in Section 3. An example with simulations is provided in the same section. A summary is given in Section 4. All proofs are given in the appendix. For brevity, whenever there is no ambiguity we drop function arguments.

2 Preliminaries

The sets of real numbers and non-negative real numbers are respectively denoted as $\mathbb{R}$ and $\mathbb{R}^+$. A set of functions that are r times continuously differentiable is denoted as C^r. A continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}^\infty$ if it is zero at zero, strictly increasing and unbounded. Given an arbitrary nonempty set $\mathcal{S}$, we denote its convex hull and interior respectively as $\mathrm{con}\mathcal{S}$ and $\mathrm{int}\{\mathcal{S}\}$.

We consider systems with a vector relative degree $\{r_1, \ldots, r_m\}$ that can be transformed by a global coordinate and feedback transformation into[1]

$$\begin{aligned} \dot{z} &= f(z, y), && (1)\\ \dot{x}^i &= A_{r_i} x^i + B_{r_i} u_i, \; y_i = C_{r_i} x^i, && (2) \end{aligned}$$

where $z \in \mathbb{R}^{n-r}$, $x^i \triangleq [x_1^i \ldots x_{r_i}^i]^T \in \mathbb{R}^{r_i}$, $y \triangleq [y_1 \; \ldots \; y_m]^T \in \mathbb{R}^m$, $u \triangleq [u_1 \ldots u_m]^T \in \mathbb{R}^m$, and $r \triangleq \sum_{i=1}^m r_i$. We also define $r^\star \triangleq \max_i r_i$ which is the maximal relative degree among all output components y_i in (2). We let the function $f : \mathbb{R}^{n-r} \times \mathbb{R}^m \to \mathbb{R}^{n-r}$, $f(0,0) = 0$, be locally Lipschitz, and define the matrices $A_{r_i} \in \mathbb{R}^{r_i \times r_i}$, $B_{r_i}^T, C_{r_i} \in \mathbb{R}^{1 \times r_i}$ by

$$A_{r_i} = \begin{bmatrix} 0 & I_{r_i-1} \\ 0 & 0 \end{bmatrix}, B_{r_i}^T = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}, C_{r_i} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}.$$

The subsystem (2) consists of m integrator chains relating the input u with the output y, where the i^{th} chain has r_i integrators and its states are denoted by x^i. In the sequel, we also use the notation $x \triangleq [(x^1)^T \ldots (x^m)^T]^T \in \mathbb{R}^r$.

The subsystem (1) represents a possibly unstable zero dynamics of the plant driven by its output y. We assume that the zero dynamics can be input-to-state stabilized (ISS) in an appropriate sense when y is regarded as a control input to the subsystem (1). In particular, we assume the following:

Assumption 1 *Suppose that there exist:*

(i) a $C^{r^\star}$ function $\sigma : \mathbb{R}^{n-r} \to \mathbb{R}^m$ with $\sigma(0) = 0$;
(ii) a C^1 Lyapunov function $V_z : \mathbb{R}^{n-r} \to \mathbb{R}^+$;
(iii) class $\mathcal{K}^\infty$ functions $\alpha_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2, 3$;
(iv) a locally Lipschitz function $\pi : \mathbb{R}^{n-r} \to \mathbb{R}^m$, with $\pi(0) = 0$

such that the following holds:

A1: $\alpha_1(\|z\|) \le V_z(z) \le \alpha_2(\|z\|)$ *for all* $z \in \mathbb{R}^{n-r}$*;*

A2: $\frac{\partial V_z}{\partial z} f(z, \sigma(z) + d) \le -\alpha_3(\|z\|) + \pi^T(z) d$ *for all* $z \in \mathbb{R}^{n-r}$ *and* $d \in \mathbb{R}^m$*;*

A3: $\lim_{\|z\| \to \infty} \frac{\|\pi(z)\|}{\alpha_3(\|z\|)} = 0.$ □

While we allow the zero dynamics (1) with $y \equiv 0$ to be unstable, Assumption 1 implies that the zero dynamics (1) with the "control input" $y = \sigma(z) + d$ is ISS with respect to the disturbance d. More

[1] The class of systems that are globally diffeomorphic to the system (1)-(2) is characterized in [17]-[18]. For simplicity we assume that this transformation is valid globally, but we stress that it needs to exist only in a set containing the path that needs to be followed.

precisely, **A2** and **A3** imply that there exists a function $\varrho \in \mathcal{K}^\infty$ and a positive definite function α_z such that (for more details see [12]):

$$\|z\| \geq \varrho^{-1}(\|d\|) \implies \frac{\partial V_z}{\partial z} f(z, \sigma(z) + d) \leq -\alpha_z(\|z\|) \ . \tag{3}$$

When the subsystem (1) is a controllable linear system, that is $f(z, y) = A_z z + B_z y$ with (A_z, B_z) controllable, Assumption 1 is automatically satisfied. Namely, it is straightforward to show that **A1-A3** hold if we take: $\sigma(z) = -K_z z$ where K_z is such that the matrix $A_z - B_z K_z$ is Hurwitz; $V_z(z) = z^T P_z z$ where the matrix $P_z = P_z^T > 0$ solves the Lyapunov equation $(A_z - B_z K_z)^T P_z + P_z(A_z - B_z K_z) = -I$; and $\pi(z) = B_z^T P_z z$.

It is well known (see [14]) that the presence of unstable zero dynamics prevents asymptotic tracking of arbitrary reference signals $y_d(t)$ and in such cases we may not be able to achieve $\limsup_{t\to\infty} \|y(t) - y_d(t)\| \leq \epsilon$ with arbitrarily small $\epsilon > 0$. The path following problem reformulates and relaxes the tracking problem in order to achieve arbitrarily small errors between the output and a given desired "path" in the output space. We assume that the desired path $\mathcal{Y}_d$ is a closed and bounded one-dimensional smooth manifold $\mathcal{Y}_d \subset \mathbb{R}^m$. Moreover, for our problem formulation it is useful to consider paths that are parameterized with a scalar parameter θ, that is

$$\mathcal{Y}_d \triangleq \{y_d(\theta) = [y_{d1}(\theta) \cdots y_{dm}(\theta)]^T : \theta \geq 0\} \ . \tag{4}$$

A parameterized path $\mathcal{Y}_d$ is periodic in its parameter θ if there exists a constant $\theta_M > 0$ such that for all $\theta \geq 0$, $y_d(\theta) = y_d(\theta + \theta_M)$. We use the following assumption:

Assumption 2 *The parameterized path $\mathcal{Y}_d$ is periodic in its parameter with the period $\theta_M > 0$, and it satisfies $0 \in \mathrm{int}\{\mathrm{con}\mathcal{Y}_d\}$.* □

The path following problem that we consider is stated next.

Problem statement (path following):
Given a path $\mathcal{Y}_d$ parameterized with θ and an arbitrary $\epsilon > 0$ design a control law u as well as an appropriate signal $\theta = \theta(t)$ so that the following requirements hold for the closed-loop system:

R1 : Practical convergence to the path $\mathcal{Y}_d$: $\limsup_{t\to\infty} \|y(t) - y_d(\theta(t))\| \leq \epsilon$,

R2 : Forward motion along the path $\mathcal{Y}_d$: $\dot{\theta}(t) \geq 0$ and $\lim_{t\to\infty} \theta(t) = \infty$,

R3 : State boundedness: $\forall t \geq 0$, $\|(z(t), x(t))\| \leq n(\|(z(0), x(0))\|)$, $\|[\dot{\theta}(t) \ldots \theta^{(r^\star-1)}(t)]\| \leq M_\Theta$,

where $n : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $M_\Theta > 0$ is a positive constant. □

Note that path-following differs from the set stabilization problem where we would only require R1 but not R2 and the second bound in R3. We impose the requirement R2 to appropriately generalize reference tracking. Indeed, by requiring $\lim_{t\to\infty} \theta(t) = \infty$ in conjunction with R1 we forbid that the output y converges to a specific point on the path $\mathcal{Y}_d$ (hence, the path following problem is not the same as stabilization of the set $\mathcal{Y}_d$). Also, we forbid a backward motion along the path $\mathcal{Y}_d$ by requiring $\dot{\theta}(t) \geq 0$.

3 Main result

In this section we first provide a novel solution to the path-following problem and then we state our main result (Theorem 1). The proofs of all results are given in the appendix. Before we state the main result, we detail the construction of the control law u and $\theta(t)$ to satisfy conditions R1-R3.

The first step in our approach is to find a function $\sigma(\cdot)$ that satisfies Assumption 1. Note that this function can be interpreted as a control law that input-to-state stabilizes zero dynamics with respect to control additive disturbances. We consider only geometric paths $y_d(\cdot)$ that satisfy Assumption 2. Using such $\sigma(\cdot)$ and $y_d(\cdot)$, we construct u as follows

$$u = \varphi(z, x, \Theta, \omega) \triangleq [\varphi_1^T \dots \varphi_m^T]^T \ , \tag{5}$$

where

$$\varphi_i(z, x, \Theta, \omega) = y_{di}^{(r_i)}(\theta) + \sigma_i^{(r_i)}(z) - K_i \tilde{e}^i \ , \tag{6}$$

y_{di} come from (4), σ_i are appropriate scalar components of σ from Assumption 1, K_i are such that $A_{r_i} - B_{r_i} K_i$ are Hurwitz and we use the definitions:

$$\tilde{e}^i \triangleq [x_1^i - \sigma_i - y_d \dots x_{r_i}^i - \sigma_i^{(r_i-1)} - y_d^{(r_i-1)}]^T, \tilde{e} \triangleq [\tilde{e}^1 \dots \tilde{e}^m]^T, \tag{7}$$

This completes the controller design. To design $\theta(t)$, we first augment the system (1)-(2) with the following dynamics

$$\dot{\Theta} = A_{r^\star}\Theta + B_{r^\star}\omega \ , \tag{8}$$

where $\Theta \triangleq [\theta \dots \theta^{(r^\star-1)}]^T$ represents $r^\star$ additional states stemming from the path parameter θ and its first $r^\star - 1$ derivatives, and ω is an additional control input representing the highest derivative of θ, that is $\theta^{r^\star} \triangleq \omega$. A key feature of the path following is the possibility to design a control law for ω and determine the path parameter θ as a function of time and system states[2]. The fact that we can choose ω and, hence, $\theta(t)$ (subject to R2 and R3) in the path following problem formulation gives us extra degrees of freedom that turn out to be useful for non-minimum phase plants. Indeed, as already pointed out under our Assumption 1 our problem may not be solvable with the choice $\theta(t) = t$ used for the standard reference tracking and another choice is needed.

A direct consequence of Assumption 2 is the existence of a periodic function ϕ_δ parameterized with a positive scalar parameter δ and satisfying certain conditions that we need to construct ω in (8).

Lemma 1 *Suppose that the path $\mathcal{Y}_d$ satisfies Assumption 2. Then, for any $\delta > 0$ and $r \in \mathbb{N}$ there exists a locally Lipschitz function $\phi_\delta : [0, 1] \to \mathbb{R}$ such that*

$$\begin{aligned} \phi_\delta(0) &= \phi_\delta(1), & (9)\\ [\Phi_r(1)\ \Phi_{r-1}(1) \dots \Phi_1(1)]^T &= [\theta_M\ 0 \dots 0]^T \ , & (10)\\ \Phi_{r-1}(\tau) &\geq 0,\ \forall \tau \in [0, 1], & (11)\\ \left\| \int_0^1 y_d(\Phi_r(\tau)) d\tau \right\| &\leq \delta \ , & (12) \end{aligned}$$

where $\Phi_0(\tau) \triangleq \phi_\delta(\tau)$ and $\Phi_i(\tau) \triangleq \int_0^\tau \Phi_{i-1}(\tau_1) d\tau_1$ represents the i^{th} integral of the function ϕ_δ. ■

Now we can construct ω. Given a fixed $\delta > 0$ we let

$$\omega = \frac{1}{T^{r^\star}} \phi_\delta \left(\frac{t}{T} \right) \ , \tag{13}$$

where ϕ_δ comes from Lemma 1 and T is a positive scalar parameter. Both parameters δ and T in (13) will depend on the required tracking accuracy ϵ and they will be determined later (see the appendix). It is useful to discuss the reasons behind the conditions in Lemma 1. The condition (9) ensures continuity of the control signal ω with respect to t. The condition (10) requires that the path parameter θ traverses exactly one period of the path $\mathcal{Y}_d$ over the interval $[0, 1]$. Moreover, it requires that at $\tau = 1$ all derivatives of θ are equal to zero, which combined with $\Theta(0) = [\theta_0\ 0 \dots 0]^T$ implies that $y_d(\theta(\tau)) = y_d(\theta(\tau + 1))$ for all $\tau \geq 0$. The condition (11) is due to the requirement $\dot{\theta} \geq 0$ in R2, while the condition (12) bounds the effect of the motion of $y_d(\theta(\tau))$ over one period.

Our main result is stated next:

[2]The standard reference tracking, in which $\theta(t) = t$, is thus a special case of the path following where $\omega = 0$ and $\Theta(0) = [0\ 1\ 0 \dots 0]^T$.

Theorem 1 *Suppose that Assumptions 1 and 2 hold. Then, for any $\epsilon > 0$ there exist $\delta = \delta(\epsilon)$ and $T^\star = T^\star(\epsilon) > 0$ such that for all $T \in (0, T^\star]$ the solutions of the system (1), (2), (5), (8), (13) starting from the set of initial conditions*

$$\mathcal{X}_0 \triangleq \{[z^T \ x^T \ \Theta^T]^T : \ \Theta = [\theta_0 \ 0 \dots 0]^T, \theta_0 \geq 0\}, \tag{14}$$

satisfy the requirements R1 − R3. ■

The proof of Theorem 1 is given in the appendix. In order to explain the intuition behind our result we introduce a new state variable:

$$\Theta_T \triangleq [\theta_1 \ \theta_2 \ \dots \ \theta_{r^\star}]^T \triangleq [\theta \ T\dot{\theta} \ \dots \ T^{r^\star - 1}\theta^{(r^\star - 1)}]^T. \tag{15}$$

and using (7) we rewrite the closed loop system (1), (2), (5), (8), (13) in a form that is more amenable for analysis:

$$\begin{aligned} \dot{z} &= f(z, \sigma(z) + y_d(\theta_1) + \tilde{e}_y), & (16)\\ \dot{\tilde{e}}^i &= A_i \tilde{e}^i, \ i = 1, \dots, m, & (17)\\ T\dot{\Theta}_T &= A_{r^\star}\Theta_T + B_{r^\star}\phi_\delta\left(\frac{t}{T}\right), & (18) \end{aligned}$$

where $A_i \triangleq A_{r_i} - B_{r_i}K_{r_i}$ are Hurwitz by design and we used the definition

$$\tilde{e}_y \triangleq y - \sigma - y_d \triangleq [\tilde{e}_1^1 \dots \tilde{e}_m^1]^T .$$

The main idea behind our proof (see the appendix) is to first find an auxiliary output $\tilde{y} \triangleq y - \sigma(z)$ which is selected so that the resulting zero dynamics of the system (1)-(2) is ISS when the auxiliary output $\tilde{y}$ is treated as their input[3]. In other words, the system is rendered strongly minimum phase with respect to the auxiliary output. Note that the control law (5), (6) is designed to achieve the path following for the auxiliary output. Indeed, since we have that $||\tilde{y} - y_d|| = ||\tilde{e}_y|| \leq ||\tilde{e}||$ ($\tilde{e}$ comes from (7)) and for all $i = 1, \dots, m$ (17) are stable by design, we have that

$$\limsup_{t\to\infty} \|\tilde{y}(t) - y_d(\theta(t))\| = \limsup_{t\to\infty} ||\tilde{e}_y(t)|| = 0 . \tag{19}$$

Note that this holds irrespective of stability of the zero dynamics subsystem (16). The last step is to show that by adjusting δ and T in (13) we can force $z(t)$ to converge to an arbitrarily small neigbourhood of the origin. Since $\sigma(\cdot)$ was assumed to be locally Lipschitz and zero at zero (Assumption 1), this implies that for arbitrarily small $\epsilon > 0$ we can adjust δ and T so that

$$\limsup_{t\to\infty} \|\sigma(z(t))\| \leq \epsilon . \tag{20}$$

This immediately implies via (19) that the actual output $y(t)$ satisfies the requirement R1 in our problem statement. In order to show that (20) holds, we note that Assumption 1 implies that

$$\dot{V}_z = \frac{\partial V_z}{\partial z} f(z, \sigma(z) + d) \leq -\alpha_3(\|z\|) + \pi^T(z)d , \tag{21}$$

where we can now think of $d \triangleq y_d(\theta_1) + \tilde{e}_y$ (see (16)). Note that (19) implies that $e_y(t)$ converges to zero and hence $d(t) \approx y_d(\theta(t))$ for sufficiently large t. Moreover, our construction of ω can be used to show that (20) holds. Indeed, by assuming $d(t) \approx y_d(\theta(t))$ and integrating (21) over interval $[t, t+T]$, we obtain

$$V_z(z(t+T)) - V_z(z(t)) \leq -\int_t^{t+T} \alpha_3(\|z(s)\|)ds + \int_t^{t+T} \pi^T(z(s))y_d(\theta(s))ds . \tag{22}$$

[3]The idea of replacing the original by an auxiliary output is reminiscent to the flatness approach, see [15]-[16] and references therein. The key difference here is that instead of searching for a flat output which approximates the original output, we construct a control law for the path parameter θ to reduce the difference between the two outputs.

Our construction of ω (see (12) in Lemma 1), guarantees that $\int_t^{t+T} \pi^T(z(s))y_d(\theta(s))ds$ can be made arbitrarily small by choosing δ and T sufficiently small. Hence, we conclude that

$$V_z(z(t+T)) - V_z(z(t)) \leq -\int_t^{t+T} \alpha_3(\|z(s)\|)ds + \epsilon_1 \tag{23}$$

holds, where $\epsilon_1 > 0$ can be made arbitrarily small. Finally, using the averaging proof techniques (see [13]) we can show that (20) holds, which immediately gives us the requirement R1. The requitement R2 holds by construction of ϕ_δ in Lemma 1 and the requirement R3 is not hard to show using the fact that the zero dynamics are ISS and (17) are stable.

Remark 1 *It is useful to compare our results to [11], [12] that are most closely related to our work.*

Theorem 1 in [11] provides sufficient conditions for existence of feedback control laws for u and ω that ensure the requirements $\mathrm{R1}-\mathrm{R3}$ *for controllable linear systems and $\epsilon = 0$. Along with an additional technical condition, Theorem 1 in [11] requires that $0 \in \mathrm{int}\{\mathrm{con}\mathcal{U}\}$, where the set $\mathcal{U}$ is the repeatable path of the path $\mathcal{Y}_d$ defined by $\mathcal{U} \triangleq \{s \in \mathcal{Y}_d : \forall\theta_1 \geq 0,\ \exists\theta_2 > \theta_1,\ y_d(\theta_2) = s\}$. Since periodic paths satisfy $\mathcal{U} = \mathcal{Y}_d$, it follows that Assumption 2 implies the conditions of Theorem 1 in [11].*

Theorem 1 in [12] provides sufficient conditions (Assumptions 1-2) for existence of feedback control laws for u and ω that ensure the requirements $\mathrm{R1}-\mathrm{R3}$ *for system (1)-(2). While Assumption 1 is the same as here, Assumption 2 requires existence of constants $\bar{\theta}_M > \bar{\theta}_m > 0$ such that for all $z \in \mathbb{R}^{n-r}$ and $\theta \in \mathbb{R}^+$ it holds that $\min_{s\in[\theta+\bar{\theta}_m,\theta+\bar{\theta}_M]} \pi^T(z)y_d(s) \leq 0$. We note that Assumption 2 here implies the corresponding one in [12].* □

Remark 2 *While the design from [12] applies to systems and paths considered in this paper, the two designs are different and they lead to completely different control laws for ω, which results in different motions on the geometric path. For instance, we achieve periodic motion on the path in this paper and the period can be reduced by reducing T in (13). On the other hand, if we apply the design from [12] to the same class of periodic paths, the motion on the geometric path may not be periodic. Hence, the design in this paper provides an alternative design to [12] that may be more appropriate in situations when periodic motion on the path is required.* □

Remark 3 *Theorem 1 can be proved with a relaxed Assumption 1 but we do not pursue this direction here. Indeed, our main result can be proved if we replace* **A2**, **A3** *with the following assumptions:*
A2': $\frac{\partial V_z}{\partial z} f(z,\sigma(z)+d) \leq -\alpha_3(\|z\|) + \bar{\pi}^T(z)\varpi(d)$;
A3': $\lim_{\|z\|\to\infty} \frac{\|\bar{\pi}(z)\|}{\alpha_3(\|z\|)} = 0$, *where $\bar{\pi} : \mathbb{R}^{n-r} \to \mathbb{R}^l$, $\bar{\pi}(0) = 0$, and $\varpi : \mathbb{R}^m \to \mathbb{R}^l$ are locally Lipschitz functions.*

Remark 4 *If ω is allowed to depend on the path parameter θ then we can prove our result without requiring periodicity of the path $\mathcal{Y}_d$ in Assumption 2. Indeed, Assumption 2 can be replaced with the following condition*

$$\exists\theta_M > \theta_m > 0,\ \forall\theta \geq 0,\ 0 \in \mathrm{int}\{\mathrm{con}\mathcal{Y}_d^\theta\}\ ,$$

where $\mathcal{Y}_d^\theta \triangleq \{y_d(s) : s \in [\theta+\theta_m, \theta+\theta_M]\}$. Furthermore, if ω is allowed to depend on both the path parameter θ and the zero dynamics states z, then Assumption 2 can be replaced with a less demanding condition from [12]

$$\exists\theta_M > \theta_m > 0,\ \forall z \in \mathbb{R}^{n-r},\ \forall\theta \geq 0, \quad \min_{s\in[\theta+\theta_m,\theta+\theta_M]} \pi^T(z)y_d(s) \leq 0.$$

However, this is not anymore a purely geometric condition but a relationship between path geometry and stabilizability of zero dynamics. □

Example 1 *We apply our design to the system*

$$\begin{aligned} \dot{z}_1 &= z_1 - z_2^3 \sin y_2 + y_1, & \dot{z}_2 &= z_2 + z_1^3 \sin y_1 + y_2, \\ \dot{y}_1 &= u_1, & \dot{y}_2 &= u_2, \end{aligned} \tag{24}$$

and the path $\mathcal{Y}_d^\star = \left\{ [2\cos\theta - 1 \;\; 2\sin\theta - 1]^T : \theta \geq 0 \right\}$. *The output of interest is* $y = [y_1 \; y_2]^T$, *with respect to which the resulting zero dynamics* $\dot{z}_1 = z_1, \; \dot{z}_2 = z_2$ *are exponentially unstable.*

We select the auxiliary output $\tilde{y} \triangleq [\tilde{y}_1 \; \tilde{y}_2]^T = y - \sigma(z_1, z_2)$, *where* $\sigma(z_1, z_2) = -[4z_1 + z_1^3 \; 4z_2 + z_2^3]^T$, *with which system (24) becomes*

$$\begin{array}{ll} \dot{z}_1 = -4z_1 - z_1^3 - z_2^3 \sin(\tilde{y}_2 - 4z_2 - z_2^3) + \tilde{y}_1, & \dot{z}_2 = -4z_2 - z_2^3 + z_1^3 \sin(\tilde{y}_1 - 4z_1 - z_1^3) + \tilde{y}_2, \\ \dot{\tilde{y}}_1 = u_1 - \dot{\sigma}_1(z), & \dot{\tilde{y}}_2 = u_2 - \dot{\sigma}_2(z). \end{array} \tag{25}$$

Differentiating the Lyapunov function $V_z(z) = \frac{1}{2}\|z\|^2$ *along the solutions of system (25) we get* $\dot{V}_z \leq -4\|z\|^2 + \pi^T(z)\tilde{y}$, *where* $\pi(z) \triangleq [z_1 \; z_2]^T$, *hence Assumption 1 is satisfied. The path* $\mathcal{Y}_d^\star$ *is periodic with the period* $\theta_M = 2\pi$ *and satisfies* $0 \in \mathrm{int}\{\mathrm{con}\mathcal{Y}_d^\star\}$, *thus Assumption 2 holds and our design is applicable.*

Figure 1: a) The constructed function $\phi(\tau)$, and b) output trajectory $y(t)$ versus the path $\mathcal{Y}_d^\star$ in $y_1 - y_2$ plane.

We construct the feedback for u *using (6)-(7), set* $T = \frac{1}{4}$, *and compute the periodic function* ϕ *using Lemma 2. We omit the explicit expression for the resulting function* ϕ, *but show it on Fig. 1a. We simulate the resulting closed-loop system (24) for* $10s$ *from initial conditions* $[z_1(0) \; z_2(0) \; y_1(0) \; y_2(0)]^T = [0 \; 0 \; 2 \; -2]^T$, *and show the obtained behavior on Fig. 1b.* □

4 Conclusion

We considered a path-following problem in which the goal is to ensure that the error between the system output and the path be asymptotically smaller than a given constant, while guaranteeing output's forward motion along the path and boundedness of all states. For a class of parameterized periodic paths and a class of nonlinear systems with input-to-state stabilizable zero dynamics we used averaging tools to construct an open-loop control law for the path parameter and a feedback control law for the original control variable which solve the problem. This paper can be viewed as a sequel to [12], in which we develop an alternative, much simpler control design under more stringent conditions on path geometry.

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5 Appendix

Proof of Lemma 1: We construct the function $\theta^\star : [0,1] \to \mathbb{R}^+$, $\theta^\star \in C^r$, satisfying

$$\left[\theta^\star \ \frac{d\theta^\star}{d\tau} \ \ldots \ \frac{d^r\theta^\star}{d\tau^r}\right]^T |_{\tau=0} \quad = \quad \left[\theta^0 \ 0 \ldots 0\right]^T, \tag{26}$$

$$\left[\theta^\star \ \frac{d\theta^\star}{d\tau} \ \ldots \ \frac{d^r\theta^\star}{d\tau^r}\right]^T |_{\tau=1} \quad = \quad \left[\theta^0 + \theta_M \ 0 \ldots 0\right]^T, \tag{27}$$

$$\dot{\theta}^\star(\tau) \quad \geq \quad 0, \ \forall \tau \in [0,1], \tag{28}$$

$$\left\| \int_0^1 y_d(\theta^\star(\tau)) d\tau \right\| \quad \leq \quad \delta. \tag{29}$$

Then the claim of Lemma 2 follows by setting $\phi \triangleq \frac{d^r\theta^\star}{d\tau^r}$.

If the path $\mathcal{Y}_d$ satisfies Assumption 2, then from Caratheodory's theorem [20, pg. 155] it follows that there exist $y_i \in \mathcal{Y}_d$ and $\alpha_i \geq 0$, $\sum_{i=1}^{m+1} \alpha_i = 1$, such that $\sum_{i=1}^{m+1} \alpha_i y_i = 0$. From the path periodicity we deduce existence of the values $\theta_i \in [0, \theta_M]$, $i = 1, \dots m$, such that $y_d(\theta_i) = y_i$. Without loss of generality we assume that $\theta_{i+1} > \theta_i$, $i = 1, \dots, m$, and $\alpha_i > 0$, $i = 1, \dots, m+1$.

We define $T_\delta \triangleq \frac{1}{6}\frac{\delta}{M_y}$, $\tau_{2i} \triangleq \sum_{j=1}^{i} \alpha_j$, $\tau_{2i-1} \triangleq \sum_{j=1}^{i} \alpha_j - \min\{\frac{\alpha_i}{2}, \frac{1}{m+1}T_\delta\}$, and

$$\theta_c(\tau) \triangleq \begin{cases} \theta_i, & \tau \in [\tau_{2i-2}, \tau_{2i-1}], \\ \theta_i + \frac{\theta_{i+1}-\theta_i}{\min\{\frac{\alpha_i}{2}, \frac{1}{m+1}T_\delta\}}(\tau - \tau_{2i-1}), & \tau \in [\tau_{2i-1}, \tau_{2i}], \end{cases}$$

where $i = 1, \dots, m+1$, $\tau_0 = 0$, and $\theta_{2m+2} = \theta_M + \theta_1$. Note that the function θ_c is C^0, piecewise linear, nondecreasing, $\theta_c(0) = \theta_1$, $\theta_c(1) = \theta_M + \theta_1$, and satisfies

$$\begin{array}{l} \left\| \int_0^1 y_d(\theta_c(\tau))d\tau \right\| = \left\| \sum_{i=1}^{m+1} \alpha_i y_i - \sum_{i=1}^{m+1}(\tau_{2i} - \tau_{2i-1})y_i + \int_{\tau_{2i-1}}^{\tau_{2i}} y_d(\theta_c(\tau))d\tau \right\| = \\ \left\| \int_{\tau_{2i-1}}^{\tau_{2i}} [y_d(\theta_c(\tau)) - y_i]d\tau \right\| \leq 2M_y \sum_{i=1}^{m+1}(\tau_{2i} - \tau_{2i-1})y_i \leq \frac{1}{3}M_y T^\delta = \frac{1}{3}\delta. \end{array}$$

Thus the function θ_c satisfies conditions (26)-(29) with $\theta^0 = \theta_1$, but it is not sufficiently smooth, that is, $\theta_c \in C^0$ instead of $\theta_c \in C^r$. We smoothen the function θ_c by redefining it on sufficiently small neighborhoods containing the "kinks", that is, containing the points $\tau = \tau_i$, $i = 1, \dots, 2m+2$. Select the points τ^k, $k = 1, \dots, 4m+4$, $\tau_k > \tau_{k-1}$, $\tau^{4m+4} = 1$, such that $\tau_i \in (\tau^{2i-1}, \tau^{2i})$ and $\tau^{2i} - \tau^{2i-1} \leq \frac{1}{m+1}T^\delta$, $i = 1, \dots, 2m+2$. Let the function $\tilde{\theta}_i : [\tau^{2i-1}, \tau^{2i}] \to \mathbb{R}^+$, $\tilde{\theta}_i \in C^r$, satisfy the following conditions

$$\begin{array}{l} \dot{\theta}_i(\tau) \geq 0, \ \forall \tau \in [\tau^{2i-1}, \tau^{2i}], \\ \tilde{\theta}_i^{(j)}(\tau^{2i-1}) = \theta_c^{(j)}(\tau^{2i-1}), \\ \tilde{\theta}_i^{(j)}(\tau^{2i}) = \theta_c^{(j)}(\tau^{2i}), \ j = 0, \dots, r, \ i = 1, \dots, 2m+2, \end{array}$$

where $\tilde{\theta}_i^{(0)} = \tilde{\theta}_i$. Utilizing the functions $\tilde{\theta}_i$ we define the function $\theta^\star \in C^r$ by

$$\theta^\star(\tau) \triangleq \begin{cases} \theta_c(\tau), & \tau \notin [\tau^{2i-1}, \tau_{2i}], \ \forall i, \\ \tilde{\theta}_i(\tau), & \tau \in [\tau^{2i-1}, \tau^{2i}], \end{cases}$$

which by construction satisfies (26)-(28), but it also satisfies (29) due to

$$\begin{array}{l} \left\| \int_0^1 y_d(\theta^\star(\tau))d\tau \right\| \leq \left\| \int_0^1 y_d(\theta_c(\tau))d\tau \right\| + \left\| \int_0^1 (y_d(\theta^\star(\tau)) - y_d(\theta_c(\tau)))\, d\tau \right\| \\ \leq \frac{\delta}{3} + \left\| \sum_{i=1}^{2m+2} \int_{\tau^{2i-1}}^{\tau^{2i}} \left(y_d(\tilde{\theta}_i(\tau)) - y_d(\theta_c(\tau)) \right) \right\| \leq \frac{\delta}{3} + 2M_y \sum_{i=1}^{2m+2}(\tau^{2i} - \tau^{2i-1}) \leq \delta. \end{array} \tag{30}$$

■

Proof of Theorem 1: Let all conditions of Theorem 1 hold and let ϕ come from Lemma 1.

Let the path-following accuracy $\epsilon > 0$ be given. In order to construct $\delta, T^\star > 0$ we introduce $M_y \triangleq \sup_{\theta \geq 0} \|y_d(\theta)\|$, $c^\star \triangleq \alpha_2 \circ \varrho^{-1}(M_y) + 1$, $\Omega(c^\star) \triangleq \{z \in \mathbb{R}^{n-r} : V_z(z) \leq c^\star\}$, $M_\pi \triangleq \sup_{z \in \Omega(c^\star)} \|\pi(z)\|$ and

$$\begin{array}{lcl} L_\pi & \triangleq & \sup\limits_{z_1, z_2 \in \Omega(c^\star),\ z_1 \neq z_2} \dfrac{\|\pi(z_1) - \pi(z_2)\|}{\|z_1 - z_2\|}, \ L_\sigma \triangleq \sup\limits_{z_1, z_2 \in \Omega(c^\star),\ z_1 \neq z_2} \dfrac{\|\sigma(z_1) - \sigma(z_2)\|}{\|z_1 - z_2\|}, \\ L_{f_1} & \triangleq & \sup\limits_{z_1, z_2 \in \Omega(c^\star),\ z_1 \neq z_2,\ \|\tilde{y}\| \leq M_y + 1} \dfrac{\|f(z_1, \sigma(z_1) + \tilde{y}) - f(z_2, \sigma(z_2) + \tilde{y})\|}{\|z_1 - z_2\|}, \\ L_{f_2} & \triangleq & \sup\limits_{z \in \Omega(c^\star),\ \|\tilde{y}_1\|, \|\tilde{y}_2\| \leq M_y + 1,\ \tilde{y}_1 \neq \tilde{y}_2} \dfrac{\|f(z, \sigma(z) + \tilde{y}_1) - f(z, \sigma(z) + \tilde{y}_2)\|}{\|\tilde{y}_1 - \tilde{y}_2\|}, \end{array} \tag{31}$$

where $y_d(\cdot)$ is the parameterized path, $\rho(\cdot)$ comes from (3) and $\alpha_2(\cdot)$, $V_z(\cdot)$, $\sigma(\cdot)$ and $\pi(\cdot)$ come from Assumption 1. All constants in (31) are finite, since all the functions are assumed to be locally Lipschitz and the sets over which the suprema are taken are compact. Let $\delta \in (0, \epsilon)$ be arbitrary and choose $T^\star > 0$ such that for all $T \in (0, T^\star]$ we have that the following holds:

$$
\begin{aligned}
e^{-L_{f1}T}\left(\alpha_1^{-1}\left(\alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - 2M_\pi(M_y+1)T\right) + \frac{L_{f2}(M_y+1)}{L_{f1}}\right) - \frac{L_{f2}(M_y+1)}{L_{f1}} \quad &\geq \quad \frac{\delta}{L_\sigma}, && (32)\\
-\frac{1}{4}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T + M_y L_\pi\left(\alpha_1^{-1}(c^\star) + \frac{L_{f2}(M_y+1)}{L_{f1}}\right)\left(\frac{1}{L_{f1}}(e^{L_{f1}T}-1) - T\right) \quad &\leq \quad 0\ , && (33)
\end{aligned}
$$

where $\alpha_1(\cdot)$ comes from Assumption 1. We show next that such $T^\star$ always exists. Indeed, for $T = 0$ condition (32) reduces to $\epsilon \geq \delta$, hence it holds. Since all functions in (32) are continuous, this implies existence of a sufficiently small $T_1 > 0$ such that all $T \in (0, T_1]$ satisfy (32). Condition (33) can be rewritten as $g(T) \triangleq -c_1 T + c_2\left(\frac{1}{L_{f1}}(e^{L_{f1}T}-1) - T\right) \leq 0$, where $c_1, c_2 > 0$, and the function g satisfies $g(0) = 0$, $\frac{dg}{dT}|_{T=0} = -c_1 < 0$. This implies existence of $T_2 > 0$ such that all $T \in (0, T_2]$ satisfy $g(T) < 0$. Taking $T^\star \triangleq \min\{T_1, T_2\}$ proves the claim. For arbitrary $\delta_1 \in (0, \epsilon)$, we also introduce

$$\delta \triangleq \frac{1}{4M_\pi}\alpha_3\left(\frac{\delta_1}{L_\sigma}\right)\ .$$

Let $T \in (0, T^\star]$ be fixed and let ω be defined using the defined δ via Lemma 1 and (13).

Next we prove that R1-R3 hold for the closed loop system (16)-(18). First, we analyze behavior of the subsystem (17). Taking the derivative of Lyapunov function $V_e(\tilde{e}) \triangleq \tilde{e}^T P \tilde{e}$, where $P = \mathrm{diag}\{P_1, \ldots, P_m\}$, and $A_i^T P_i + P_i A_i \leq -I$, along solutions of subsystem (17), we get that $\dot{V}_e \leq -\|\tilde{e}\|^2$, and hence the errors $\tilde{e}$ and $\tilde{e}_y$ converge to zero, since

$$\|\tilde{e}_y(t)\| \leq \|\tilde{e}(t)\| \leq \frac{p_M}{p_m}\|\tilde{e}(0)\|e^{-\frac{t}{p_M}}, \qquad \forall t \geq 0\ , \tag{34}$$

where $p_m I \leq P \leq p_M I$. Next, we consider the behavior of the subsystem (16). Note that by Assumption 1 we have that (16) is input-to-state stable from input $d \triangleq y_d(\theta) + \tilde{e}_y$ to state z. Moreover, due to (34) and the definition of M_y we have that

$$\limsup_{t\to\infty} V_z(z(t)) < c^\star = \alpha_2 \circ \varrho^{-1}(M_y) + 1. \tag{35}$$

We will show that there exists time $t^\star > 0$ such that

$$V_z(z(t)) \leq \alpha_1\left(\tfrac{\epsilon}{L_\sigma}\right) \qquad \forall t \geq t^\star\ . \tag{36}$$

Note that (36) can be used to complete the proof. Indeed, using the fact that $||y_d(\theta) - y|| - ||\sigma(z)|| \leq ||y_d(\theta) - \sigma(z) - y|| = ||\tilde{e}_y||$, that $||\sigma(z)|| \leq L_\sigma ||z||$, (34) and since (36) implies that $\limsup_{t\to\infty} ||z(t)|| \leq \frac{\epsilon}{L_\sigma}$, we have that

$$\limsup_{t\to\infty} ||y_d(\theta(t)) - y(t)|| \leq \limsup_{t\to\infty} ||\sigma(z(t))|| \leq L_\sigma \limsup_{t\to\infty} ||z(t)|| \leq \epsilon\ ,$$

which completes the proof of R1. The requirement R2 is satisfied by construction due to (10)-(11), while the requirement R3 follows from (34)-(35), (10), and periodicity of the function ϕ.

The last thing we need to do is prove that (36) holds. We use the bounds

$$
\begin{aligned}
\dot{V}_z \leq -\alpha_3(\|z\|) + \pi^T(z)(y_d(\theta_1) + \tilde{e}_y) \quad &\leq \quad \|\pi(z)\|(\|y_d(\theta_1)\| + \|\tilde{e}_y\|), && (37)\\
L_{f1}\|z\| + L_{f2}(M_y+1) \quad &\geq \quad \|\dot{z}\| \ \geq\ -L_{f1}\|z\| - L_{f2}(M_y+1) && (38)
\end{aligned}
$$

which combined with $z(t_0) \in \Omega(c^\star)$, $\|\tilde{e}_y(t_0)\| \leq 1$ result in the following estimates

$$
\begin{aligned}
V_z(z(t)) \quad &\leq \quad V_z(z(t_0)) + (t - t_0)M_\pi(1 + M_y), && (39)\\
\|z(t) - z(t_0)\| \quad &\leq \quad (e^{L_{f1}(t-t_0)} - 1)\left(\|z(t_0)\| + \frac{L_{f2}}{L_{f1}}(1+M_y)\right), && (40)\\
\|z(t)\| \quad &\geq \quad e^{-L_{f1}(t-t_0)}\left(\|z(t_0)\| + \frac{L_{f2}}{L_{f1}}(1+M_y)\right) - \frac{L_{f2}}{L_{f1}}(1+M_y). && (41)
\end{aligned}
$$

From (34)-(35) it follows that there exists time $t_1 > 0$ such that $\forall t \geq t_1$, $V_z(z(t)) \leq c^\star$ and $\|\tilde{e}_y(t)\| \leq \min\left\{1, \frac{1}{4M_\pi}\alpha_3\left(\frac{\delta}{L_\sigma}\right)\right\}$. If $\forall t \geq t_1$ it holds that $V_z(z(t)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - 2M_\pi(M_y+1)T$, then condition (36) is satisfied for $t^\star = t_1$. Suppose that there exists $t_2 \geq t_1$ for which $V_z(z(t_2)) \in \left[\alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - 2M_\pi(M_y+1)T, c^\star\right]$ which combined with (32) and (41) implies

$$\begin{array}{l} \|z(t_2+T)\| \geq \frac{\delta}{L_\sigma}, \\ \forall t \in [t_2, t_2+T],\ \frac{1}{4}\alpha_3(\|z(t)\|) \geq \pi^T(z(t))\tilde{e}_y(t). \end{array} \tag{42}$$

Integrating (37) over the period $t \in [t_2, t_2+T]$, and utilizing (42) we get

$$\begin{array}{l} V_z(z(t_2+T)) - V_z(z(t_2)) \leq -\int_{t_2}^{t_2+T}\alpha_3(\|z(t)\|)dt + \int_{t_2}^{t_2+T}\pi^T(z(t))(y_d(\theta_1(t)) + \tilde{e}_y(t))dt \\ \leq -\frac{3}{4}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T + \int_{t_2}^{t_2+T}\pi^T(z(t))y_d(\theta_1(t))dt \leq -\frac{1}{4}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T, \end{array} \tag{43}$$

where we bounded the term $\int_{t_2}^{t_2+T}\pi^T(z(t))y_d(\theta_1(t))dt$ by combining (12), (40) and (33),

$$\begin{array}{l} \int_{t_2}^{t_2+T}\pi^T(z(t))y_d(\theta_1(t))dt \leq \pi^T(z(t_2))\int_{t_2}^{t_2+T}y_d(\theta_1(t))dt + M_y\int_{t_2}^{t_2+T}\|\pi(z(t)) - \pi(z(t_2))\|dt \\ \leq \frac{1}{4}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T + M_yL_\pi\left(\|z(t_2)\| + \frac{L_{f2}}{L_{f1}}(1+M_y)\right)\int_{t_2}^{t_2+T}(e^{L_{f1}(t-t_2)} - 1)dt \\ \leq \frac{1}{4}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T + M_yL_\pi\left(\alpha_1^{-1}(c^\star) + \frac{L_{f2}}{L_{f1}}(1+M_y)\right)\left(\frac{1}{L_{f1}}(e^{L_{f1}T} - 1) - T\right) \leq \frac{1}{2}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T. \end{array} \tag{44}$$

From (43) we deduce that there exists $t_3 = t_2 + mT$, $m \in \mathbb{N}$, such that $V_z(z(t_3)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - 2M_\pi(M_y+1)T$. We show that there exists time $t_4 \triangleq t_3 + lT$, $l \in \mathbb{N}$, such that

$$V_z(z(t_4)) \quad \leq \quad \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - 2M_\pi(M_y+1)T, \tag{45}$$

$$\forall t \quad \in \quad [t_3, t_4],\ V_z(z(t)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right). \tag{46}$$

Bound (39) guarantees that $V_z(z(t)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - M_\pi(M_y+1)T$, $\forall t \in [t_3, t_3+T]$. If $V_z(z(t_3+T)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - 2M_\pi(M_y+1)T$, then (45)-(46) hold for $t_4 = t_3 + T$. If however, $V_z(z(t_3+T)) \geq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - 2M_\pi(M_y+1)T$ then from (39) and (43) we have $V_z(z(t_3+2T)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - M_\pi(M_y+1)T - \frac{1}{2}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T$ and $V_z(z(t)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right)$, $\forall t \in [t_3+T, t_3+2T]$. Repeating this argument l times, where $l \geq \frac{2M_\pi(M_y+1)}{\alpha_3\left(\frac{\delta}{L_\sigma}\right)}$, we get $V_z(z(t_3+(l+1)T)) \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right) - M_\pi(M_y+1)T - \frac{l}{2}\alpha_3\left(\frac{\delta}{L_\sigma}\right)T \leq \alpha_1\left(\frac{\epsilon}{L_\sigma}\right)$. Thus conditions (45)-(46) hold for $t_4 = t_3 + (l+1)T$. Finally, by induction we have that the condition (36) holds with $t^\star = t_3$.

■