Topological semantics for intuitionistic modal logics I: completeness, Gödel translations, and bisimulations

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Abstract
We take the well-known intuitionistic modal logic of Fischer Servi with semantics in bi-relational Kripke frames, and give the natural extension to topological Kripke frames. The two frame conditions relating the intuitionistic partial order (or pre-order) with the modal accessibility relation are shown to generalise to the requirement that the relation and its converse be lower semi-continuous with respect to the topology. We establish the topological completeness of the Fischer Servi axiomatization by exhibiting a canonical model that does not correspond to any bi-relational frame. Fischer Servi’s extension of the Gödel translation maps intuitionistic modal logic into the bi-modal (or multi-modal) fusion of classical S4 with K (or its modal or tense extensions), plus two further axioms expressing interaction between the S4 and K modalities which characterize the lower semi-continuity of the K accessibility relation and its converse with respect to the topology interpreting the S4 modalities. Analogously, we prove topological completeness of these classical bi-modal and multi-modal logics by exhibiting a canonical model whose topology does not correspond to any S4 relational frame. We then relate the intuitionistic and classical canonical models via a semantic map derived from the extended Gödel translation, and show it to be a topological bisimulation, where the conditions beyond the usual for bisimulations are that the map and its converse be lower semi-continuous. We conclude the paper by using the canonical model to give a Hennessy-Milner type result on maximal topological bisimulations preserving the intuitionistic semantics. The Hennessy-Milner class we identify includes models of continuous dynamics within Euclidean space, which are the focus of work in the formal analysis of hybrid and other complex systems.

**Keywords:** topological semantics, intuitionistic modal logics, Gödel translation, axiomatic completeness, bisimulations.

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1 Introduction and motivation

Topological semantics for intuitionistic logic and for the classical modal logic S4 have a long history going back to Tarski and co-workers in the 1930s and 40s, predating the relational Kripke semantics for both [25, 30]. A little earlier again is the 1933 Gödel translation $\text{GT} \, [\varphi]$ of intuitionistic logic into classical S4. The translation makes perfect sense within the topological semantics: $\square$ is interpreted by topological interior, and the translation $\text{GT}(\neg \varphi) = \square \neg \text{GT}(\varphi)$ says that intuitionistic negation calls for the interior of the complement, and not just the complement.

In the topological semantics, the basic semantic object is the denotation set $[\varphi]^M$ of a formula $\varphi$, consisting of the set of all states/worlds of the model $\mathcal{M}$ at which the formula is true, and the semantic clauses of the logic are given in terms of operations on sets of states. The intuitionistic requirement on the semantics is that all formulae must denote open sets; that is, sets that are equal to their own interior. Any formula $\varphi$ partitions the state space $X$ into three disjoint sets: $[\varphi]^M$ and $[\neg \varphi]^M$ and $bd_T([\varphi]^M)$, with the points in the topological boundary set $bd_T([\varphi]^M)$ falsifying the law of excluded middle, since they neither satisfy nor falsify $\varphi$.

For the extension to intuitionistic modal logics, Fischer Servi developed semantics over bi-relational Kripke frames in the late 1970s, which has generated a good deal of research [13, 14, 15, 12, 19, 28, 31, 33, 34]. What is surprising is that there seems to be little in the literature on combining the two: keeping the topology to interpret the intuitionistic base logic, adding a binary relation to interpret the modal operators, and then characterizing frame conditions on the interaction between modal and intuitionistic semantics in terms of topological properties of the modal accessibility relation.

In this paper, we present semantics for intuitionistic modal logic over topological frames $\mathcal{F} = (X, T, R)$ where $(X, T)$ is a topological space and $R \subseteq X \times X$ is a binary relation. We show that over topological frames, the two Fischer Servi bi-relational frame conditions generalize to a semi-continuity property of the relation, and of its converse, with respect to the topology. As for the base logic, Fischer Servi’s extension of the Gödel translation reads as a direct transcription of the topological semantics. Where $\square$ is the box modality for the accessibility relation, and $\Box$ is topological interior, the translation $\text{GT}(\square \varphi) = \Box \text{GT}(\varphi)$ says that the intuitionistic box requires the interior of the classical box operator, since the latter is defined by an intersection and may fail to preserve open sets. In contrast, the translation clause $\text{GT}(\Diamond \varphi) = \Diamond \text{GT}(\varphi)$ says that, semantically, the operator $\Diamond$ preserves open sets, and this condition is exactly the lower semi-continuity (l.s.c.) condition on the accessibility relation. Similarly, Fischer Servi’s other frame
condition characterizes the lower semi-continuity of the converse relation\(^1\). The symmetry of the frame conditions on the modal relation and its converse becomes much more transparent in their topological form. Moreover, it soon becomes clear that when we lift the topological semantics to intuitionistic tense logics extending Fischer Servi’s modal logic (introduced by Ewald in [12]), with modalities in pairs \(\Box, \mathcal{E}\), and \(\Diamond, \mathcal{D}\), for future and past along the accessibility relation, the resulting semantics and metatheoretic results such as completeness come out cleaner and simpler for the tense logic than they do for the modal logic. We can often streamline arguments involving the box modality \(\Box\) by using its adjoint diamond \(\Diamond\), which like \(\Box\), preserves open sets. In addition, the extra expressive power is used in an intended application, discussed more below, and in the sequel paper \(^2\).

As is now well-known (see, for example [20, 18, 26, 32, 1]), the relational Kripke semantics for intuitionistic logic and for S4, based on partial orders or pre-orders, can be seen as a special case of the topological semantics for these logics by considering the one-to-one correspondence between pre-orders \(\preceq\) on a space \(X\), and topologies \(\mathcal{T}\) on \(X\) known as Alexandroff\(^3\). Such topologies are characterized by the property that for every state \(x \in X\), there is a smallest open set \(U \in \mathcal{T}\) such that \(x \in U\), or equivalently, \(\mathcal{T}\) is closed under arbitrary intersections. In particular, any topology with only a finite number of opens sets (and thus all topologies on finite spaces) are Alexandroff.

One of the main contributions of this paper is that we prove completeness of the Fischer Servi and Ewald axiomizations of intuitionistic modal and tense logics with respect to topological semantics by exhibiting a non-Alexandroff topology on the canonical model, over the space of prime theories of the logic, and thus properly extend the semantics of the logics beyond the existing classes of bi-relational models. By analogous (but much simpler) means, we prove topological completeness of the classical multi-modal logic consisting of the fusion of S4 and

\(^1\)The only other work we know of which gives topological semantics for intuitionistic modal logics is that of Wijesekera [33], which is further investigated in Hilken [22, 4]: this logic is properly weaker than that of Fischer Servi. Wijesekera’s motivation was to develop a constructive base for concurrent dynamic logic, and in this setting, the intuitionistic diamond could fail to be normal (i.e. fail to distribute over disjunction). In the bi-relational and topological semantics in [33], or in the relational spaces in [22, 4], there are no frame conditions relating the intuitionistic partial order or topology with the modal accessibility relation, and the semantic clauses for both box and diamond require application of the interior operator to guarantee open sets.

\(^2\)Topological semantics for intuitionistic modal logics II: applications to approximate model-checking in modal and tense logics\(^5\), by the same authors, which we will refer to as Part II.

\(^3\)Terminology varies: Grzegorczyk uses the term “totally distributive” in [20], and Mints uses “teratological” in [26]; they are also sometimes termed “digital” topologies in the study of digital images, represented as subsets of \(\mathbb{Z} \times \mathbb{Z}\) [23]. This class of topologies was first identified by Alexandroff in a 1937 paper, under the name “Diskrete Räume”, and studied in his later text [2].

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tense logic, and our canonical model there also has a non-Alexandroff topology. We then relate the intuitionistic and classical canonical models via a semantic map derived from the extended Gödel translation, and show it to be a topological bisimulation, where the conditions beyond the usual for bisimulations are that the map and its converse be lower semi-continuous. Finally, we conclude with a Hennessy-Milner type result on maximal topological bisimulations that preserve the intuitionistic semantics, where the proof takes its route via our intuitionistic canonical model.

The Hennessy-Milner class of models that we identify is quite substantial, and includes models of continuous dynamics within Euclidean space, and hybrid (mixed continuous-discrete) systems with product spaces of discrete space and Euclidean space where the modal accessibility relation is that of reachability with respect to the hybrid dynamics of the system. These models are the focus of work in the formal analysis of hybrid control systems, which are ubiquitous in contemporary engineering practice, as they arise whenever a computer or digital device is used to control or regulate a physical process whose state changes continuously with time. For continuous dynamics, a quite general mathematical model is that of differential inclusions, which are the non-deterministic analog of differential equations. While differential equations uniquely determine the value of the derivative at each state, the more general solutions of differential inclusions have their derivatives constrained to lie within a set of values, where that set is a function of the state. Such models can arise by starting with a deterministic model, and then explicitly modeling any known factors of disturbance, noise or imprecision, to create robustness in the mathematical modeling [5]. So consider topological frames $\mathcal{F}$ where $X \subseteq \mathbb{R}^n$, $\mathcal{T}$ is the standard Euclidean topology, and $R$ is the reachability relation of a differential inclusion $\frac{d}{dt}\gamma(t) \in F(\gamma(t))$ where $F : X \to 2^{\mathbb{R}^n}$ is a set-valued map; i.e. $(x, x') \in R$ iff there exists a solution curve $\gamma : [0, \tau] \to X$ of the system that starts at state $x$ and leads to $x'$, with time interval $[0, \tau] \subseteq \mathbb{R}^+$ in the non-negative reals. Under some standard assumptions on $F$ [5, 6], the reachability relation $R$ and its converse will be l.s.c. (as well as reflexive and transitive). The formula $\Diamond p$ denotes the set of states reachable from the $p$ states, with $p$ considered as a source or initial state set, while the forward modal diamond formula $\Box p$ denotes the set of states from which $p$ states can be reached, here $p$ denoting a target or goal state set. This class of frames also provides primary motivation for the sequel Part II, where we address the task of approximate model-checking of modal and tense formulas in models where the exact evaluation of denotation sets is not possible – such as most models over uncountable state spaces. We take up this example in more detail in the latter part of Section 4.

The rest of the paper is organized as follows. Section 2 tersely reviews the necessary material from general topology. In Section 3, we recall the topological semantics for intuitionistic propo-
sitional logic and classical S4, and the Gödel translation from the former to the latter. Section 4 develops topological semantics for intuitionistic modal and tense logics, generalizing known results on bi-relational Kripke semantics. The main results are in Sections 5, 6 and 7. Section 5 addresses the topic of topological bisimulations, and we identify intuitionistic and topological analogs of modal saturation properties in order to demarcate our candidate Hennessy-Milner class. In Section 6, we prove completeness with respect to the topological semantics of Ewald’s axiomatization of intuitionistic tense logic, and of the classical multi-modal logic consisting of the fusion of classical S4 and tense logics. In the following section, we study their relationship via the Gödel translation, and show it to be that of a topological bisimulation, and then prove our Hennessy-Milner type result for topological bisimulations. The paper concludes with Section 8, consisting of a brief discussion of decidability and open problems.

2 Preliminaries: some general topology

We adopt the notation from set-valued analysis [5] in writing \( r : X \leadsto Y \) to mean both that \( r : X \to 2^Y \) is a set-valued map, with (possibly empty) set-values \( r(x) \subseteq Y \) for each \( x \in X \), and equivalently, that \( r \subseteq X \times Y \) is a relation. The expressions \( y \in r(x) \), \( (x, y) \in r \) and \( x \circ y \) are synonymous, and we use the terms “map” and “relation” interchangeably. Every map \( r : X \leadsto Y \) has an inverse or converse \( r^{-1} : Y \leadsto X \) given by: \( x \in r^{-1}(y) \) iff \( y \in r(x) \). The domain of a set-valued map is \( \text{dom}(r) := \{ x \in X \mid r(x) \neq \emptyset \} \), and the range is \( \text{ran}(r) := \text{dom}(r^{-1}) \subseteq Y \). A map \( r : X \leadsto Y \) is injective if \( x \neq x' \) implies \( r(x) \neq r(x') \), for all \( x, x' \in \text{dom}(r) \), and is surjective if \( \text{ran}(r) = Y \). A map \( r : X \leadsto Y \) is total on \( X \) if \( \text{dom}(r) = X \). We will write (as usual) \( r : X \to Y \) to mean \( r \) is a function, i.e. a single-valued map total on \( X \) with values written \( r(x) = y \) (rather than \( r(x) = \{ y \} \))\(^4\). For \( r_1 : X \leadsto Y \) and \( r_2 : Y \leadsto Z \), we write their relational/sequential composition as \( r_1 \circ r_2 : X \leadsto Z \) given by \( (r_1 \circ r_2)(x) := \{ z \in Z \mid (\exists y \in Y) [(x, y) \in r_1 \land (y, z) \in r_2] \} \), in sequential left-to-right application order; this is the reverse of the usual order for composition of functions (and we refrain from using the symbol \( \circ \) to avoid any confusion), but is better suited for our purposes. Recall that \( (r_1 \circ r_2)^{-1} = r_2^{-1} \circ r_1^{-1} \).

Recall that a pre-order is a reflexive and transitive binary relation, and a partial order is a pre-order that is also antisymmetric. On notation, for partial orders or pre-orders \( \subseteq, \leq, \sqsubseteq \), we write \( \subset, \prec, \sqsubset \) for the corresponding strict partial orders or pre-orders; i.e. \( x \prec x' \) iff \( x \leq x' \)
and not $x' \preceq x$. Likewise, we write $\supseteq, \triangleright, \supset$ for the corresponding converse partial orders or pre-orders.

A relation $r : X \leadsto Y$ determines two pre-image operators (predicate transformers). The lower or existential pre-image function $r^{-\exists} : 2^Y \to 2^X$ is given by

$$r^{-\exists}(W) := \{x \in X \mid (\exists y \in Y)[(x, y) \in r \land y \in W]\} = \{x \in X \mid W \cap r(x) \neq \emptyset\}$$  \hspace{1cm} (1)

for $W \subseteq Y$. The upper or universal pre-image $r^{-\forall} : 2^Y \to 2^X$ is dual w.r.t. set-complement:

$$r^{-\forall}(W) := X - r^{-\exists}(Y - W) = \{x \in X \mid r(x) \subseteq W\}$$  \hspace{1cm} (2)

In words, $x \in r^{-\exists}(W)$ iff some $r$-successor of $x$ lies in $W$, while $x \in r^{-\forall}(W)$ iff all $r$-successors of $x$ lie in $W$, including $x \notin \text{dom}(r)$. The operator $r^{-\exists}$ distributes over arbitrary unions, while $r^{-\forall}$ distributes over arbitrary intersections: $r^{-\exists}(\emptyset) = \emptyset, r^{-\exists}(Y) = \text{dom}(r), r^{-\forall}(\emptyset) = X - \text{dom}(r)$, and $r^{-\forall}(Y) = X$. Note that when $r : X \to Y$ is a function, the pre-image operators reduce to the standard inverse-image operator; i.e. $r^{-\exists}(W) = r^{-\forall}(W) = r^{-1}(W)$. The pre-image operators respect relational inclusions: if $r_1 \subseteq r_2 \subseteq X \times Y$, then for all $W \subseteq Y$, we have $r_1^{-\exists}(W) \subseteq r_2^{-\exists}(W)$, but reversing to $r_2^{-\forall}(W) \subseteq r_1^{-\forall}(W)$ for the universal operators. For binary relations $r : X \leadsto X$ on a space $X$, the pre-images express in operator form the standard relational Kripke semantics for the (future) diamond and box modal operators determined by $r$.

The operators on sets derived from the converse relation $r^{-1}$ are usually called the post-image operators $r^{\exists}, r^{\forall} : 2^X \to 2^Y$ defined by $r^{\exists} := (r^{-1})^{-\exists}$ and $r^{\forall} := (r^{-1})^{-\forall}$. These operators arise in the relational Kripke semantics for the post diamond and box modal operators in tense and temporal logics. The fundamental relationship between pre- and post-images is the adjoint property: for all $W \subseteq X$ and $V \subseteq Y$,

$$W \subseteq r^{-\forall}(V) \iff r^{\exists}(W) \subseteq V.$$  \hspace{1cm} (3)

Note that for compositions of relations, with $r_1 : X \leadsto Y$ and $r_2 : Y \leadsto Z$, the pre- and post-image operators satisfy $(r_1 \bullet r_2)^{-Q}(V) = r_1^{-Q}(r_2^{-Q}(V))$ and $(r_1 \bullet r_2)^Q(W) = r_2^Q(r_1^Q(W))$ for quantifiers $Q \in \{\exists, \forall\}$, and sets $V \subseteq Z$ and $W \subseteq X$.

Recall that a topology $\mathcal{T} \subseteq 2^X$ on a set $X$ is a family of subsets of $X$ that is closed under arbitrary unions and finite intersections. So $\mathcal{T}$ is a distributive lattice of sets. The extreme cases are the discrete topology $\mathcal{T}_D = 2^X$, and the trivial topology $\mathcal{T}_\emptyset = \{\emptyset, X\}$. The interior operator $\text{int}_r : 2^X \to 2^X$ determined by $\mathcal{T}$ is given by $\text{int}_r(W) := \bigcup \{U \in \mathcal{T} \mid U \subseteq W\}$. Sets $W \in \mathcal{T}$ are called open w.r.t. $\mathcal{T}$, and this is so iff $W = \text{int}_r(W)$. Let $\neg \mathcal{T}$ denote the dual lattice under...
set-complement; i.e., \( \neg \mathcal{T} := \{ V \subseteq X \mid (X - V) \in \mathcal{T} \} \). Sets \( W \in \neg \mathcal{T} \) are called \textit{closed} w.r.t. \( \mathcal{T} \), and this is so iff \( W = \text{cl}_\mathcal{T}(W) \), where the dual \textit{closure operator} \( \text{cl}_\mathcal{T} : 2^X \rightarrow 2^X \) is given by \( \text{cl}_\mathcal{T}(W) := \bigcap \{ V \in \neg \mathcal{T} \mid W \subseteq V \} \). The topological \textit{boundary} is \( \text{bd}_\mathcal{T}(W) := \text{cl}_\mathcal{T}(W) - \text{int}_\mathcal{T}(W) \).

A family of open sets \( \mathcal{B} \subseteq \mathcal{T} \) constitutes a \textit{basis} for a topology \( \mathcal{T} \) on \( X \) if every open set \( W \in \mathcal{T} \) is a union of basic opens in \( \mathcal{B} \), and for every \( x \in X \) and every pair of basic opens \( U_1, U_2 \in \mathcal{B} \) such that \( x \in U_1 \cap U_2 \), there exists \( U_3 \in \mathcal{B} \) such that \( x \in U_3 \subseteq (U_1 \cap U_2) \). A family of sets \( \{ W_i \}_{i \in I} \) in \( X \) has the \textit{finite intersection property} if the intersection of every finite sub-family is non-empty; i.e., for every finite subset \( F \subseteq I \) of indices, \( \bigcap_{i \in F} W_i \neq \emptyset \). An elementary result we use is that a topological space \((X, \mathcal{T})\) is \textit{compact} iff for every family of sets \( \{ W_i \}_{i \in I} \) with the finite intersection property, the intersection of all the closures is non-empty: \( \bigcap_{i \in I} \text{cl}_\mathcal{T}(W_i) \neq \emptyset \).

The purely topological notion of \textit{continuity} for a function \( f : X \rightarrow Y \) is that the inverse image \( f^{-1}(U) \) is open whenever \( U \) is open. Analogously, the pre-image operators can be used to characterize purely topological notions of continuity for relations/set-valued maps, as introduced by Kuratowski and Bouligand in the 1920s. Given two topological spaces \((X, \mathcal{T})\) and \((Y, \mathcal{S})\), a map \( R : X \rightrightarrows Y \) is called: \textit{lower semi-continuous (l.s.c.)} if for every \( \mathcal{S}\)-open set \( U \) in \( Y \), \( R^{-\mathcal{S}}(U) \) is \( \mathcal{T}\)-open in \( X \); \textit{upper semi-continuous (u.s.c.)} if for every \( \mathcal{S}\)-open set \( U \) in \( Y \), \( R^{-\mathcal{S}}(U) \) is \( \mathcal{T}\)-open in \( X \); and simply \textit{continuous} if it is both l.s.c. and u.s.c. \cite{24, 5}. The u.s.c. condition is equivalent to \( R^{-\mathcal{S}}(V) \) is \( \mathcal{T}\)-closed in \( X \) whenever \( V \) is \( \mathcal{S}\)-closed in \( Y \). From a modal logic perspective, the semi-continuity conditions are best appreciated as inclusions of operators on sets: \( R : X \rightrightarrows Y \) is l.s.c. iff \( R^{-\mathcal{S}}(\text{int}_\mathcal{S}(W)) \subseteq \text{int}_\mathcal{T}(R^{-\mathcal{S}}(W)) \) for all \( W \subseteq Y \); and \( R : X \rightrightarrows Y \) is u.s.c. iff \( R^{-\mathcal{S}}(\text{int}_\mathcal{S}(W)) \subseteq \text{int}_\mathcal{T}(R^{-\mathcal{S}}(W)) \) for all \( W \subseteq Y \) \cite{24, Vol. I, §181, p.173}. Obviously, both of the semi-continuity conditions reduce to the standard notion of continuity for functions \( R : X \rightarrow Y \). The semi-continuity properties are preserved under relational composition, and also under finite unions of relations.

As noted in the introduction, we have particular interest in \textit{Alexandroff topologies} because of their correspondence with Kripke semantics. A topology \( \mathcal{T} \) on \( X \) is called Alexandroff if for every \( x \in X \), there is a \textit{smallest} open set \( U \in \mathcal{T} \) such that \( x \in U \). In particular, every \textit{finite} topology (i.e. only finitely many open sets) is Alexandroff. There is a one-to-one correspondence between pre-orders on \( X \) and Alexandroff topologies on \( X \). Any pre-order \( \preceq \) on \( X \) induces an Alexandroff topology \( \mathcal{T}_{\preceq} \) by taking \( \text{int}_{\mathcal{T}_{\preceq}}(W) := (\preceq)^{-\mathcal{T}}(W) \), which means \( U \in \mathcal{T}_{\preceq} \) iff \( U \) is upwards-\( \preceq \)-closed, and \( V \in \neg \mathcal{T}_{\preceq} \) iff \( V \) is downwards-\( \preceq \)-closed, and \( d_{\mathcal{T}_{\preceq}}(W) = (\preceq)^{-\mathcal{T}}(W) \). In particular, \( \mathcal{T}_{\preceq} \) is closed under arbitrary intersections as well as arbitrary unions, and \( \neg \mathcal{T}_{\preceq} = \mathcal{T}_{\succeq} \); i.e. the converse \( \succeq \) of the pre-order generates the closed sets of \( \preceq \). Conversely, for any topology, define a pre-order \( \preceq_{\mathcal{T}} \) on \( X \), known as the \textit{specialisation pre-order}: \( x \preceq_{\mathcal{T}} y \) iff \( (\forall U \in \mathcal{T})[x \in U \Rightarrow y \in U] \). For any
pre-order, $\preceq_{\mathcal{T}_1} = \preceq$, and for any topology, $\mathcal{T} \preceq \mathcal{T}$ iff $\mathcal{T}$ is Alexandroff\(^5\).

For any topology $\mathcal{T}$, the relation $\approx$ on $X$ of topological equivalence under $\mathcal{T}$ (or the Stone $T_0$ quotient), is given by $\approx := (\preceq \cap \approx)$, so $x \approx y$ means $x$ and $y$ belong to all the same $\mathcal{T}$-open sets. A topology $\mathcal{T}$ has $T_0$ separation iff the pre-order $\preceq$ is a partial order, which is the case iff $\approx$ is identity. In the hierarchy of separation properties for topologies, $T_0$ is the weakest. A topology $\mathcal{T}$ has $T_1$ separation iff every singleton set $\{x\}$ for $x \in X$ is closed, and is Hausdorff or has $T_2$ separation iff for every pair of distinct points $x_1, x_2 \in X$, there exists disjoint open sets $U_1, U_2 \in \mathcal{T}$ such that $x_1 \in U_1$ and $x_2 \in U_2$. So $T_2$ implies $T_1$ implies $T_0$, and the only Alexandroff topology that is $T_2$ or $T_1$ is the discrete topology $\mathcal{T}_D$. In our study of a Hennessy-Milner class, we have use for yet stronger separation properties. A topology $\mathcal{T}$ has regular or $T_3$ separation iff it is $T_1$ and for each pair consisting of a point $x \in X$ and a closed set $C$ not containing $x$, there exist disjoint open sets $U_1, U_2 \in \mathcal{T}$ such that $x \in U_1$ and $C \subseteq U_2$. A topology $\mathcal{T}$ has normal or $T_4$ separation iff it is $T_1$ and for each pair of disjoint closed sets $C_1$ and $C_2$, there exist disjoint open sets $U_1, U_2 \in \mathcal{T}$ such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$. So $T_4$ implies $T_3$ implies $T_2$. If $(X, \mathcal{T})$ is a compact Hausdorff space, or is a regular space with a countable basis, then $\mathcal{T}$ is normal.

3 Classical S4 and intuitionistic propositional logic

Fix a countably infinite set $AP$ of atomic propositions. The propositional language $\mathcal{L}_0$ is generated from $p \in AP$ by the grammar:

$$\varphi ::= p \mid \bot \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \rightarrow \varphi_2$$

As usual, further connectives are defined: $\neg \varphi ::= \varphi \rightarrow \bot$ and $\varphi_1 \leftrightarrow \varphi_2 ::= (\varphi_1 \rightarrow \varphi_2) \land (\varphi_2 \rightarrow \varphi_1)$, and the constant $\top ::= \bot \rightarrow \bot$. Let $\mathcal{L}_{0, \Box}$ be the mono-modal language extending $\mathcal{L}_0$ with the addition of the unary modal operator $\Box$. A further modal operator $\Diamond$ can be defined as the classical dual: $\Diamond \varphi ::= \neg \Box \neg \varphi$.

Let $\text{IPC} \subseteq \mathcal{L}_0$ be the set of intuitionistic propositional theorems, and abusing notation, let $\text{IPC}$ also denote one's favourite axiomatisation for intuitionistic propositional logic. Likewise, let $\text{S4} \Box \subseteq \mathcal{L}_{0, \Box}$ be the set of theorems of classical S4, and let $\text{S4} \Box$ also denote any standard axiomatisation of classical S4. To be concrete, let $\text{S4} \Box$ contain all instances of classical propositional

\(^5\)This correspondence is set out in [18], pp. 135-136; in [1], §3.1.1; and also in [32].
tautologies in the language \( \mathcal{L}_{0, \Box} \), and the axiom schemes:

\[
\begin{align*}
\mathbf{N} & : \quad \Box T \\
\mathbf{R} & : \quad \Box (\varphi_1 \land \varphi_2) \leftrightarrow \Box \varphi_1 \land \Box \varphi_2 \\
\mathbf{4} & : \quad \Box \varphi \rightarrow \Box \Box \varphi
\end{align*}
\]

and be closed under the inference rules of \textit{modus ponens} (\( \text{MP} \)) and \( \Box \)-monotonicity (\( \text{Mono}\Box \)):

from \( \varphi_1 \rightarrow \varphi_2 \) infer \( \Box \varphi_1 \rightarrow \Box \varphi_2 \).

On notation, for any axiomatically presented logic \( \Lambda \) in a language \( \mathcal{L} \), set of formulae \( \mathcal{A} \subseteq \mathcal{L} \) and formula \( \varphi \in \mathcal{L} \), we write \( \mathcal{A} \vdash_{\Lambda} \varphi \) to mean that there exists a finite set \( \{\psi_1, \ldots, \psi_n\} \subseteq \mathcal{A} \) of formulae such that \( (\psi_1 \land \cdots \land \psi_n) \rightarrow \varphi \) is a theorem of \( \Lambda \) (allowing \( n = 0 \) and \( \varphi \) is a theorem of \( \Lambda \)). The relation \( \vdash_{\Lambda} \subseteq 2^\mathcal{L} \times \mathcal{L} \) is the consequence relation of \( \Lambda \). We will abuse notation (as we have with \( \text{IPC} \) and \( \text{S4}\Box \)) and identify \( \Lambda \) with its set of theorems, i.e. \( \Lambda = \{\varphi \in \mathcal{L} \mid \mathcal{A} \vdash_{\Lambda} \varphi \} \).

We consider a variant of the Gödel translation \( \text{GT} : \mathcal{L}_0 \rightarrow \mathcal{L}_{0, \Box} \), used in [16, 15]:

\[
\begin{align*}
\text{GT}(p) & := \Box p & \text{for atomic } p \in AP \\
\text{GT}(\bot) & := \bot \\
\text{GT}(\varphi_1 \lor \varphi_2) & := \text{GT}(\varphi_1) \lor \text{GT}(\varphi_2) \\
\text{GT}(\varphi_1 \land \varphi_2) & := \text{GT}(\varphi_1) \land \text{GT}(\varphi_2) \\
\text{GT}(\varphi_1 \rightarrow \varphi_2) & := \Box (\text{GT}(\varphi_1) \rightarrow \text{GT}(\varphi_2))
\end{align*}
\]

In particular, intuitionistic negation comes out as \( \text{GT}(\neg \varphi) = \text{GT}(\varphi \rightarrow \bot) = \Box (\text{GT}(\varphi) \rightarrow \bot) = \Box \neg \text{GT}(\varphi) \) and for double negation, \( \text{GT}(\neg \neg \varphi) = \Box \neg \Box \neg \text{GT}(\varphi) = \Box \Box \text{GT}(\varphi) \). Reading the \( \text{S4}\Box \) as topological interior, we can read off the intuitionistic topological semantics directly from the clauses of the Gödel translation. The original Gödel translation [17] actually prefixes \( \Box \) to every subformula of a given intuitionistic formula, and is shown to be equivalent to the one given here in [16] (Ch. 9, #20).

**Definition 3.1** Given a topological space \( \mathcal{F} = (X, T) \), a model over \( \mathcal{F} \) is a structure \( \mathcal{M} = (X, T, v) \) where \( v : AP \sim X \) is the atomic valuation relation. \( \mathcal{M} \) is an open model if for each \( p \in AP \), the set \( v(p) \) is open in \( T \). For open models \( \mathcal{M} \), the intuitionistic denotation map \( \llbracket \cdot \rrbracket^\mathcal{M}_i : \mathcal{L}_0 \sim X \) is defined by:

\[
\begin{align*}
\llbracket p \rrbracket^\mathcal{M}_i & := v(p) \\
\llbracket \bot \rrbracket^\mathcal{M}_i & := \emptyset \\
\llbracket \varphi_1 \lor \varphi_2 \rrbracket^\mathcal{M}_i & := \llbracket \varphi_1 \rrbracket^\mathcal{M}_i \cup \llbracket \varphi_2 \rrbracket^\mathcal{M}_i \\
\llbracket \varphi_1 \land \varphi_2 \rrbracket^\mathcal{M}_i & := \llbracket \varphi_1 \rrbracket^\mathcal{M}_i \cap \llbracket \varphi_2 \rrbracket^\mathcal{M}_i \\
\llbracket \varphi_1 \rightarrow \varphi_2 \rrbracket^\mathcal{M}_i & := \text{int}_{\cdot} (X - \llbracket \varphi_1 \rrbracket^\mathcal{M}_i) \cup \llbracket \varphi_2 \rrbracket^\mathcal{M}_i
\end{align*}
\]

8
A formula $\varphi \in \mathcal{L}_0$ is intuitionistically topologically valid in an open model $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if $\mathcal{M} \models X$, and it is int-top valid in $\mathcal{F} = (X, \mathcal{T})$, written $\mathcal{F} \models \varphi$, if $\mathcal{M} \models \varphi$ for all open models $\mathcal{M}$ over $\mathcal{F}$. Define $\mathbb{T}$ to be the set of all $\varphi \in \mathcal{L}_0$ such that $\mathcal{F} \models \varphi$ in every topological space $\mathcal{F}$.

It is immediate that in an open model $\mathcal{M}$, the denotation set $\mathcal{D}_{\mathcal{M}}(\varphi)$ is open in $\mathcal{T}$ for all $\varphi \in \mathcal{L}_0$, and this corresponds to the $\preccurlyeq$-persistence or upward-$\preccurlyeq$-closed property in the relational Kripke semantics. The topological reading of the Gödel translation is further illuminated in the following equations, for any formula $\varphi$:

\[
\begin{align*}
\mathcal{D}_{\mathcal{M}}(\neg \varphi) & = \text{int}_T(X - \mathcal{D}_{\mathcal{M}}(\varphi)) \quad \mathcal{D}_{\mathcal{M}}(\varphi) & = X - \mathcal{D}_{\mathcal{M}}(\neg \varphi) \\
\mathcal{D}_{\mathcal{M}}(\neg \neg \varphi) & = \text{int}_T(\mathcal{D}_{\mathcal{M}}(\neg \varphi)) \quad \mathcal{D}_{\mathcal{M}}(\neg \varphi) & = (X - \mathcal{D}_{\mathcal{M}}(\varphi)) \cap (X - \mathcal{D}_{\mathcal{M}}(\neg \varphi))
\end{align*}
\]  

(5)

The classical topological semantics for the language $\mathcal{L}_{0 \Box}$ gives explicit representation to the topology, in contrast to the implicit representation in the intuitionistic semantics.

**Definition 3.2** For the modal language $\mathcal{L}_{0 \Box}$, we define the (classical) denotation map $\mathcal{D} : \mathcal{L}_{0 \Box} \rightarrow X$ with respect to arbitrary topological models $\mathcal{M} = (X, \mathcal{T}, v)$, where $v : AP \rightarrow X$ is unrestricted. The map $\mathcal{D} : \mathcal{L}_{0 \Box} \rightarrow X$ is defined the same way as $\mathcal{D}$ for atomic $p \in AP$, $\bot$, $\lor$ and $\land$, but differs on the clauses:

\[
\begin{align*}
\mathcal{D}(\varphi_1 \rightarrow \varphi_2) & := (X - \mathcal{D}(\varphi_1)) \cup \mathcal{D}(\varphi_2) \\
\mathcal{D}(\Box \varphi) & := \text{int}_T(\mathcal{D}(\varphi))
\end{align*}
\]

A formula $\varphi \in \mathcal{L}_{0 \Box}$ is topologically valid in $\mathcal{F} = (X, \mathcal{T})$, written $\mathcal{F} \models \varphi$, if $\mathcal{D}(\varphi) = X$ for all $\mathcal{M} = (\mathcal{F}, v)$. Let $\mathbb{T}$ be the set of all $\varphi \in \mathcal{L}_{0 \Box}$ such that $\mathcal{F} \models \varphi$ for every topological space $\mathcal{F}$.

Topological soundness is a simple verification, and topological completeness can be obtained cheaply from Kripke completeness with respect to frames $\mathcal{F} = (X, \preccurlyeq)$ where $\preccurlyeq$ is a pre-order on $X$, using the correspondence between pre-orders and Alexandroff topologies.

**Proposition 3.3** [Topological soundness and completeness]

- For all $\varphi \in \mathcal{L}_0$, $\varphi$ is a theorem of IPC iff $\varphi \in \mathbb{T}$.
- For all $\psi \in \mathcal{L}_{0 \Box}$, $\psi$ is a theorem of $S4 \Box$ iff $\psi \in \mathbb{T}$.

**Proposition 3.4** [Gödel translation] For all $\varphi \in \mathcal{L}_0$,

- $\varphi$ is a theorem of IPC iff $\text{GT}(\varphi)$ is a theorem of $S4 \Box$.
- $\text{GT}(\varphi) \leftrightarrow \Box \text{GT}(\varphi)$ is a theorem of $S4 \Box$. 


While we can get topological completeness cheaply, it can also be obtained at greater expense with respect to particular classes of topological spaces. The classic McKinsey and Tarski result [25] is topological completeness of $\textbf{S4}$ in separable dense-in-itself metric spaces; in particular, $\mathbb{R}^n$ equipped with the standard Euclidean topology $T_E$. The recent thesis of Aiello [1] gives new proofs of the McKinsey and Tarski theorem, and of topological completeness for an extension of $\textbf{S4}$ interpreted over finite unions of convex sets in $\mathbb{R}$. Aiello also investigates topological semantics for $\textbf{S4}_u$, the extension with a universal modality, and the work of Bennett and others on spatial reasoning which uses $\textbf{S4}_u$ to encode a decidable fragment of the Region Connection Calculus (RCC), introduced by Randell, Cui and Cohn; see [1], Chapter 4.

4 Intuitionistic modal and tense logics, and their classical analogs

For the syntax of intuitionistic modal and tense logics, let $\mathcal{L}^m_\square (\mathcal{L}^t_\square)$ be the modal (tense) language extending $\mathcal{L}_0$ with the addition of two (four) modal operators $\Diamond$ and $\Box$ (and $\Diamond$ and $\Box$), generated by the grammar:

$$\varphi ::= p \mid \bot \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \Diamond \varphi \mid \Box \varphi \mid ( \Diamond \varphi \mid \Box \varphi )$$

for $p \in AP$. Likewise, for the classical topological modal and tense logics, let $\mathcal{L}^m_\Diamond (\mathcal{L}^t_\Diamond)$ be the modal (tense) language extending $\mathcal{L}_0_\Diamond$ with the addition of $\Diamond$ and $\Box$ (and $\Diamond$ and $\Box$).

We give a quite straightforward topological extension of the bi-relational semantics of Fischer Servi [13, 14], and Plotkin and Stirling [28, 31] over Kripke frames $\mathcal{F} = (X, \preceq, R)$, where $\preceq$ is a pre-order on $X$, and $R : X \rightharpoonup X$ is any binary relation. In the bi-relational semantics, the central concern is the connection between the two relations: the pre-order $\preceq$ as the intuitionistic information ordering, and the relation $R$ as the modal accessibility relation. Using the induced Alexandroff topology $T_\preceq$, a bi-relational Kripke frame $\mathcal{F}$ is equivalent to the topological frame $(X, T_\preceq, R)$. The four bi-relational conditions identified in [28] can be cleanly transcribed as semi-continuity conditions on the relations $R : X \rightharpoonup X$ and $R^{-1} : X \rightharpoonup X$ with respect to the topology $T_\preceq$, which then generalize to arbitrary topologies.

**Definition 4.1** Let $\mathcal{F} = (X, \preceq, R)$ be a bi-relational frame. Four conditions expressing interaction relationships between $\preceq$ and $R$ are identified as follows:

10
Zig($\preceq$, $R$) : if $x \preceq y$ and $xR x'$ then $(\exists y' \in X) \left[ y R y' \text{ and } x' \preceq y' \right]$

Zag($\preceq$, $R$) : if $x \preceq y$ and $y R y'$ then $(\exists x' \in X) \left[ x R x' \text{ and } x' \preceq y' \right]$

Zig($\preceq$, $R^{-1}$) : if $x \preceq y$ and $x'R x$ then $(\exists y' \in X) \left[ y' R y \text{ and } x' \preceq y' \right]$

Zag($\preceq$, $R^{-1}$) : if $x \preceq y$ and $y'R y$ then $(\exists x' \in X) \left[ x'R x \text{ and } x' \preceq y' \right]$

We use the names “Zig” and “Zag” because Zig($\preceq$, $R$) and Zag($\preceq$, $R$) are exactly the well-known forth and back conditions on $\preceq$ being a bisimulation on the frame $(X, R)$, that are conventionally known by those names (for a review, see [7], Ch. 2). From earlier work [8], we know the bisimulation conditions correspond to semi-continuity properties with respect to the Alexandroff topology $T_{\preceq}$.

**Proposition 4.2** ([8]) Let $F = (X, \preceq, R)$ be a bi-relational frame, with $T_{\preceq}$ it induced topology. In the following table, the conditions listed along each row are equivalent.

<table>
<thead>
<tr>
<th></th>
<th>Zig($\preceq$, $R$)</th>
<th>Zag($\preceq$, $R$)</th>
<th>Zig($\preceq$, $R^{-1}$)</th>
<th>Zag($\preceq$, $R^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(\triangleright \bullet R) \subseteq (R \bullet \triangleright)$</td>
<td>$(\preceq \bullet R) \subseteq (R \bullet \preceq)$</td>
<td>$R$ is l.s.c. in $T_{\preceq}$</td>
<td>$R$ is u.s.c. in $T_{\preceq}$</td>
</tr>
<tr>
<td>2</td>
<td>$(\triangleright \bullet R) \subseteq (R \bullet \triangleright)$</td>
<td>$(\preceq \bullet R) \subseteq (R \bullet \preceq)$</td>
<td>$(R \triangleright \preceq) \subseteq (\preceq \bullet R)$</td>
<td>$(R \triangleright \preceq) \subseteq (\preceq \bullet R)$</td>
</tr>
<tr>
<td>3</td>
<td>$(\triangleright \bullet R) \subseteq (R \bullet \triangleright)$</td>
<td>$(\preceq \bullet R) \subseteq (R \bullet \preceq)$</td>
<td>$R^{-1}$ is l.s.c. in $T_{\preceq}$</td>
<td>$R^{-1}$ is u.s.c. in $T_{\preceq}$</td>
</tr>
<tr>
<td>4</td>
<td>$(\triangleright \bullet R) \subseteq (R \bullet \triangleright)$</td>
<td>$(\preceq \bullet R) \subseteq (R \bullet \preceq)$</td>
<td>$(R \triangleright \preceq) \subseteq (\preceq \bullet R)$</td>
<td>$(R \triangleright \preceq) \subseteq (\preceq \bullet R)$</td>
</tr>
</tbody>
</table>

In the bi-relational semantics introduced by Fischer Servi in [14] and used in [15, 28, 12, 31, 19], it is the first and third frame conditions Zig($\preceq$, $R$) and Zig($\preceq$, $R^{-1}$) that are identified as those needed to give an intuitionistic semantics for modalities based on $R$. In frames meeting these conditions, one can give semantic clauses for the diamond and box that are natural under the intuitionistic reading of the restricted $\exists$ and $\forall$ quantification with respect to $R$-successors. More precisely, the resulting logic is faithfully embedded into intuitionistic first-order logic by the standard modal to first-order translation, and a natural extension of the Gödel translation faithfully embeds it into the classical bi-modal logic combining $S_4 \Box$ with $K$ or extensions. Various further motivation for these two frame conditions are discussed in the literature [15, 28, 12, 31].
From Proposition 4.2, we see that these two frame conditions correspond to the lower semi-continuity of $R$ and $R^{-1}$ in $\mathcal{T}_\mathcal{L}$, and we use this observation to generalize the intuitionistic semantics to arbitrary topologies. Moreover, the symmetry between $R$ and $R^{-1}$ of the frame conditions means that, having fixed a suitable intuitionistic semantics for the forward or future modal operators $\star$ and $\boxdot$ for $R$, we can re-use exactly the same conditions to give intuitionistic semantics for the backward or past modal operators $\star$ and $\boxdot$ for $R^{-1}$. Practically, we have in mind applications where the additional expressivity from having both future and past modalities is required (in the formal verification and design of hybrid systems), but also pragmatically, one discovers that the semantics and metatheory may well be cleaner and simpler for the tense logic than it is for the modal logic. In particular, arguments involving the box modality $\boxdot$ become somewhat simpler when one can pass to the adjoint diamond $\star$, which – symmetrically with $\star$ – preserves open sets.

The semi-continuity properties of maps/relations are not widely known outside specialist areas like set-valued analysis, and there, the focus quickly narrows to the metric versions of the semi-continuity properties [5]. As further endorsement for the naturalness of the conditions, consider Aiello’s notion of a topological bisimulation between topological S4 models $\mathcal{M} = (X, \mathcal{T}, v)$ and $\mathcal{M}' = (X', \mathcal{T}', v')$, in Definition 2.1.2 of [1], which extends the well-known relational concept of bisimulation, and likewise, preserves truth and satisfaction between models (see [7], Ch. 2). A brief study of his definition reveals that a relation $B : X \leadsto X'$ satisfies the forth and back conditions of being a topological bisimulation, exactly when both $B$ and $B^{-1}$ are l.s.c. with respect to $\mathcal{T}$ and $\mathcal{T}'$. We shall come back to the topic of bisimulations in the next section, and again in Section 7, but we turn now to the semantics of our logics.

**Definition 4.3** A topological frame is a structure $\mathcal{F} = (X, \mathcal{T}, R)$ where $(X, \mathcal{T})$ is a topological space and $R : X \leadsto X$ is a binary relation. We say $\mathcal{F}$ is an l.s.c. topological frame if both $R$ and $R^{-1}$ are l.s.c. with respect to $\mathcal{T}$. A model over $\mathcal{F}$ is a structure $\mathcal{M} = (\mathcal{F}, v)$ where $v : AP \leadsto X$ is an atomic valuation relation. As before, a model $\mathcal{M}$ will be called an open model if for each $p \in AP$, the denotation set $v(p)$ is open in $\mathcal{T}$. For open models $\mathcal{M}$ over l.s.c. frames $\mathcal{F}$, the intuitionistic denotation map $\llbracket \cdot \rrbracket^\mathcal{M} : \mathcal{L} \leadsto X$ (or $\llbracket \cdot \rrbracket^\mathcal{M} : \mathcal{L}^m \leadsto X$) is defined the same way as for $\mathcal{L}_0$, with the additional clauses:

\[
\begin{align*}
\llbracket \star \varphi \rrbracket^\mathcal{M}_i & := R^{-3} (\llbracket \varphi \rrbracket^\mathcal{M}_i) \\
\llbracket \boxdot \varphi \rrbracket^\mathcal{M}_i & := R^{3} (\llbracket \varphi \rrbracket^\mathcal{M}_i) \\
\llbracket \star \varphi \rrbracket^\mathcal{M} & := \text{int}_R (R^{-\gamma} (\llbracket \varphi \rrbracket^\mathcal{M} )) \\
\llbracket \boxdot \varphi \rrbracket^\mathcal{M} & := \text{int}_R (R^{\gamma} (\llbracket \varphi \rrbracket^\mathcal{M} ))
\end{align*}
\]

A formula $\varphi \in \mathcal{L}_i$ (or $\varphi \in \mathcal{L}^m$) will be called int-modal-top valid in an open model $\mathcal{M}$, written
\[ \mathcal{M} \models \varphi, \text{ if } \llbracket \varphi \rrbracket_\mathcal{M}^* = X, \text{ and is called int-modal-top valid in an l.s.c. frame } \mathcal{F} = (X, \mathcal{T}, R), \text{ written } \mathcal{F} \models \varphi, \text{ if } \mathcal{M} \models \varphi \text{ for all open models } \mathcal{M} \text{ over } \mathcal{F}. \text{ We say } \varphi \text{ is satisfiable in } \mathcal{M} \text{ if } \llbracket \varphi \rrbracket_\mathcal{M}^* \neq \emptyset, \text{ and } \varphi \text{ is falsifiable in } \mathcal{M} \text{ if } \llbracket \varphi \rrbracket_\mathcal{M}^* \neq X. \text{ Let } \mathbb{K}^\mathcal{T} \text{ (} \mathbb{K}^{\mathcal{M}\mathcal{T}} \text{) be the set of all } \varphi \in \mathcal{L}^e (\varphi \in \mathcal{L}^m) \text{ such that } \mathcal{F} \models \varphi \text{ in every l.s.c. topological frame } \mathcal{F}. \]

The property that every denotation set \( \llbracket \varphi \rrbracket_\mathcal{M}^* \) is open in \( \mathcal{T} \) follows immediately from the openness condition on \( v(p) \), the l.s.c. properties of \( R^{-3} \) and \( R^3 \), and the extra interior operation in the semantics for \( \Box \) and \( \Diamond \). The semantic clauses are exactly as one would expect, given the standard modal to first-order translation, and the topological semantics for intuitionistic first-order logic, where \( \forall \) is evaluated by the interior of an intersection, and \( \exists \) is evaluated by a union. We now turn to the semantics for the classical analogs of modal and tense logics.

**Definition 4.4** For the tense (modal) language \( \mathcal{L}_e^\Box (\mathcal{L}_m^\Box) \), we define the classical denotation map \( \llbracket \cdot \rrbracket^\mathcal{M} : \mathcal{L}_e^\Box \rightharpoonup X \) ( \( \llbracket \cdot \rrbracket^\mathcal{M} : \mathcal{L}_m^\Box \rightharpoonup X \) ) with respect to arbitrary topological models \( \mathcal{M} = (X, \mathcal{T}, R, v) \), where \( v : AP \rightharpoonup X \) is unrestricted; the map is defined the same way as for \( \llbracket \cdot \rrbracket_{\Omega, \Box} \), with the additional clauses:

\[
\begin{align*}
\llbracket \diamond \varphi \rrbracket^\mathcal{M} & := R^{-3} (\llbracket \varphi \rrbracket^\mathcal{M}) & \llbracket \lozenge \varphi \rrbracket^\mathcal{M} & := R^3 (\llbracket \varphi \rrbracket^\mathcal{M}) \\
\llbracket \Box \varphi \rrbracket^\mathcal{M} & := R^{-\forall} (\llbracket \varphi \rrbracket^\mathcal{M}) & \llbracket \square \varphi \rrbracket^\mathcal{M} & := R^{\exists} (\llbracket \varphi \rrbracket^\mathcal{M})
\end{align*}
\]

A formula \( \varphi \in \mathcal{L}_e^\Box \) (or \( \varphi \in \mathcal{L}_m^\Box \) ) will be called modal-top valid in a model \( \mathcal{M} \), written \( \mathcal{M} \models \varphi \), if \( \llbracket \varphi \rrbracket_\mathcal{M}^* = X \), and is called modal-top valid in a topological frame \( \mathcal{F} = (X, \mathcal{T}, R) \), written \( \mathcal{F} \models \varphi \), if \( \mathcal{M} \models \varphi \) for all models \( \mathcal{M} \) over \( \mathcal{F} \). Let \( \mathbb{K}^\mathcal{T} \) (\( \mathbb{K}^{\mathcal{M}\mathcal{T}} \)) be the set of all \( \varphi \in \mathcal{L}_e^\Box \) (\( \varphi \in \mathcal{L}_m^\Box \)) such that \( \mathcal{F} \models \varphi \) for every topological frame \( \mathcal{F} \). Let \( \mathbb{K}^{\mathcal{T}\mathcal{L}^e} \) (\( \mathbb{K}^{\mathcal{M}\mathcal{L}^e} \)) be the set of all \( \varphi \in \mathcal{L}_e^\Box \) (\( \varphi \in \mathcal{L}_m^\Box \)) such that \( \mathcal{F} \models \varphi \) in every l.s.c. topological frame \( \mathcal{F} \).

The semi-continuity frame conditions also have clean characterizations in the classical multimodal logics, as we have observed in [10].

**Proposition 4.5** [Modal characterization of semi-continuity conditions]

Let \( \mathcal{F} = (X, \mathcal{T}, R) \) be a topological frame and let \( p \in AP \). In the following table, the conditions listed across each row are equivalent.

<table>
<thead>
<tr>
<th>(1.)</th>
<th>( R \text{ is l.s.c. in } \mathcal{T} )</th>
<th>( \mathcal{F} \models \lozenge \square p \rightarrow \square \lozenge p )</th>
<th>( \mathcal{F} \models \lozenge \square p \leftrightarrow \square \lozenge p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.)</td>
<td>( R \text{ is u.s.c. in } \mathcal{T} )</td>
<td>( \mathcal{F} \models \Box \diamond p \rightarrow \diamond \Box p )</td>
<td></td>
</tr>
<tr>
<td>(3.)</td>
<td>( R^{-1} \text{ is l.s.c. in } \mathcal{T} )</td>
<td>( \mathcal{F} \models \Box \lozenge p \rightarrow \lozenge \Box p )</td>
<td>( \mathcal{F} \models \lozenge \square p \leftrightarrow \square \lozenge p )</td>
</tr>
<tr>
<td>(4.)</td>
<td>( R^{-1} \text{ is u.s.c. in } \mathcal{T} )</td>
<td>( \mathcal{F} \models \lozenge \Box p \rightarrow \Box \lozenge p )</td>
<td></td>
</tr>
</tbody>
</table>
**Proof.** The equivalences in rows (1.) and (2.) are proved in [10]. For row (3.), let (a), (b) and (c) denote the three conditions. The equivalence (a) $\iff$ (c) is immediate by symmetry of $R$ and $R^{-1}$ from row (1.), and likewise for the equivalence of (c) and (c'): $\mathcal{F} \models \Diamond \Box p \rightarrow \Box \Diamond p$.

For (b) $\Rightarrow$ (c'), observe that by the adjoint property (Assertion (3)), $\Box p \rightarrow \Box \Diamond p$ is modal-top valid, hence if $\mathcal{F} \models \Box \Diamond p \rightarrow \Box p$, then by substitution of $\Diamond p$ for $p$ in (b), we get that $\mathcal{F} \models \Box p \rightarrow \Box \Diamond p$. Applying adjointness again, we can conclude that $\mathcal{F} \models \Diamond \Box p \rightarrow \Box \Diamond p$. For the converse (c') $\Rightarrow$ (b), the substitution of $\Box p$ for $p$ in (c') gives that $\mathcal{F} \models \Box \Diamond p \rightarrow \Diamond \Box p$.

Hence $\mathcal{F} \models \Box \Diamond p \rightarrow \Box p$. Applying adjointness, we can conclude that $\mathcal{F} \models \Diamond \Box p \rightarrow \Box \Diamond p$. The verification for row (4.) can be established by showing the equivalence of (d': $\mathcal{F} \models \Box \Diamond p \rightarrow \Diamond \Box p$, and (e): $\mathcal{F} \models \Box \Diamond p \rightarrow \Box \Diamond p$, similarly using adjointness.

For naturally occurring l.s.c. topological frames with non-Alexandrov topologies, consider frames $\mathcal{F}$ where $X \subseteq \mathbb{R}^n$ is a finite dimensional vector space, with norm $\| \cdot \|$ inducing the standard Euclidean topology $T_E$ on $X$ (as a subspace of $\mathbb{R}^n$). Considering relations on $X$, start by defining:

$$AC(X) := \{ \gamma : [0, \tau] \rightarrow X \mid \tau \in \mathbb{R}_0^+ \land \gamma \text{ is absolutely continuous on the interval } [0, \tau] \}$$

A **differential inclusion** is given by a set-valued map $F : X \rightrightarrows \mathbb{R}^n$, and **solutions** to the differential inclusion $\dot{x} \in F(x)$ starting at a state $x \in X$ are defined by:

$$\text{Sol}_F(x) := \{ \gamma \in AC(X) \mid \gamma(0) = x \land \left( \frac{d}{dt} \gamma(s) \right) \in F(\gamma(s)) \text{ for almost all } s \in [0, \tau] \}$$

In general, the solution set $\text{Sol}_F(x)$ is partially ordered by inclusion (considering solutions as subsets $\gamma \subseteq \mathbb{R}_0^+ \times X$). To ensure the existence of non-trivial solutions from each $x \in cl(dom(F))$, one needs to impose regularity assumptions on $F : X \rightrightarrows \mathbb{R}^n$, such as the **Marchaud** conditions [6]: (a) $F$ is total on $X$; (b) $F \subseteq X \times \mathbb{R}^n$ is a closed set; (c) the image set $F(x)$ is convex and compact in $\mathbb{R}^n$ for every $x \in dom(F)$; and (d) the growth of $F$ is linear, in that there exists a real constant $c > 0$ such that $\sup \{ \| y \| \mid y \in F(x) \} \leq c(\| x \| + 1)$ for all $x \in dom(F)$. Now consider the **reachability relation** $R_F : X \rightrightarrows X$ defined by $(x, x') \in R_F$ iff there exists $\gamma \in \text{Sol}_F(x)$ such that $\gamma(t) = x'$ for some $t \in [0, \tau] = dom(\gamma)$. Clearly, $R_F$ is reflexive and transitive, so the $\Diamond$ and $\Box$ modalities will satisfy the axioms of S4.

The extension from modal to tense logics is of particular utility in this application area, as the tense formula $\Diamond p$ denotes the set of states reachable from $p$ states, and formulae of this type are of key interest in the formal analysis and verification of hybrid and continuous dynamical systems. Suppose $p$ denotes a set of initial states and $q$ a set of bad or error states. To prove
a safety condition \( p \rightarrow \Box \neg q \), which is equivalent by the adjoint property to \( \Diamond p \rightarrow \neg q \), one attempts to compute the reachable region \([\Diamond p]\) in order to show it has empty intersection with the set of error states \([q]\) [3, 11]. Alternatively, suppose that \( p \) denotes a set of target states. Then the future diamond \( \Diamond p \) denotes the set of states from which the target \( p \) states can be reached, according to the dynamics described by the reachability relation \( R \). In a multi-modal language, formulae of this type are of interest in the design and synthesis of hybrid and switching control systems, where a common task is to choose from among a finite number of control actions in order to drive the system state toward a target region [27]. Under the Marchaud conditions (and weaker assumptions), both the forwards and backwards relations \( R_F \) and \( R_F^{-1} \) will be lower semi-continuous, thus \( \mathcal{F} = (X, \mathcal{T}_E, R_F) \) will be an l.s.c. topological frame. Note that if \( F : X \rightarrow \mathbb{R}^n \) is actually a function and the differential equation \( \frac{d}{dt} \nu(t) = F(x(t)) \) has a unique maximal solution \( \eta : [0, \tau_x) \rightarrow X \) starting from each \( x \in X \), with \( \tau_x \in \mathbb{R}_0^+ \cup \{\infty\} \), then \( \text{Solv}_F(x) = \{ \eta \mid [0,\tau] \mid t \in [0,\tau_x) \} \) is linearly ordered. That is, deterministic continuous dynamics described by differential equations are a special case of general, non-deterministic continuous dynamics described by differential inclusions\(^6\).

5 Topological bisimulations

We now turn to the topic of bisimulations between topological models, and their semantic preservation properties. As noted above, Aiello’s notion of a topological bisimulation \( B : X_1 \sim X_2 \) between topological S4 models \( \mathcal{M}_1 = (X_1, \mathcal{T}_1, v_1) \) and \( \mathcal{M}_2 = (X_2, \mathcal{T}_2, v_2) \), in Definition 2.1.2 of [1], can be simply transcribed into the equivalent condition that \( B \) and \( B^{-1} \) are l.s.c. with respect to \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \).

**Definition 5.1** Let \( \mathcal{M}_1 = (X_1, \mathcal{T}_1, R_1, v_1) \) and \( \mathcal{M}_2 = (X_2, \mathcal{T}_2, R_2, v_2) \) be two topological models, and let \( B : X_1 \sim X_2 \) be any relation. Two conditions expressing interaction relationships between \( B \) and \( R_1 \) and \( R_2 \) are identified as follows:

\(^6\)Differential inclusions form a subclass (over non-negative real time) of a broad class of dynamical systems called general flow systems, introduced in [9], where they are shown to give an adequate semantics for a generalization of the branching or non-deterministic temporal logic CTL* in the language with a 2-place “UNTIL” operator, and universal and existential path quantifiers. Box and diamond 1-place operators are definable; they correspond to the \( \Box \) and \( \Diamond \) modalities of the reachability relation of the dynamical system, and satisfy the axioms of S4.
\[
\text{Zig}(B,R_1,R_2) : \forall x,x' \in X_1, \forall y \in X_2, \quad \text{if} \quad x \top y \quad \text{and} \quad x R_1 x' \\
\text{then} \quad (\exists y' \in X_2)[ y \top R_2 y' \quad \text{and} \quad x' \top B y']
\]
\[
\text{Zag}(B,R_1,R_2) : \forall x \in X_1, \forall y,y' \in X_2, \quad \text{if} \quad x \top y \quad \text{and} \quad y R_2 y' \\
\text{then} \quad (\exists x' \in X_1)[ x \top R_1 x' \quad \text{and} \quad x' \top B y']
\]

We say \(B : X_1 \sim X_2\) is a modal topo-bisimulation between \(\mathcal{M}_1\) and \(\mathcal{M}_2\) if:

(i) \(B\) and \(B^{-1}\) are l.s.c. with respect to \(\mathcal{T}_1\) and \(\mathcal{T}_2\); and

(ii) the conditions \(\text{Zig}(B,R_1,R_2)\) and \(\text{Zag}(B,R_1,R_2)\) both hold; and

(iii) the inclusions \(B^3(v_1(p)) \subseteq v_2(p)\) and \(B^{-3}(v_2(p)) \subseteq v_1(p)\) both hold for all \(p \in \text{AP}\).

We say \(B : X_1 \sim X_2\) is a tense topo-bisimulation if it is a modal topo-bisimulation, and:

(iv) the conditions \(\text{Zig}(B,R_1^{-1},R_2^{-1})\) and \(\text{Zag}(B,R_1^{-1},R_2^{-1})\) both hold.

We can readily transcribe these two extended zig-zag conditions in (ii) into relational inclusions: \((B^{-1} \bullet R_1) \subseteq (R_2 \bullet B^{-1})\), and \((B \bullet R_2) \subseteq (R_1 \bullet B)\), respectively. Together with the additional tense conditions in (iv), one has the two equalities \((R_1 \bullet B) = (B \bullet R_2)\) and \((R_2 \bullet B^{-1}) = (B^{-1} \bullet R_1)\).

The set-operator form of the semantic preservation conditions are:

\[
B^3([\varphi]_{\mathcal{T}_1}^\mathcal{M}_1) \subseteq [\varphi]_{\mathcal{T}_1}^\mathcal{M}_2 \quad \text{and} \quad B^{-3}([\varphi]_{\mathcal{T}_1}^\mathcal{M}_2) \subseteq [\varphi]_{\mathcal{T}_1}^\mathcal{M}_1
\]

(6)

and likewise for classical denotation maps \([\varphi]^{\mathcal{M}_1}\). We will also use the dual versions under the adjoint equivalence (3). These are:

\[
[\varphi]_{\mathcal{T}_1}^\mathcal{M}_2 \subseteq B^{-\top}([\varphi]_{\mathcal{T}_1}^\mathcal{M}_1) \quad \text{and} \quad [\varphi]_{\mathcal{T}_1}^\mathcal{M}_1 \subseteq B^{\top}([\varphi]_{\mathcal{T}_1}^\mathcal{M}_2)
\]

(7)

and likewise for classical denotation maps \([\varphi]^{\mathcal{M}_2}\). Note also that \(B^{-1} : X_2 \sim X_1\) being l.s.c. has a further equivalent characterization: \(\text{int}_{\mathcal{T}_2}(B^{-\top}(W)) \subseteq B^{-\top}(\text{int}_{\mathcal{T}_2}(W))\), for all \(W \subseteq X_2\); this is a generalization of the characterization for binary relations on a single space \(X\) that is formalized in Proposition 4.5, Row (3).

What we discover is that exactly the same notion of a bisimulation between models yields the same semantic preservation property for both the intuitionistic and the classical semantics. Otherwise put, the specifically topological requirement that the operators \(B^{-3}\) and \(B^3\) preserve open sets is enough to push through the result for intuitionistic modal and tense logics.

**Theorem 5.2** [Semantic preservation for tense topo-bisimulations]

Let \(\mathcal{M}_1 = (X_1, \mathcal{T}_1, R_1, v_1)\) and \(\mathcal{M}_2 = (X_2, \mathcal{T}_2, R_2, v_2)\) be any two topological models, and let \(B : X_1 \sim X_2\) be a tense topo-bisimulation between \(\mathcal{M}_1\) and \(\mathcal{M}_2\).
(1.) If $\mathcal{M}_1$ and $\mathcal{M}_2$ are open and l.s.c. topological models, then for all $x \in X_1$ and $y \in X_2$:

$$xB \ y \ \text{implies} \ \ (\forall \varphi \in \mathcal{L}^t)[x \in \lbrack \varphi \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1} \iff y \in \lbrack \varphi \rbrack_{\mathcal{M}_2}^{\mathcal{M}_2}]$$

(2.) For all $x \in X_1$ and $y \in X_2$:

$$xB \ y \ \text{implies} \ \ (\forall \varphi \in \mathcal{L}^t)[x \in \lbrack \varphi \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1} \iff y \in \lbrack \varphi \rbrack_{\mathcal{M}_2}^{\mathcal{M}_2}]$$

**Proof.** For each part, the proof proceeds as usual, by induction on the structure of formulae, to establish the two inclusions displayed in equation (6), or their analogs for the classical denotation maps. The base case for atomic propositions is given by condition (iii). For the classical semantics in Part (2.), the argument is completely standard for the propositional and modal/tense operators, and the case for topological $\Box$ is given in [1], Ch. 2. For the intuitionistic semantics in Part (1.), we give the cases for implication $\to$ and for box $\Box$. Assume the result holds for $\varphi_1$ and $\varphi_2$ in $\mathcal{L}^t$. In particular, from Assertions (6) and (7), we have: $(X_1 - \lbrack \varphi_1 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}) \subseteq (X_1 - B^{-3}([\varphi_1]_{\mathcal{M}_2}^{\mathcal{M}_2}))$, and $[\varphi_2]_{\mathcal{M}_1}^{\mathcal{M}_1} \subseteq B^{-\Box}([\varphi_2]_{\mathcal{M}_2}^{\mathcal{M}_2})$. Now:

$$B^{-3}(\lbrack \varphi_1 \to \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1})$$

$$\subseteq B^{-3}(\lbrack \varphi_1 \to \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1} \cup \lbrack \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1})$$

$$\subseteq B^{-3}(\lbrack \varphi_1 \to \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1} \cup \lbrack \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1})$$

by induction hypothesis

$$\subseteq B^{-3}(\lbrack \varphi_1 \to \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1} \cup \lbrack \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1})$$

by duality of $B^{-\Box}$ and $B^{-3}$

$$\subseteq \Box \ l_s.c.$$ by monotonicity of $B^{-\Box}$

$$\subseteq \Box \ l_s.c.$$ by adjoint property

$$= \lbrack \varphi_1 \to \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}$$

The verification for $B^{-3}([\varphi_1 \to \varphi_2]_{\mathcal{M}_1}^{\mathcal{M}_1}) \subseteq \lbrack \varphi_1 \to \varphi_2 \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}$ proceeds similarly, with the induction hypothesis: $(X_2 - [\varphi_1]_{\mathcal{M}_2}^{\mathcal{M}_2}) \subseteq (X_2 - B^-([\varphi_1]_{\mathcal{M}_2}^{\mathcal{M}_2}))$, and $[\varphi_2]_{\mathcal{M}_1}^{\mathcal{M}_1} \subseteq B^\Box([\varphi_2]_{\mathcal{M}_2}^{\mathcal{M}_2})$.

For the $\Box$ case:

$$\lbrack \Box \varphi \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}$$

$$\subseteq \lbrack \Box \varphi \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}$$

by induction hypothesis

$$\subseteq \lbrack \Box \varphi \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}$$

since $R_1 \cdot B = B \cdot R_2$

$$\subseteq B^{-\Box} \lbrack \Box \varphi \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}$$

by $B^{-1}$ being l.s.c. (dual $B^\Box$ form)

$$= B^{-\Box} \lbrack \varphi \rbrack_{\mathcal{M}_1}^{\mathcal{M}_1}$$

The argument for $\bullet$ is analogous, and appeals to $B$ being l.s.c. (dual $B^\Box$ form).
In Section 7, Theorem 7.3, we give a partial converse (Hennessy-Milner type result) by proving that a certain class of open l.s.c. models has the property that for any two models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) in the class, there is a tense topo-bisimulation \( B \) between them that is maximal w.r.t. the property of respecting the intuitionistic semantics; i.e. for all \( x \in X_1 \) and \( y \in X_2 \):

\[
x B y \iff (\forall \varphi \in \mathcal{L}^1) \left[ x \in [\varphi]^\mathcal{M}_1 \iff y \in [\varphi]^\mathcal{M}_2 \right]
\]

We will prove the result by setting up a tense topo-bisimulation between each of the models \( \mathcal{M}_i \) and the canonical model produced in the completeness proof in Section 6.

In classical modal (tense) logics, the basic Hennessy-Milner result restricts to models whose accessibility relations \( R \) are image-finite (bi-image-finite); i.e. for all \( x \in X \), the image set \( R(x) \) (and also \( R^{-1}(x) \)) must be of finite cardinality. A natural generalization of the image-finite condition is that of a model \( \mathcal{M} \) being modally saturated: for every state \( x \in X \) and every set of formulae \( \mathcal{A} \subseteq \mathcal{L}^m \), if for every finite subset \( \{ \varphi_1, \ldots, \varphi_n \} \subseteq \mathcal{A} \), there is an \( R \)-successor \( x' \in R(x) \) that satisfies each formula \( \varphi_k \) for \( 1 \leq k \leq n \), then there is an \( R \)-successor \( x_* \in R(x) \) that satisfies every formula in \( \mathcal{A} \) (and likewise for \( \mathcal{A} \subseteq \mathcal{L}^t \) and \( R^{-1} \) for tense logics); see [7], Ch. 2. For a diamond modality \( \Box \) interpreted by \( R \), the classical modal saturation property is equivalent to the condition that: \( x R x' \) iff for all \( \varphi \in \mathcal{L}^m \), if \( x' \in [\varphi]^\mathcal{M} \) then \( x \in [\Box \varphi]^\mathcal{M} \).

Informally, this condition says that the relation \( R \) is “recoverable” from the algebra of denotation sets \( \{[\varphi]^\mathcal{M} \mid \varphi \in \mathcal{L}^m \} \) in the model, in the same way that the usual relation in the canonical Kripke model for a logic is definable in terms of the Lindenbaum algebra of the logic.

In our study of saturation concepts for intuitionistic semantics, we first identify a simple topological property of models that captures the idea of the topology being “recoverable” from the algebra of intuitionistic denotation sets in the model.

**Definition 5.3** For any open and l.s.c. topological model \( \mathcal{M} = (X, \mathcal{T}, R, v) \), and for any topological model \( \mathcal{M}' = (X', \mathcal{T}', R', v') \), define sub-families of open sets as follows:

\[
\mathcal{O}_t(\mathcal{M}) := \\{ [\varphi]^\mathcal{M} \mid \varphi \in \mathcal{L}_t^t \} \quad \mathcal{O}_d(\mathcal{M}') := \\{ [\Box \varphi]^\mathcal{M}' \mid \varphi \in \mathcal{L}_d^d \}
\]

Let \( \mathcal{T}_t^\mathcal{M} \) be the sub-topology of \( \mathcal{T} \) generated by the family \( \mathcal{O}_t(\mathcal{M}) \), and let \( \mathcal{T}_d^\mathcal{M}' \) be the sub-topology of \( \mathcal{T}' \) generated by the family \( \mathcal{O}_d(\mathcal{M}') \). We will say that the topology \( \mathcal{T} \) is manifest in an open and l.s.c. model \( \mathcal{M} \) if \( \mathcal{T}_t^\mathcal{M} = \mathcal{T} \), and we say that the topology \( \mathcal{T}' \) is \( \Box \)-manifest in an arbitrary topological model \( \mathcal{M}' \) if \( \mathcal{T}_d^\mathcal{M}' = \mathcal{T}' \).

It is readily seen that the families \( \mathcal{O}_t(\mathcal{M}) \) and \( \mathcal{O}_d(\mathcal{M}') \) are closed under finite intersections, just by taking conjunctions of formulae, and thus they constitute a basis for the topologies \( \mathcal{T}_t^\mathcal{M} \) and \( \mathcal{T}_d^\mathcal{M}' \).
and $\mathcal{T}_E^\delta$ respectively. A topology $\mathcal{T}$ is manifest in an open and l.s.c. model $\mathcal{M}$ if there are no other open sets in $\mathcal{T}$ besides the ones you get by taking unions of intuitionistic denotation sets of formulae.

From our example class of frames $\mathcal{F}$ over $X \subseteq \mathbb{R}^n$, the Euclidean topology $\mathcal{T}_E$ has as a basis the countable family of all metric $\delta$-balls $B_\delta(x)$ where $\delta > 0$ is rational and the centers $x \in X$ have rational coordinates; here, $B_\delta(x) := \{ y \in \mathbb{R}^n \mid d(x, y) < \delta \}$. Thus we can make the topology $\mathcal{T}_E$ manifest in a model $\mathcal{M}$ over $\mathcal{F}$ if the atomic valuation of $\mathcal{M}$ maps surjectively onto this family; i.e. for every pair $(x, \delta) \in (X \cap \mathbb{Q}^n) \times \mathbb{Q}^+$, there is an atomic proposition $p \in AP$ such that $\llbracket p \rrbracket_\mathcal{M} = B_\delta(x) \cap X$. (Recall that there are several different equivalent metrics on $\mathbb{R}^n$ that all generate the Euclidean topology.) More generally, if the topological space $(X, \mathcal{T})$ has a countable basis, and the maps $R, R^{-1} : X \leadsto X$ are l.s.c. then we can form from it an open and l.s.c. model $\mathcal{M}$ in which $\mathcal{T}$ is manifest.

Requiring the topology to be manifest in a model is of course only one half of the story; we also need to restrict to models in which the relations $R$ and $R^{-1}$ are suitably recoverable from the algebra of intuitionistic denotation sets in the model. In particular, we need a concept of relational satisfaction that addresses not only the satisfiability of formulae in a given set by $R$-successors (and $R$-predecessors), but also the falsifiability of formulae not in the given set.

**Definition 5.4** A set of formulae $\mathcal{A} \subseteq \mathcal{L}^i$ will be called negation-consistent if there is no formula $\varphi \in \mathcal{L}^i$ such that both $\varphi \in \mathcal{A}$ and $\neg \varphi \in \mathcal{A}$. A set $\mathcal{A} \subseteq \mathcal{L}^i$ is said to have the disjunction property if for all formulae $\varphi, \psi \in \mathcal{L}^i$, we have the disjunction $(\varphi \lor \psi) \in \mathcal{A}$ iff at least one of $\varphi \in \mathcal{A}$ or $\psi \in \mathcal{A}$. A set $\mathcal{A} \subseteq \mathcal{L}^i$ will be called $\texttt{IT}$-semantically-closed if, whenever there is a validity $((\varphi_1 \land \cdots \land \varphi_n) \rightarrow \psi) \in \texttt{IT}$ and $\varphi_k \in \mathcal{A}$ for every $k \in \{1, \ldots, n\}$, then the consequent $\psi \in \mathcal{A}$. Finally, a set of formulae $\mathcal{A} \subseteq \mathcal{L}^i$ will be called $\texttt{IT}$-semantically-prime if it is negation-consistent, it has the disjunction property, and it is $\texttt{IT}$-semantically-closed.

Given an open and l.s.c. topological model $\mathcal{M} = (X, \mathcal{T}, R, v)$, we call a set of formulae $\mathcal{A}$ realizable as a theory in $\mathcal{M}$ if there exists $x \in X$ such that $\varphi \in \mathcal{A}$ iff $x \in \llbracket \varphi \rrbracket_\mathcal{M}$, for all formulae $\varphi \in \mathcal{L}^i$. In this case, we will say the state $x$ realizes $\mathcal{A}$ as a theory in $\mathcal{M}$; let $\llbracket \mathcal{A} \rrbracket_\mathcal{M}$ denote the set of all such states $x \in X$, which we will call the realization set for $\mathcal{A}$.

**Lemma 5.5** Let $\mathcal{M}$ be any open and l.s.c. topological model, and let $\mathcal{A} \subseteq \mathcal{L}^i$ be any set of formulae. If $\mathcal{A}$ is realizable in $\mathcal{M}$, then $\mathcal{A}$ is $\texttt{IT}$-semantically-prime.

**Proof.** Suppose $\mathcal{A}$ is realizable in $\mathcal{M}$, and choose any state $z \in \llbracket \mathcal{A} \rrbracket_\mathcal{M} \neq \emptyset$. Thus for all formulae $\varphi \in \mathcal{L}^i$, we have $z \in \llbracket \varphi \rrbracket_\mathcal{M}$ iff $\varphi \in \mathcal{A}$. Then $\mathcal{A}$ must be negation-consistent,
otherwise \( (\mathcal{A})^{\wedge} = \emptyset \). For the disjunction property, observe that \( (\varphi \lor \psi) \in \mathcal{A} \) if \( z \in \varphi_i \lor_i \psi \) if \( \varphi \in \mathcal{A} \) or \( \psi \in \mathcal{A} \). For the \( \mathcal{K} \text{-T} \) semantic-closure, suppose \(((\varphi_1 \land \cdots \land \varphi_n) \rightarrow \psi) \in \mathcal{K} \text{-T} \) and \( \varphi_k \in \mathcal{A} \) for every \( k \in \{1, \ldots, n\} \). Then \( \varphi_1 \land \cdots \land \varphi_n \in \mathcal{A} \) as required.

Direct from the definition of the realization set, we have for any set of formulae \( \mathcal{A} \):

\[
(\mathcal{A})^{\wedge} = \left( \bigcap_{\varphi \in \mathcal{A}} \varphi_i \right) \cap \left( \bigcap_{\psi \in \mathcal{A}} (X - \psi_i) \right) \tag{8}
\]

Define \( \mathcal{A}^\bot := \{ \varphi \in \mathcal{A} \mid (\exists \psi \in \mathcal{L}) \varphi = \neg \psi \} \) to be the subset of negated formulae in \( \mathcal{A} \), and \( \mathcal{A}^\bot := \mathcal{A} - \mathcal{A}^\bot \). So \( \mathcal{A} \) is negation-consistent iff \( \neg \psi \in \mathcal{A}^\bot \) implies \( \psi \notin \mathcal{A} \) and \( \neg \neg \psi \notin \mathcal{A}^\bot \), and \( \varphi \in \mathcal{A}^\bot \) implies \( \neg \varphi \notin \mathcal{A}^\bot \), for all formulae \( \varphi, \psi \). Define \( \sim \mathcal{A} := \{ \psi \in \mathcal{L} \mid \neg \psi \notin \mathcal{A}^\bot \} \) and \( \partial \mathcal{A} := \{ \psi \in \mathcal{L} \mid \neg \psi \notin \mathcal{A} \} \) to partition the complement of \( \mathcal{A} \). Now \( \psi \in \sim \mathcal{A} \) implies \( \neg \psi \in \mathcal{A}^\bot \), and \( \psi \in \sim \mathcal{A} \) iff \( \neg \psi \in \mathcal{A}^\bot \) when \( \mathcal{A} \) is negation-consistent. Then since \( \neg \psi \in \mathcal{A}^\bot \), we get \( \bigcap_{\varphi \in \mathcal{A}^\bot} \varphi_i \subseteq \bigcap_{\psi \in \partial \mathcal{A}} (X - \psi_i) \). From Assertions (5) and (8), it then follows that:

\[
(\mathcal{A})^{\wedge} = \left( \bigcap_{\varphi \in \mathcal{A}} \varphi_i \right) \cap \left( \bigcap_{\psi \in \partial \mathcal{A}} \text{bd}_T(\psi_i) \right) \tag{9}
\]

Otherwise put, the falsifiability of formulae in \( \mathcal{A} \) is already accounted for by the satisfiability of formulae in \( \mathcal{A}^\bot \), so on the falsifiability side, the remaining issue is with the boundaries for formulae in \( \partial \mathcal{A} \). Observe that for any subset \( B \subseteq \text{bd}_T(\psi_i) \), we have \( \text{int}_T(\text{cl}_T(B)) \subseteq \text{int}_T(\text{cl}_T(\text{bd}_T(\psi_i))) = \text{int}_T(\text{bd}_T(\psi_i)) = \emptyset \). Hence, unless the boundary set \( \partial \mathcal{A} = \emptyset \), the realization set \( (\mathcal{A})^{\wedge} \) will be nowhere-dense, i.e. its closure has empty interior, or equivalently, the complement of its closure is dense, which in turn means the closure of that set is the whole space. If \( \mathcal{A} \) is negation-consistent, then \( \partial \mathcal{A} = \{ \psi \in \mathcal{L} \mid \psi \notin \mathcal{A} \} \) exactly when \( \mathcal{A} \) is negation-maximal, in the sense that for every formula \( \varphi \in \mathcal{L} \), either \( \varphi \in \mathcal{A} \) or else \( \neg \varphi \in \mathcal{A} \).

**Lemma 5.6** Let \( \mathcal{M} \) be any open and l.s.c. topological model, and let \( \mathcal{A} \subseteq \mathcal{L} \) be any negation-consistent set of formulae. If the topology \( \mathcal{T} \) is manifest in \( \mathcal{M} \), then:

\[
\text{cl}_T((\mathcal{A})^{\wedge}) = \left( \bigcap_{\varphi \in \mathcal{A}} \text{cl}_T(\varphi_i) \right) \cap \left( \bigcap_{\psi \in \partial \mathcal{A}} \text{bd}_T(\psi_i) \right)
\]

**Proof.** To begin with, set:

\[
Z := \bigcap_{\varphi \in \mathcal{A}} \varphi_i \quad \text{and} \quad C := \bigcap_{\varphi \in \mathcal{A}} \text{cl}_T(\varphi_i) \quad \text{and} \quad D := \bigcap_{\psi \in \partial \mathcal{A}} \text{bd}_T(\psi_i)
\]

20
So by Assertion (9), we have \( (\mathcal{A})^{\mathbb{M}} = Z \cap D \), and hence the inclusions \( (\mathcal{A})^{\mathbb{M}} \subseteq cl_\mathbb{T}(\mathcal{A})^{\mathbb{M}} \subseteq (C \cap D) \). We need to show that \( cl_\mathbb{T}(\mathcal{A})^{\mathbb{M}} = (C \cap D) \). (Note that in the extremal case when \( \partial A = \emptyset \), the intersection over an empty family gives \( D = X \) and hence \( (\mathcal{A})^{\mathbb{M}} = Z \), so we don’t need to address this case separately.)

Now since \( \mathcal{T} \) is manifest in the model \( \mathcal{M} \), we can take as a basis the family of all denotation sets \( \mathcal{Q}_1(\mathcal{M}) \). Hence:

\[
cl_\mathbb{T}(\mathcal{A})^{\mathbb{M}} = \bigcup \{ (X - [\psi]_1^{\mathbb{M}}) \mid \psi \in \mathcal{L} \text{ and } (\mathcal{A})^{\mathbb{M}} \subseteq (X - [\psi]_1^{\mathbb{M}}) \}
\]

Thus we can conclude that \( cl_\mathbb{T}(\mathcal{A})^{\mathbb{M}} \neq (C \cap D) \) iff there is a formula \( \psi \in \mathcal{L} \) such that \( (\mathcal{A})^{\mathbb{M}} = (Z \cap D) \subseteq (X - [\psi]_1^{\mathbb{M}}) \) and \( (C - Z) \cap D \cap [\psi]_1^{\mathbb{M}} \neq \emptyset \). Observe that for any \( \psi \in \mathcal{L} \), we have:

\[
(C - Z) \cap D \cap [\psi]_1^{\mathbb{M}} = \bigcup_{B \subseteq A, B \neq \emptyset} \left( D \cap [\psi]_1^{\mathbb{M}} \cap \bigcap_{\varphi \in A - B} [\varphi]_1^{\mathbb{M}} \cap \bigcup_{\psi \in B} bd_\mathbb{T}([\psi]_1^{\mathbb{M}}) \right)
\]

So suppose, for a contradiction, that \( cl_\mathbb{T}(\mathcal{A})^{\mathbb{M}} \neq (C \cap D) \). Then there is a formula \( \psi \) satisfying the two conditions \( (D \cap [\psi]_1^{\mathbb{M}}) \subseteq (X - Z) = \bigcup_{\varphi \in A} (X - [\varphi]_1^{\mathbb{M}}) \) and \( (C - Z) \cap D \cap [\psi]_1^{\mathbb{M}} \neq \emptyset \) simultaneously, and this can occur iff \( [\psi]_1^{\mathbb{M}} \cap (\mathcal{A})^{\mathbb{M}} = \emptyset \) and we have \( B = \mathcal{A} \) and the non-empty intersection \( B : = D \cap [\psi]_1^{\mathbb{M}} \cap \bigcap_{\varphi \in A} bd_\mathbb{T}([\varphi]_1^{\mathbb{M}}) \). Note that for any formula \( \psi \), we have \( bd_\mathbb{T}([-\psi]_1^{\mathbb{M}}) = bd_\mathbb{T}([-\psi]_1^{\mathbb{M}}) \subseteq bd_\mathbb{T}([\psi]_1^{\mathbb{M}}) \), and hence \( [\psi]_1^{\mathbb{M}} \) is disjoint from the boundaries \( bd_\mathbb{T}([\psi]_1^{\mathbb{M}}) \) and \( bd_\mathbb{T}([-\psi]_1^{\mathbb{M}}) \). Now the non-empty intersection \( B \neq \emptyset \) implies that:

(i) \( : [\psi]_1^{\mathbb{M}} \cap D \neq \emptyset \), and it also implies that:

(ii) \( : \bigcap_{\varphi \in A} ([\psi]_1^{\mathbb{M}} \cap bd_\mathbb{T}([\varphi]_1^{\mathbb{M}})) \neq \emptyset \). Condition (ii) in turn implies that \( \psi \notin \mathcal{A} \) and \( \neg \psi \notin \mathcal{A} \), and hence \( \psi \in \partial \mathcal{A} \). But \( D = \bigcap_{\chi \in \partial \mathcal{A}} bd_\mathbb{T}([\chi]_1^{\mathbb{M}}) \), hence \( \psi \in \partial \mathcal{A} \) implies that \( [\psi]_1^{\mathbb{M}} \cap D = \emptyset \), which contradicts condition (i). Hence \( cl_\mathbb{T}(\mathcal{A})^{\mathbb{M}} = (C \cap D) \), as required.

**Definition 5.7** Let \( \mathcal{M} = (X, \mathcal{T}, R, v) \) be any open and l.s.c. topological model.

The relation \( R \) (or \( R^{-1} \)) is image-closed with respect to \( \mathcal{T} \) if for each state \( x \in X \), the set \( R(x) \) (or \( R^{-1}(x) \)) is closed in \( \mathcal{T} \).

The relation \( R \) (or \( R^{-1} \)) will be called boundary-closed in \( \mathcal{M} \) if for all finite sets of formulae \( \{\psi_1, \ldots, \psi_m\} \subseteq \mathcal{L} \), if \( D := \bigcap_{1 \leq j \leq m} bd_\mathbb{T}([\psi_j]_1^{\mathbb{M}}) \) then \( R^{-1}(D) \) (or \( R^{-1}(D) \)) is closed in \( \mathcal{T} \).

We say that the relation \( R \) (or \( R^{-1} \)) has negative saturation in \( \mathcal{M} \) if for every set of formulae \( \mathcal{B} \subseteq \mathcal{L} \) and for every \( x \in X \), the following condition holds:

if, for every finite subset \( \{\theta_1, \ldots, \theta_m\} \subseteq \mathcal{B} \), there is an \( R \)-successor \( x' \in R(x) \)

( an \( R \)-predecessor \( x' \in R^{-1}(x) \)) such that \( x' \notin [\psi_j]_1^{\mathbb{M}} \) for each \( j \in \{1, \ldots, m\} \),

then there exists an \( x^* \in R(x) \) (an \( x^* \in R^{-1}(x) \)) such that \( x^* \in \bigcap_{\theta \in \mathcal{B}} (X - [\theta]_1^{\mathbb{M}}) \).
We say that the relation $R$ (or $R^{-1}$) has realization saturation in $\mathcal{M}$ if for every set of formulae $\mathcal{A} \subseteq \mathcal{L}^1$ that is $\mathbb{K}^{1T}$-semantically-prime, and for every $x \in X$, the following condition holds:

- if, for every finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathcal{A}$, there is an $R$-successor $x_1 \in R(x)$
  - (an $R$-predecessor $x_1 \in R^{-1}(x)$) such that $x_1 \in \llbracket \varphi_k \rrbracket_1^{\mathcal{A}}$ for each $k \in \{1, \ldots, n\}$, and
- for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \partial \mathcal{A}$, there is an $R$-successor $x_2 \in R(x)$
  - (an $R$-predecessor $x_2 \in R^{-1}(x)$) such that $x_2 \in \text{bd}_\tau(\llbracket \psi_j \rrbracket_1^{\mathcal{A}})$ for each $j \in \{1, \ldots, m\}$,

then $x \in R^{-3}(\llbracket \mathcal{A} \rrbracket_1^{\mathcal{A}})$ (or $x \in R^3(\llbracket \mathcal{A} \rrbracket_1^{\mathcal{A}})$); i.e. there is an $R$-successor $x^* \in R(x)$

- (an $R$-predecessor $x^* \in R^{-1}(x)$) that realizes $\mathcal{A}$ as a theory in $\mathcal{M}$.

We have identified two distinct notions of relational saturation for the intuitionistic semantics. In the following series of lemmas, we establish that the first implies the second in the presence of the further conditions of the topology being manifest in the model, and having regular $(T_3)$ separation, and the relations $R$ and $R^{-1}$ being image-closed. Also observe that $R$ being u.s.e. implies that $R$ is boundary-closed, but not in general conversely. These results will motivate our subsequent demarcation of a candidate Hennessy-Milner class, and the identification of interesting sub-classes within it.

**Lemma 5.8** Let $\mathcal{M}$ be any open and l.s.e. topological model, and let $\mathcal{A} \subseteq \mathcal{L}^1$ be any negation-consistent set of formulae. If $\mathcal{T}$ is manifest in $\mathcal{M}$, and $R$ has negative saturation in $\mathcal{M}$, then for all $x \in X$, the following condition holds:

- if, for every finite subset $\{\theta_1, \ldots, \theta_m\} \subseteq (\mathcal{L}^1 - \mathcal{A})$, there is $x' \in R(x)$ such that $x' \notin \llbracket \theta_j \rrbracket_1^{\mathcal{A}}$
  - for each $j \in \{1, \ldots, m\}$, then $x \in R^{-3}(\text{cl}_\tau(\llbracket \mathcal{A} \rrbracket_1^{\mathcal{A}}))$.

**Proof.** Fix $x \in X$, and suppose that for every finite subset $\{\theta_1, \ldots, \theta_m\} \subseteq (\mathcal{L}^1 - \mathcal{A})$, there is an $x' \in R(x)$ such that $x' \notin \llbracket \theta_j \rrbracket_1^{\mathcal{A}}$ for each $j \in \{1, \ldots, m\}$. Since $R$ has negative saturation, there exists an $x^* \in R(x)$ such that $x^* \in \bigcap_{\varphi \in \mathcal{A}} (X - \llbracket \varphi \rrbracket_1^{\mathcal{A}})$.

Now for any formula $\varphi \in \mathcal{A}$, we must have $\neg \varphi \notin \mathcal{A}$, since $\mathcal{A}$ is negation-consistent, and hence $x^* \in (X - \llbracket \neg \varphi \rrbracket_1^{\mathcal{A}}) = \text{cl}_\tau(\llbracket \varphi \rrbracket_1^{\mathcal{A}})$. For any boundary formula $\psi \in \partial \mathcal{A}$, we have both $\psi \notin \mathcal{A}$ and $\neg \psi \notin \mathcal{A}$, and hence $x^* \in (X - \llbracket \psi \rrbracket_1^{\mathcal{A}}) \cap (X - \llbracket \neg \psi \rrbracket_1^{\mathcal{A}}) = \text{bd}_\tau(\llbracket \psi \rrbracket_1^{\mathcal{A}})$. Applying Lemma 5.6, we can conclude that:

$$x^* \in R(x) \cap \text{cl}_\tau(\llbracket \mathcal{A} \rrbracket_1^{\mathcal{A}}) = R(x) \cap \left( \bigcap_{\varphi \in \mathcal{A}} \text{cl}_\tau(\llbracket \varphi \rrbracket_1^{\mathcal{A}}) \right) \cap \left( \bigcap_{\psi \in \partial \mathcal{A}} \text{bd}_\tau(\llbracket \psi \rrbracket_1^{\mathcal{A}}) \right)$$

Thus $x \in R^{-3}(\text{cl}_\tau(\llbracket \mathcal{A} \rrbracket_1^{\mathcal{A}}))$, as required. 

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Lemma 5.9 Let $\mathcal{M}$ be any open and l.s.c. topological model. If $\mathcal{T}$ is manifest in $\mathcal{M}$, $R$ has negative saturation in $\mathcal{M}$, the topology $\mathcal{T}$ has regular $(T_3)$ separation, and $R$ is image-closed, then $R$ has realization saturation in $\mathcal{M}$.

**Proof.** To prove $R$ has realization saturation in $\mathcal{M}$, fix a set of formulae $\mathcal{A} \subseteq \mathcal{L}_1$ that is $\mathbb{K}^\mathcal{T}$-semantically-prime, and fix a state $x \in X$. Suppose that for every finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathcal{A}$, there is an $x_1 \in R(x)$ such that $x_1 \in \{\varphi_k\}^{\mathcal{A}}$ for each $k \in \{1, \ldots, n\}$, and for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \partial \mathcal{A}$, there is an $x_2 \in R(x)$ such that $x_2 \in bd_\mathcal{T}(\{\psi_j\}^{\mathcal{A}})$. Since $R$ has negative saturation in $\mathcal{M}$, consider the set $\mathcal{B} = \mathcal{L}_1 - \mathcal{A}$, and fix a finite subset $\{\theta_1, \ldots, \theta_m\} \subseteq \mathcal{B}$. For each $\theta_j$, we either have $\theta_j \in \sim \mathcal{A}$ and $-\theta_j \in \mathcal{A}$, or else $\theta_j \in \partial \mathcal{A}$ and both $\theta_j \notin \mathcal{A}$ and $-\theta_j \notin \mathcal{A}$. We need to show there is an $x' \in R(x) \cap \mathcal{A}$. We proceed by cases.

**Case I:** $\partial \mathcal{A} = \emptyset$. Then we have $-\theta_j \notin \mathcal{A}$ for every $j \in \{1, \ldots, m\}$, and hence there exists an $x_1 \in R(x) \cap \mathcal{A}$. We then claim that $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathcal{A}$, where this finite set of formulae is defined by $\varphi_i := \theta_i$ for $1 \leq i \leq q$, and $\psi_j := (\theta_i \lor \theta_j)$ for $q < j \leq m$, and hence there exists $x_2 \in R(x) \cap \mathcal{A}$. To prove the claim, there is nothing to do for $\varphi_i = \theta_i \in \partial \mathcal{A}$ for $1 \leq i \leq q$. For $\psi_j$ for $q < j \leq m$, first note that $\psi_j \notin \mathcal{A}$ since $\mathcal{A}$ has the disjunction property. Moreover, $(-\psi_j \leftrightarrow (-\theta_i \land -\theta_j)) \in \mathbb{K}^\mathcal{T}$.

**Case III:** $\partial \mathcal{A} \neq \emptyset$ and $\theta_i \in \partial \mathcal{A}$ for at least one $i \in \{1, \ldots, m\}$. Then re-number the formulae so that for some $q \in \{1, \ldots, m\}$, we have $\theta_i \in \partial \mathcal{A}$ for $1 \leq i \leq q$, and $-\theta_j \in \mathcal{A}$ for $q < j \leq m$. We then claim that $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathcal{A}$, where this finite set of formulae is defined by $\varphi_i := \theta_i$ for $1 \leq i \leq q$, and $\psi_j := (\theta_i \lor \theta_j)$ for $q < j \leq m$, and hence there exists $x_2 \in R(x) \cap \mathcal{A}$. To prove the claim, there is nothing to do for $\varphi_i = \theta_i \in \partial \mathcal{A}$ for $1 \leq i \leq q$. For $\psi_j$ for $q < j \leq m$, first note that $\psi_j \notin \mathcal{A}$ since $\mathcal{A}$ has the disjunction property. Moreover, $(-\psi_j \leftrightarrow (-\theta_i \land -\theta_j)) \in \mathbb{K}^\mathcal{T}$.

Since $\mathcal{A}$ is $\mathbb{K}^\mathcal{T}$-semantically-closed and negation-consistent, and $-\theta_i \notin \mathcal{A}$, we can conclude that $-\psi_j \notin \mathcal{A}$, and hence $\psi_j \notin \mathcal{A}$ for every $j \in \{1, \ldots, m\}$, as claimed. Now for $q < j \leq m$:

$$bd_\mathcal{T}(\{\psi_j\}^{\mathcal{A}}) = cl_\mathcal{T}(\{\theta_i \lor \theta_j\}^{\mathcal{A}}) \cap cl_\mathcal{T}(\{-\theta_i \land -\theta_j\}^{\mathcal{A}})$$

$$= (cl_\mathcal{T}(\{\theta_j\}^{\mathcal{A}}) \cap cl_\mathcal{T}(\{-\theta_i \land -\theta_j\}^{\mathcal{A}})) \cup (cl_\mathcal{T}(\{\theta_i\}^{\mathcal{A}}) \cap cl_\mathcal{T}(\{-\theta_i \land -\theta_j\}^{\mathcal{A}}))$$

$$\subseteq bd_\mathcal{T}(\{\theta_j\}^{\mathcal{A}}) \cup cl_\mathcal{T}(\{-\theta_j\}^{\mathcal{A}})$$

$$\subseteq X - \{\theta_j\}^{\mathcal{A}}$$

Hence $x_2 \in R(x) \cap \mathcal{A}$, as required.

Having satisfied the hypothesis for negative saturation w.r.t. the set of formulae $\mathcal{B} = \mathcal{L}_1 - \mathcal{A}$, we can conclude by Lemma 5.8 that $x \in R^{-3}(cl_\mathcal{T}(\{\mathcal{A}\}^{\mathcal{A}}))$. Since $R^{-1}$ is l.s.c. we have the inclusion $R^{-3}(cl_\mathcal{T}(W)) \subseteq cl_\mathcal{T}(R^{-3}(W))$, for all subsets $W \subseteq X$. Thus $x \in cl_\mathcal{T}(R^{-3}(\{\mathcal{A}\}^{\mathcal{A}}))$, and $R^{-3}(\{\mathcal{A}\}^{\mathcal{A}}) \neq \emptyset$, and hence also $\mathcal{A}^{\mathcal{A}} \neq \emptyset$.

To complete the proof, suppose, for a contradiction, that $x \notin R^{-3}(\{\mathcal{A}\}^{\mathcal{A}})$. Fix $z \in \mathcal{A}^{\mathcal{A}}$, which must lie outside the non-empty set $R(x)$, which is closed as $R$ is image-closed. Since the
topology $\mathcal{T}$ has regular separation, this means that for the point $z$ and the closed set $R(x)$, there must exist two disjoint basic open sets $[[\varphi_*]]^\mathcal{M}$ and $[[\psi_*]]^\mathcal{M}$ such that $z \in [[\varphi_*]]^\mathcal{M}$ and $R(x) \subseteq [[\psi_*]]^\mathcal{M}$. We now analyze the constraints on these formulae. By the disjointness of the open sets $[[\varphi_*]]^\mathcal{M}$ and $[[\psi_*]]^\mathcal{M}$, we can conclude that $R(x) \subseteq [[\neg \varphi_* \land \neg \psi_*]]^\mathcal{M}$, and $R(x) \cap [[\varphi_*]]^\mathcal{M} = \emptyset$, and also that $z \in [[\varphi_* \land \neg \psi_*]]^\mathcal{M}$. Then since $z \in (\mathcal{A})^\mathcal{M}$, we can conclude that $(\varphi_* \land \neg \psi_*) \in \mathcal{A}$, and hence $x \in (\Diamond(\varphi_* \land \neg \psi_*)^\mathcal{M}$, from our original assumptions on $\mathcal{A}$ and $x$. Thus we have $R(x) \cap [[\varphi_* \land \neg \psi_*]]^\mathcal{M} \neq \emptyset$, and so $R(x) \cap [[\varphi_*]]^\mathcal{M} \neq \emptyset$, which yields our desired contradiction. So we can conclude that $x \in R^{-3}(\mathcal{A})^\mathcal{M}$ and $R(x) \cap (\mathcal{A})^\mathcal{M} \neq \emptyset$, as required. This completes the proof, and we are done.

**Definition 5.10** Let $\mathcal{D}_0$ denote the class of all open l.s.c. models $\mathcal{M} = (X, \mathcal{T}, R, v)$ such that:
(a) the topology $\mathcal{T}$ is manifest in $\mathcal{M}$; (b) the relations $R$ and $R^{-1}$ both have realization saturation in $\mathcal{M}$; and (c) both $R$ and $R^{-1}$ are boundary-closed in $\mathcal{M}$.

In Theorem 7.3, we will prove that the class $\mathcal{D}_0$ has the Hennessy-Milner property for the intuitionistic semantics. The following result establishes that $\mathcal{D}_0$ contains many interesting and naturally occurring topological models.

**Theorem 5.11** Let $\mathcal{M} = (X, \mathcal{T}, R, v)$ be any topological model such that $(X, \mathcal{T})$ is a compact Hausdorff space, the topology $\mathcal{T}$ has a countable basis $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ with the atomic valuation $v(p_n) = U_n$ for some enumeration of atomic propositions $\mathcal{AP} = \{p_n \mid n \in \mathbb{N}\}$, and both the relations $R$ and $R^{-1}$ are both l.s.c. and image-closed with respect to $\mathcal{T}$, and both are boundary-closed in $\mathcal{M}$. Then $\mathcal{M} \in \mathcal{D}_0$.

**Proof.** The topology $\mathcal{T}$ is clearly manifest in the model $\mathcal{M}$, and by assumption, we have the l.s.c., image-closed and boundary-closed properties for $R$ and $R^{-1}$, and an open atomic valuation. Since $(X, \mathcal{T})$ is compact Hausdorff, the topology has normal and hence regular separation. We claim that from compactness plus the image-closed properties, we can prove negative saturation. Then by Lemma 5.9, we have realization saturation, and so $\mathcal{M} \in \mathcal{D}_0$.

To see that $R$ (and symmetrically, $R^{-1}$) has negative saturation, fix a set of formulae $\mathcal{B} \subseteq \mathcal{L}^i$ and a state $x \in X$, and suppose that for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \mathcal{B}$, there is an $x' \in R(x)$ such that $x' \notin [[\psi_j]]^\mathcal{M}$ for each $j \in \{1, \ldots, m\}$. Then since $R(x)$ is closed, we know that the family of closed sets $\{R(x) \cap (X - [[\psi]]^\mathcal{M}) \mid \psi \in \mathcal{B}\}$ has the finite intersection property. Then by compactness of the topology $\mathcal{T}$ on $X$, the intersection of the whole family is non-empty, so there exists an $x^* \in R(x)$ such that $x^* \in \bigcap_{\psi \in \mathcal{B}} (X - [[\psi]]^\mathcal{M})$. Thus $R$ has negative saturation, as claimed. 

\[\]
As a more concrete example of models fulfilling the hypotheses of Theorem 5.11, we can look to a particularly simple subclass of differential inclusion models, namely the *rectangular differential inclusions*. In spite of and/or because of their simplicity, they are of particular interest in the formal analysis of hybrid systems [29, 3]. Hybrid automata with continuous dynamics in each discrete mode given by a rectangular differential inclusion (or more restricted systems) have been the focus of investigations on boundaries between decidability and undecidability of exact symbolic model-checking of temporal and modal logics interpreted over transition system representations of hybrid automata [3, 21]. Rectangular differential inclusions are also used to approximate the dynamics of more complex systems which are not amenable to exact symbolic model-checking [29]. We will return to rectangular differential inclusions in their latter capacity in the sequel *Part II*, as part of an investigation of approximate model-checking.

So as an example of a model in $\mathcal{D}_0$, consider a topological model $\mathcal{M} = (X, T, R, u)$ such that $X \subseteq \mathbb{R}^n$ is a subset that is compact in Euclidean topology on $\mathbb{R}^n$; $T$ is the Euclidean (subspace) topology on $X$; $R : X \rightsquigarrow X$ is the reachability relation of a constant differential inclusion $\frac{d}{dt}x(t) \in U$, where $U := [a_1, b_1] \times \cdots \times [a_n, b_n]$ is an $n$-rectangle, with derivative bounds $a_i \leq b_i \in \mathbb{R}$ for $1 \leq i \leq n$; and $u : AP \rightsquigarrow X$ is an atomic valuation such that $v(p(x, \delta)) = B_\delta(x) \cap X$, where $AP = \{ p(x, \delta) \mid (x, \delta) \in (X \times \mathbb{Q}^+)^\times \}$ is an enumeration of $AP$ indexed by pairs of points $x \in X$ with rational coordinates and rational distances $\delta > 0$. For definiteness, choose the square metric $d(x, y) = \max\{|x_i - y_i| \mid 1 \leq i \leq n\}$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. So each $\delta$ “ball” $B_\delta(x)$ will be an open $n$-cube with side length $2\delta$. The reachability relation $R : X \rightsquigarrow X$ can be explicitly characterized as follows: for $x \in X$, and $y, u \in \mathbb{R}^n$,

$$R(x) = \{ y \in X \mid (\exists t \geq 0)(\exists u \in U) \ y = x + tu \}$$

The content of this characterization is that any point you can reach by an arbitrary absolutely continuous function $\gamma$ whose derivative lies $U$, you must in fact be able to reach by a straight line with slope vector $u \in U$. Moreover, for each $x \in X$, the successor set $R(x)$ is closed in $T$. Now for subsets $W \subseteq X$, we have:

$$R^{-1}(W) = \{ x \in X \mid (\exists t \geq 0)(\exists u \in U) \ x + tu \in W \}$$

$$R(W) = \{ y \in X \mid (\exists t \geq 0)(\exists u \in U) \ y - tu \in W \}$$

One can directly verify that for each basic open ball $B_\delta(x)$, the sets $R^{-1}(B_\delta(x))$ and $R(B_\delta(x))$ are open, and hence $R$ and $R^{-1}$ are l.s.c. Alternatively, we can use the fact that, when $X$ is a metric space, $R : X \rightsquigarrow X$ is l.s.c. iff for every $x \in \text{dom}(R)$, if $(x_n)_{n\in\mathbb{N}}$ is any sequence in $\text{dom}(R)$ converging to $x$ and $x R y$, then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ converging to $y$ with $x_n R y_n$ for
all \( n \in \mathbb{N} \). [5]. When \( X \) is a compact metric space (as we have here), \( R : X \rightarrow X \) is u.s.c. iff \( R \subseteq X \times X \) is a closed set in the product topology, and this is readily verified in our case. Since the u.s.c. property implies the boundary-closed property, we have \( \mathcal{M} \in \mathcal{D}_0 \). More generally, there is a significant subclass of differential inclusion models over compact spaces that lies within the class \( \mathcal{D}_0 \).

6 Axiomatization and topological completeness

Let \( \mathbf{IK} \) be the axiomatic system of Fischer Servi [15, 12, 19], which is equivalent to an alternative axiomatisation given in [28, 31]; \( \mathbf{IK} \) also goes by the name \( \mathbf{FS} \) in [19] and [34, 35]. \( \mathbf{IK} \) has as axioms all instances in the language \( \mathcal{L}^m \) of intuitionistic propositional theorems, and further axiom schemes:

\[
\begin{align*}
\mathbf{R} \otimes & : \diamond (\varphi \lor \psi) \leftrightarrow (\diamond \varphi \lor \diamond \psi) & \quad \mathbf{N} \otimes & : \neg \diamond \bot \\
\mathbf{R} \square & : \Box (\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi) & \quad \mathbf{N} \square & : \Box \top \\
\mathbf{F1} \circ & : \Box (\varphi \rightarrow \psi) \leftrightarrow (\Box \varphi \rightarrow \diamond \psi) & \quad \mathbf{F2} \circ & : (\Box \varphi \rightarrow \Box \psi) \rightarrow \Box (\varphi \rightarrow \psi)
\end{align*}
\]

and is closed under the inference rules of *modus ponens* (\( \mathbf{MP} \)) and the rule (\( \mathbf{Mono} \)): from \( \varphi_1 \rightarrow \varphi_2 \) infer \( \Box \varphi_1 \rightarrow \Box \varphi_2 \), and likewise (\( \mathbf{Mono} \)): from \( \varphi_1 \rightarrow \varphi_2 \) infer \( \Box \varphi_1 \rightarrow \Box \varphi_2 \).

The theorems of \( \mathbf{IK} \) include some semi-duality properties between \( \diamond \) and \( \Box \): the two implications \( \diamond \varphi \rightarrow \neg \Box \neg \varphi \) and \( \Box \neg \varphi \rightarrow \neg \diamond \varphi \), and the equivalence \( \neg \diamond \varphi \leftrightarrow \Box \neg \varphi \) [28].

With regard to notation for combinations of modal logics, we follow that used in the work of Wolter and Zakharyaschev [34, 35], and elsewhere. If \( \Lambda_1 \) and \( \Lambda_2 \) are axiomatically presented modal logics in languages \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) respectively, then the fusion \( \Lambda_1 \otimes \Lambda_2 \) is the smallest multi-modal logic in the language \( \mathcal{L}_1 \otimes \mathcal{L}_2 \) containing \( \Lambda_1 \) and \( \Lambda_2 \), and closed under all the inference rules of \( \Lambda_1 \) and \( \Lambda_2 \), where \( \mathcal{L}_1 \otimes \mathcal{L}_2 \) denotes the least common extension of the languages \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). If \( \Lambda \) is a logic in language \( \mathcal{L} \), and \( \Gamma \) is a finite list of schema in \( \mathcal{L} \), then the extension \( \Lambda \oplus \Gamma \) is the smallest logic in \( \mathcal{L} \) extending \( \Lambda \), containing the schema in \( \Gamma \) as additional axioms, and closed under the rules of \( \Lambda \).

As a point of comparison, the basic system of intuitionistic modal logic in [34], under the name \( \mathbf{IntK} \), is a sub-system in the sense that \( \mathbf{IK} = \mathbf{IntK} \oplus \mathbf{F1} \circ \otimes \oplus \mathbf{F2} \circ \otimes \). The latter two schemes were identified by Fischer Servi in [15]. The intuitionistic modal logics considered in [33] and [4, 22] are yet weaker sub-systems: they have the normality schemes \( \mathbf{R} \square \) and \( \mathbf{N} \square \) for \( \Box \), but \( \otimes \) is sub-normal - they include the scheme \( \mathbf{N} \otimes \), but \( \mathbf{R} \otimes \) is replaced by \( (\Box \varphi \land \diamond \psi) \rightarrow \diamond (\varphi \land \psi) \), the latter being a theorem of \( \mathbf{IntK} \).
For the extension to tense logics with forwards and backwards modalities, let $\text{IK}^t$ be the deductive system presented by Ewákl [12] which is the fusion of $\text{IK} \uparrow \square := \text{IK}$ with the “mirror” system $\text{IK} \square \downarrow$ having axiom schemes $R \uparrow$, $N \uparrow$, $R \downarrow$, $N \downarrow$, $F1 \downarrow \square$, and $F2 \downarrow \uparrow$, and inference rules (Mono $\uparrow$) and (Mono $\downarrow$), which is then further extended with four axiom schemes expressing the adjoint property (Assertion (3)) of the operators interpreting the tense modalities:

$$\text{Ad1} : \varphi \rightarrow \square \varphi \quad \text{Ad2} : \varphi \rightarrow \square \varphi$$

$$\text{Ad3} : \varphi \rightarrow \square \varphi \quad \text{Ad4} : \square \varphi \rightarrow \varphi$$

Thus $\text{IK}^t := (\text{IK} \downarrow \square \text{IK} \uparrow \square) \oplus \text{Ad1} \oplus \text{Ad2} \oplus \text{Ad3} \oplus \text{Ad4}$. Note that classically, the two axiom schemes Ad1 and Ad2 suffice, since the other two are derivable by classical duality between box and diamond.

Before turning to completeness of the intuitionistic logics, we first identify various related classical logics. Let $\text{K} \square$ be the minimal normal modal logic (over a classical propositional base), and let $(\text{S4} \square \otimes \text{K} \square)$ be the bi-modal fusion of $\text{S4} \square$ and $\text{K} \square$, and let $\text{K}^t \text{LSC} := (\text{S4} \square \otimes \text{K} \square) \oplus (\downarrow \square \varphi \rightarrow \square \downarrow \varphi) \oplus (\square \square \varphi \rightarrow \square \square \varphi)$ be the extension of $(\text{S4} \square \otimes \text{K} \square)$ with characteristic modal schema for the $R$-l.s.c. and $R^{-1}$-l.s.c. frame conditions, from Proposition 4.5 (and as identified by Fischer Servi in [13]). Likewise, define $\text{K}^t := (\text{K} \square \otimes \text{K} \square) \oplus \text{Ad1} \oplus \text{Ad2}$ as the minimal normal tense logic, and define $\text{K}^t \text{LSC} := (\text{S4} \square \otimes \text{K}^t) \oplus (\downarrow \square \varphi \rightarrow \square \downarrow \varphi) \oplus (\square \varphi \rightarrow \square \varphi)$, here using instead the tense schema for $R^{-1}$-l.s.c. from Proposition 4.5. A related logic is the extension $\text{S4LSC} := (\text{S4} \square \otimes \text{S4} \downarrow \square) \oplus (\downarrow \square \varphi \rightarrow \square \downarrow \varphi)$ of the fusion of $\text{S4} \square$ and $\text{S4} \downarrow \square$, studied by Davoren and Goré [10] (where it goes under the working name of LSC). Its semantics are in topological frames where $R$ is l.s.c. and reflexive and transitive (a pre-order), and are motivated by the reachability relation of differential inclusions or equations.

In what follows, we will deal generically with extensions $\text{IK} \oplus \Gamma$ or $\text{IK}^t \oplus \Gamma$ for subsets $\Gamma$ of the five axiom schema below or their $\square \rightarrow$ mirror images:

$$\text{T} \square \varphi : (\square \varphi \rightarrow \varphi) \land (\varphi \rightarrow \square \varphi)$$

$$\text{B} \square \varphi : (\varphi \rightarrow \square \varphi) \land (\square \varphi \rightarrow \varphi)$$

$$\text{D} \varphi : \varphi \uparrow$$

$$\text{4} \square \varphi : (\square \varphi \rightarrow \square \square \varphi) \land (\downarrow \square \varphi \rightarrow \downarrow \varphi)$$

$$\text{5} \square \varphi : (\downarrow \square \varphi \rightarrow \downarrow \varphi) \land (\varphi \rightarrow \square \varphi)$$

where the schema characterize, in turn, the properties of relations $R : X \rightarrow X$ of reflexivity, symmetry, totality (seriality), transitivity and Euclideanness, and the mirror image scheme
characterize relations $R$ such that $R^{-1}$ has the property. Note that reflexivity, symmetry and transitivity are all such that $R$ has this property iff $R^{-1}$ has the property, so the mirrored tense schemes $[\Diamond \Box \Diamond], [\Box \Diamond \Box]$ and $[\Diamond \Diamond \Diamond]$ are semantically equivalent to their un-mirrored modal versions.

For a set $\Gamma$ of schema, let $\mathcal{C}(\Gamma)$ be the set of all formulae $\varphi \in \mathcal{L}^t$ that are int-modal-top valid in every l.s.c. topological frame whose relation $R$ has the properties corresponding to the schema in $\Gamma$, and let $\mathcal{C}_t(\Gamma)$ be the set of all formulae $\varphi \in \mathcal{L}^t_\Gamma$ that are modal-top valid in every topological frame whose relation $R$ has the properties corresponding to the schema in $\Gamma^t$.

The topological soundness of $\textbf{IK}^t$ and of $\textbf{K}^t\textbf{LSC}$ are easy verifications. For example, the soundness of the Fischer Servi scheme $\textbf{F1}\equiv[\Diamond \Box \Diamond]$ is equivalent to the assertion that:

$$R^{-3}(\text{int}_T(-U \cup V)) \subseteq \text{int}_T(-\text{int}_T(R^{-t}(U)) \cup R^{-3}(V))$$

for all open sets $U, V \in \mathcal{T}$. Since $R$ is l.s.c., we have the inclusion $R^{-3}(\text{int}_T(-U \cup V)) \subseteq \text{int}_T(R^{-3}(-U \cup V))$. Applying distribution over unions, duality, and monotonicity, we can get $\text{int}_T(R^{-3}(-U \cup V)) \subseteq \text{int}_T(-\text{int}_T(R^{-t}(U)) \cup R^{-3}(V))$, so we are done. For the adjoint axiom Ad3: $[\Box \varphi \rightarrow \varphi]$, soundness is equivalent to $R^{-3}(\text{int}_T(R^t(U))) \subseteq U$ for all open sets $U \in \mathcal{T}$. Since $R$ is l.s.c. we have the inclusion $R^{-3}(\text{int}_T(R^t(U))) \subseteq \text{int}_T(R^{-3}(R^t(U)))$ and then by the adjointness of $R^{-3}$ and $R^t$, and the openness of $U$, the required inclusion follows.

Recall that for a logic $\Lambda$ in a language $\mathcal{L}$ with deductive consequence relation $\vdash_\Lambda$, a set of formulae $x \subseteq \mathcal{L}$ is said to be $\Lambda$-consistent if $x \not\vdash_\Lambda \bot$; $x$ is $\Lambda$-deductively closed if $x \vdash_\Lambda \varphi$ implies $\varphi \in x$ for all formulae $\varphi \in \mathcal{L}$; and $x$ is maximal $\Lambda$-consistent if $x$ is $\Lambda$-consistent, and no proper superset of $x$ is $\Lambda$-consistent. (Hence every maximal $\Lambda$-consistent set $x$ is negation-maximal and $\partial x = \emptyset$.) A set $x \subseteq \mathcal{L}$ is a prime theory of $\Lambda$ if $\Lambda \subseteq x$, and $x$ has the disjunction property, and is $\Lambda$-consistent, and $\Lambda$-deductively closed. (A corollary of soundness and completeness for $\textbf{IK}^t$ will be that a set of formulae in $\mathcal{L}^t$ is $\mathbb{K}^t\mathbb{T}$-semantically-prime iff it is a prime theory of $\textbf{IK}^t$).

To prove topological completeness, we could get a free ride from the corresponding proofs for bi-relational frames, using the proofs from [15, 31] for $\textbf{IK}$ or from [12] for $\textbf{IK}^t$. Taking this strategy, we would build a canonical model over the state space $X_{\text{ip}}$ defined to be the set of all sets of formulae $x \subseteq \mathcal{L}^t$ that are prime theories of $\textbf{IK}^t$. The space $X_{\text{ip}}$ is partially ordered by inclusion, so we have available an Alexandroff topology $\mathcal{T}_\mathcal{C}$. One then defines the modal

\footnotetext{\textsuperscript{7}See [28] for a discussion and some results on the intuitionistic analog of correspondence theory, and see [31], \S 6.3, for a discussion of the thickness of the $[\Diamond \Box \Diamond]$ scheme $[\Diamond \Box \Diamond] \rightarrow [\Box \Diamond \Diamond]$, which classically characterizes confluent or directed relations $(x R x_1$ and $x R x_2$ implies there exists $x'$ such that $x_1 R x'$ and $x_2 R x'$), and more generally, difficulties with $(k, l, m, n)$-inconsistent relations $R$ classically characterized by $[\Box \Diamond \Diamond] \rightarrow [\Box \Diamond \Diamond].$}
accessibility relation by:
\[ x R_0 x' \iff \{ \Diamond \varphi \mid \varphi \in x' \} \subseteq x \quad \text{and} \quad \{ \Box \varphi \mid \varphi \in x \} \subseteq x' \quad \text{and} \]
\[ \{ \Diamond \varphi \mid \varphi \in x \} \subseteq x' \quad \text{and} \quad \{ \Box \varphi \mid \varphi \in x' \} \subseteq x \]  \hspace{1cm} (11)

This is the conjunction of the usual conditions defining the canonical relations determined by \( \Diamond \), \( \Box \), \( \Diamond \) and \( \Box \) (the diamond/box pairs of conditions are classically equivalent), and as verified in [15] and [31] for the modal logic, and [12] for the tense logic, the relations \( R_0 \) and \( R_0^{-1} \) satisfy the frame conditions \( \text{Zig}(\subseteq, R_0) \) and \( \text{Zig}(\subseteq, R_0^{-1}) \). So we get an l.s.c. topological frame \( \mathcal{F}_0 = (X_{ip}, \mathcal{T}_c, R_0) \), and with the canonical valuation \( u : AP \models X_{ip} \) given by \( u(p) = \{ x \in X_{ip} \mid p \in x \} \), one then proves of the model \( \mathcal{M}_0 = (\mathcal{F}_0, u) \) the “Truth Lemma”: \( x \in \llbracket \varphi \rrbracket^\circ \) iff \( \varphi \in x \), for all \( \varphi \in \mathcal{L}^t \) and \( x \in X_{ip} \).

We could settle for this, as it does formally establish completeness of \( \text{IK}^t \) with respect to topological semantics. But an Alexandroff topology is not really convincing: isn’t it just the bi-relational semantics dressed up in topological terms? Fortunately, we can do better, and the route we take illuminates the topological content of the Gödel translation.

We will first do an easier proof of topological completeness for the classical logic \( \text{K}^t \text{LSC} \) with a canonical model over the space \( Y_{\text{im}} \) of maximal \( \text{K}^t \text{LSC} \)-consistent theories (in the language with topological \( \Box \)), which we will equip with a non-Alexandroff topology. We then turn to completeness of the intuitionistic logic \( \text{IK}^t \), where we equip the space \( X_{ip} \) of \( \text{IK}^t \)-prime theories with a non-Alexandroff topology, and prove this gives a canonical model. The crucial, and non-trivial step, is the case for the diamonds \( \Diamond \) and \( \Diamond \), where we have to produce \( R \)-successors and \( R \)-predecessors to establish the inclusions \( \llbracket \Diamond \varphi \rrbracket^\circ \subseteq R^{-3} (\llbracket \varphi \rrbracket^\circ) \), and \( \llbracket \Diamond \varphi \rrbracket^\circ \subseteq R^3 (\llbracket \varphi \rrbracket^\circ) \), and simultaneously ensure the relations \( R \) and \( R^{-1} \) are l.s.c. We then return to these two canonical models in the next section, where we show the Gödel translation determines a natural relation between \( X_{ip} \) and \( Y_{\text{im}} \), and that with respect to the non-Alexandroff topologies on these spaces, this Gödel map is a tense topo-bisimulation between the intuitionistic and classical canonical models.

**Theorem 6.1 [Topological soundness and completeness]**

Let \( \Gamma \) be any finite set of axiom schemes from \( \mathcal{L}^t \) from the list in (10) above.

1. For all \( \psi \in \mathcal{L}^t_{\Box} \), \( \psi \) is a theorem of \( \text{K}^t \text{LSC} \oplus \Gamma \) iff \( \psi \in \mathbb{K}^t_{\text{LST}} \cap \mathbb{C}_{\Box}(\Gamma) \).
2. For all \( \varphi \in \mathcal{L}^t \), \( \varphi \) is a theorem of \( \text{IK}^t \oplus \Gamma \) iff \( \varphi \in \mathbb{K}^t_{\text{T}} \cap \mathbb{C}(\Gamma) \).

**Proof.** Taking soundness as established, we prove the completeness half of Part (1.), and then turn to that for Part (2.). On notation, for the remainder of this section and the next, we will
use \( \text{IL} \) and \( \text{L} \), respectively, as abbreviations for the axiomatically presented logics \( \text{IK}^t \oplus \Gamma \) and \( \text{K}^t \text{LSC} \oplus \Gamma \).

For Part (1), we define a model \( \mathcal{M} = (Y_{\text{m}}, S, Q, v) \) as follows:

\[
Y_{\text{m}} := \{ y \subseteq L | y \text{ is a maximal } \text{L}-\text{consistent theory} \}
\]

\( S \) is the topology on \( Y_{\text{m}} \) with basic open sets \( \{ V_\psi | \psi \in L \} \)

where \( V_\psi := \{ y \in Y_{\text{m}} | \Box \psi \in y \} \)

\( Q : Y_{\text{m}} \rightarrow Y_{\text{m}} \)

\( Q(y) := \{ y' \in Y_{\text{m}} | \{ \Box \psi | \psi \in y' \} \subseteq y \text{ and } \{ \Diamond \psi | \psi \in y \} \subseteq y' \} \)

\( v : AP \rightarrow Y_{\text{m}} \)

\( v(p) := \{ y \in Y_{\text{m}} | p \in y \} \)

**Claim 1:** \( \# \) \( S \) is a non-Alexandroff topology; it is compact and dense-in-itself; and it is \( \Box \)-manifest in the model \( \mathcal{M} \).

**Proof of Claim 1:** We prove directly that the topology is nowhere Alexandroff. Fix \( y \in Y_{\text{m}} \)
and suppose, for a contradiction, that there is an open set \( W \in S \) such that \( W = \bigcap \{ V \in S | y \in V \} \). Since every open set is a union of basis sets, and \( W \) is the smallest open set containing \( y \), we must have \( W = V_\psi \) for some single formula \( \Box \psi \in y \). Let \( P \subseteq AP \) be the infinite set of atomic propositions that are not subformulae of \( \psi \). Since \( P \) is infinite and \( y \) is maximal consistent, we can find one (and in fact infinitely many) atomic formula \( p_\ast \in P \) such that \( \Box p_\ast \in y \), and hence \( y \in V_{\Box p_\ast} \). But now we have \( y \in V_{\Box (\psi \land p_\ast)} = V_\psi \cap V_{\Box p_\ast} \). Since \( p_\ast \) does not occur in \( \psi \), \( V_{\Box p_\ast} - V_\psi \neq \emptyset \), so \( V_{\Box (\psi \land p_\ast)} \) is an open set containing \( y \) that is smaller than \( V_\psi \), giving a contradiction. This topology is given in Aiello’s thesis [1], §3.2.1, in a completeness proof for S4, where it is shown to be the intersection of the usual Alexandroff topology and the Stone topology which has as a basis the collection of sets \( V_\psi \) for all formulae. The compactness and dense-in-itself properties are proved in [1], §3.2.1. The property of the topology \( S \) being \( \Box \)-manifest in the model \( \mathcal{M} \) is immediate from the definition.

**Claim 2:** \( Q \) and \( Q^{-1} \) are l.s.c. with respect to \( S \), and possess the relational properties prescribed by the axiom schema in \( \Gamma \).

**Proof of Claim 2:** Since the existential pre-images distribute over unions, it suffices to show of each basic open set that \( Q^{-3}(V_\psi) \) and \( Q^3(V_\psi) \) are open in \( S \). Now,

\[
Q^3(V_\psi) = \{ y \in Y_{\text{m}} | \exists y' \in Y_{\text{m}} : \Box \psi \in y' \wedge \{ \Diamond \phi | \phi \in y' \} \subseteq y \wedge \{ \Diamond \phi | \phi \in y \} \subseteq y' \}
\]

\[
Q^{-3}(V_\psi) = \{ y \in Y_{\text{m}} | \exists y' \in Y_{\text{m}} : \Box \psi \in y' \wedge \{ \Diamond \phi | \phi \in y' \} \subseteq y \wedge \{ \Diamond \phi | \phi \in y \} \subseteq y' \}
\]

Using the characteristic formulae for l.s.c. relations in Proposition 4.5, together with the \( \text{L} \) theorems \( \Diamond \phi \rightarrow \Diamond \Diamond \phi \) and \( \Diamond \Diamond \Box \psi \leftrightarrow \Box \Diamond \Diamond \Box \psi \), and their \( \Diamond \)
mirror images, it can be shown that:

$$Q^\square_{1}(V_{\square \psi}) = V_{\square \square \psi} \quad \text{and} \quad Q^{\square}_{1}(V_{\square \psi}) = V_{\square \square \psi}$$  \hspace{1cm} (12)

Relational properties preserved by further schema in $\Gamma$ can be verified in the usual way.

Claim 3 ("Truth Lemma"): \( y \in [[\psi]]^{\alpha \varepsilon} \square \) iff \( \psi \in \mathcal{E}_{\varepsilon} \) and \( y \in Y_{\varepsilon m} \).
Hence \( \psi \) is a theorem of \( \mathbf{K}^{\varepsilon \mathbf{L} \mathbf{S} \mathbf{C}} \oplus \Gamma \) iff \( [[\psi]]^{\alpha \varepsilon} \square = Y_{\varepsilon m} \).

Proof of Claim 3: The only case departing from standard for classical Kripke semantics is that for \( \square \), which requires \( y \in \text{int}_{\varepsilon}([[\psi]]^{\alpha \varepsilon} \square) \) iff \( \square \psi \in y \). The verification is straightforward, and can be found in [1], Lemma 3.2.4. This completes the proof for Part (1).

We now turn to the intuitionistic logic, and for Part (2), define a model \( \mathcal{M}_{\varepsilon} = (X_{\varepsilon p}, T_{\varepsilon}, R_{\varepsilon}, u_{\varepsilon}) \) as follows:

\[
X_{\varepsilon p} := \{ x \subseteq \mathcal{L}_{t} \mid x \text{ is a prime IL-theory} \}
\]

\( T_{\varepsilon} \) is the topology on \( X_{\varepsilon p} \) with basic open sets \( \{ U_{\varphi} \mid \varphi \in \mathcal{L}_{t} \} \)
where \( U_{\varphi} := \{ x \in X_{\varepsilon p} \mid \varphi \in x \} \)

\( R_{\varepsilon} : X_{\varepsilon p} \leadsto X_{\varepsilon p} \quad R_{\varepsilon} := R_{0} \) i.e. \( x R_{\varepsilon} x' \) iff
\[
\begin{align*}
\{ \Diamond \psi \mid \psi \in x' \} & \subseteq x \\
\{ \psi \mid \Box \psi \in x \} & \subseteq x' \quad \text{and}
\end{align*}
\]

\( \{ \Diamond \psi \mid \psi \in x \} \subseteq x' \) and \( \{ \psi \mid \Box \psi \in x' \} \subseteq x\)

\( u_{\varepsilon} : AP \leadsto X_{\varepsilon p} \quad u_{\varepsilon}(p) := U_{p} \)

Claim 4: \( T_{\varepsilon} \) is a non-Alexandroff topology; it is \( T_{0} \), dense-in-itself, and compact; and it is manifest in the model \( \mathcal{M}_{\varepsilon} \).

Proof of Claim 4: We first examine the specialization pre-order: \( x \preceq_{\varepsilon T_{\varepsilon}} x' \) iff for all formulae \( \varphi \in \mathcal{L}_{t}, \varphi \in x \Rightarrow \varphi \in x' \). Hence \( x \preceq_{\varepsilon T_{\varepsilon}} x' \) iff \( x \subseteq x' \). So \( T_{\varepsilon} \) has the same specialization pre-order as the "default" Alexandroff topology, and each basic open \( U_{\varphi} \) of \( T_{\varepsilon} \) is upwards-\( \subseteq \)-closed, so \( T_{\varepsilon} \subseteq T_{\varepsilon} \). But \( T_{\varepsilon} \neq T_{\varepsilon} \) because there exist upwards-\( \subseteq \)-closed sets that are not open in \( T_{\varepsilon} \). For one such, let \( x_{0} \in X_{\varepsilon p} \) be any prime theory that is \( \subseteq \)-maximal (i.e. no proper superset of \( x_{0} \) is a prime theory of \( \mathbf{IL} \)). Then \( W = \{ x_{0} \} \) is trivially upwards-\( \subseteq \)-closed, but there is no basic open \( U_{\varphi} \) of \( T_{\varepsilon} \) such that \( U_{\varphi} \subseteq W \). The fact that \( T_{\varepsilon} \) is \( T_{0} \) follows from its specialization pre-order \( \preceq_{\varepsilon T_{\varepsilon}} = \subseteq \) being a partial order. For the dense-in-itself property, it suffices to show that \( \text{int}_{\varepsilon T_{\varepsilon}}(\{ x \}) = \emptyset \) for all \( x \in X_{\varepsilon p} \). This is immediate, since \( \text{int}_{\varepsilon T_{\varepsilon}}(\{ x \}) = \bigcup \{ U_{\varphi} \mid U_{\varphi} \subseteq \{ x \} \} \), and there are no basic opens \( U_{\varphi} \) such that \( U_{\varphi} \subseteq \{ x \} \). It is also immediate that \( T_{\varepsilon} \) is manifest in the model \( \mathcal{M}_{\varepsilon} \). The compactness of \( T_{\varepsilon} \) will be proved in the next section, as Part (5) of Theorem 7.2.

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Claim 5: \( R_\varepsilon^{-3}(U, \varphi) = U \boxdot \varphi \) and \( R_\varepsilon^{3}(U, \varphi) = U \boxdot \varphi \) for all \( \varphi \in \mathcal{L}^k \), hence \( R_* \) and \( R_*^{-1} \) are \( \Gamma \)-c.e. with respect to \( T_* \); moreover, they possess any relational properties prescribed by schema in \( \Gamma \).

Proof of Claim 5: Fix \( \varphi \in \mathcal{L}^k \), and suppose \( x \in R_*^{-3}(U, \varphi) \). So there exists \( x' \in X_{ip} \) such that \( x R_* x' \) and \( \varphi \in x' \). Then we must have \( \Box \varphi \in x \), and hence \( x \subseteq U \Box \varphi \). Thus \( R_*^{-3}(U, \varphi) \subseteq U \Box \varphi \).

The converse implication takes some effort, and is proved in a sequence of four sub-claims. Fix \( x \in U \Box \varphi \), so we have \( \Box \varphi \in x \). We need to exhibit an \( x' \in X_{ip} \) such that \( x R_* x' \) and \( \varphi \in x' \).

First, define:

\[
s' := \left\{ \psi \in \mathcal{L}^k \mid \Box \varphi \in x \right\} \cup \left\{ \Diamond \psi \in \mathcal{L}^k \mid \psi \in x \right\}
\]

Sub-claim 5.1: \( s' \) is \( \Box I L \)-consistent.

Proof of Sub-claim 5.1: Suppose, for a contradiction, that \( s' \) is not \( \Box I L \)-consistent. So there exist \( \psi_1, \ldots, \psi_n, \Diamond \chi_1, \ldots, \Diamond \chi_m \in s' \) such that \( \Box \psi_i \in x \) for \( 1 \leq i \leq n \) and \( \chi_j \in x \) for \( 1 \leq j \leq m \), and \( (\psi_1 \land \cdots \land \psi_n \land \Diamond \chi_1 \land \cdots \land \Diamond \chi_m) \rightarrow \bot \) is a theorem of \( I L \). Now since \( x \) is \( I L \)-deductively closed, we have \( \Box \psi \in x \) where \( \psi := \psi_1 \land \cdots \land \psi_n \), and \( \chi \in x \) where \( \chi := \chi_1 \land \cdots \land \chi_m \). Since \( \Diamond \chi \rightarrow (\Diamond \chi_1 \land \cdots \land \Diamond \chi_m) \) is an \( I L \) theorem, we can conclude that \( (\psi \land \Box \chi) \rightarrow \bot \) is a theorem.

This in turn implies that \( (\Box \psi \land \Box \Diamond \chi) \rightarrow \Box \bot \) is a theorem. Applying the adjoint axiom \( \text{Ad}1 \), we get as a theorem \( (\Box \psi \land \chi) \rightarrow (\Box \psi \land \Box \Diamond \chi) \); taken together with the \( I L \) theorem \( \Box \bot \leftrightarrow \Box \bot \), we can then conclude that \( (\Box \psi \land \chi) \rightarrow \Box \bot \) must be a theorem. Now with \( \Box \psi \in x \) and \( \chi \in x \) and the \( I L \)-deductive closure of \( x \), we get \( \Box \bot \in x \). But \( \Box \varphi \in x \) and \( \Box \varphi \rightarrow \Box \bot \) is a theorem, hence \( \Box \bot \in x \), and we have a contradiction. Hence \( s' \) is \( I L \)-consistent.

Sub-claim 5.2: Defining \( z' := s' \cup \{ \varphi \} \), the set \( z' \) is \( I L \)-consistent.

Proof of Sub-claim 5.2: Suppose, for a contradiction, that \( z' \) is not \( I L \)-consistent. Since \( s' \) is \( I L \)-consistent, the only possibility for the inconsistency in \( z' \) is that it comes from \( \varphi \), meaning there is a \( \Box \psi \in x \) and a \( \chi \in x \) such that \( (\psi \land \Box \chi) \rightarrow \Box \bot \) is an \( I L \) theorem. Applying the adjoint axiom \( \text{Ad}1 \), we get as a theorem \( (\Box \psi \land \chi) \rightarrow (\Box \psi \land \Box \Diamond \chi) \); hence we can then conclude that \( (\Box \psi \land \chi) \rightarrow \Box \Diamond \varphi \) is an \( I L \) theorem. Now the intuitionistic \( \Box \) and \( \Diamond \) are related by a “semi-duality” theorem \( \Box \Diamond \varphi \leftrightarrow \Diamond \Box \varphi \) of \( I L \) [28], hence we get that \( (\Box \psi \land \chi) \rightarrow \Box \Diamond \varphi \) is a theorem. Now with \( \Box \psi \in x \) and \( \chi \in x \), we get \( \Diamond \varphi \in x \), contradicting the assumption that \( \Diamond \varphi \in x \). Hence \( z' \) is \( I L \)-consistent.

Sub-claim 5.3: Extending the relation \( R_* \) to \( I L \)-consistent sets of formulae, we have \( x R_* z' \).

Proof of Sub-claim 5.3: We need to show, for all \( \psi \in \mathcal{L}^k \), that: (i) \( \psi \in z' \Rightarrow \Diamond \psi \in x \); (ii) \( \Box \psi \in x \Rightarrow \Diamond \psi \in z' \); (iii) \( \psi \in x \Rightarrow \Diamond \psi \in z' \); and (iv) \( \Box \psi \in z' \Rightarrow \psi \in x \). Conditions (ii) and (iii) are immediate from the construction of \( z' \) and \( s' \). For condition (i), if \( \psi \in z' \), then either (a) \( \Box \psi \in x \); or (b) \( \psi = \Diamond \chi \) and \( \chi \in x \); or (c) \( \psi = \varphi \). In case (a), we appeal to the theorem.
(□ψ ∧ ◇φ) → ◇ψ to conclude that ◇ψ ∈ x. In case (b), we use the theorem (χ ∧ ◇φ) → ◇χ to conclude that ◇χ ∈ x, and hence ◇ψ ∈ x. For the last case (c), we have ψ = φ ∈ z', and by assumption, ◇ψ = ◇φ ∈ x, as required. For condition (iv), if □φ ∈ z', then either (d) □□ψ ∈ x, or (e) □ψ = φ. In case (d), we appeal to the theorem (□□ψ ∧ ◇φ) → ◇□ψ to conclude that ◇□ψ ∈ x, and hence by the adjoint axiom Ad3, we have ψ ∈ x. In case (e), we have ◇φ ∈ x, which means ◇□ψ ∈ x; again by axiom Ad3, we have ψ ∈ x.

**Sub-claim 5.4:** There exists x' ∈ X irresponsible such that x' ≤ x' and x R∗ x'; hence x ∈ R∗3(U φ).

**Proof of Sub-claim 5.4:** We construct a sequence of IL-consistent sets z'n for n ∈ N as follows:

\[
\begin{align*}
n & = 0 \\
n + 1 = 2k + 1 & : z_{n+1}' := z_n' \cup \{ψ ∈ L^k \mid ◇ψ ∈ x \text{ and } ∃χ ∈ L^k : (ψ ∨ χ) ∈ z_n' \text{ or } (χ ∨ ψ) ∈ z_n'\} \\
n + 1 = 2k + 2 & : z_{n+1}' := z_n' \cup \{ψ ∈ L^k \mid ∃χ ∈ L^k : ⊢_{IL} χ → ψ \text{ and } χ ∈ z_n'\}
\end{align*}
\]

In words, the disjunction property is attended to the odd stages 2k + 1 = n + 1 of the construction, while deductive closure is dealt with during the even stages 2k + 2 = n + 1. We then take x' := ∪n∈N z'n, so that x' will have the disjunction property and be IL-deductively closed, hence x' ∈ X irresponsible. Finally, one verifies that x R∗ x' by establishing each the four clauses:

(i) ψ ∈ x' ⇒ ◇ψ ∈ x; (ii) □φ ∈ x ⇒ ψ ∈ x'; (iii) ψ ∈ x ⇒ ◇ψ ∈ x'; and (iv) □ψ ∈ x' ⇒ ψ ∈ x. Clauses (ii) and (iii) are already satisfied by z'0 ∈ x'. So we prove by induction on the stage n that: (i)n ψ ∈ z'n ⇒ ◇ψ ∈ x, and (iv)n □ψ ∈ z'n ⇒ ψ ∈ x.

The base case for n = 0 is Sub-claim 5.3. Assume the result holds for n, and consider first the case where n + 1 is odd. To prove (i)n+1, suppose ψ ∈ z'n+1 − z'n. Then we have ◇ψ ∈ x, so we are done. To prove (iv)n+1, suppose □φ ∈ z'n+1 − z'n. Then we have □□ψ ∈ x, and by the adjoint axiom Ad3, we can conclude that ψ ∈ x, and we are done. Now consider the case where n + 1 is even. To prove (i)n+1, suppose ψ ∈ z'n+1 − z'n. Then there is a χ ∈ z'n such that ⊢_{IL} χ → ψ. By the induction hypothesis, (i)n holds, so ◇χ ∈ x. By the monotonicity rule, we get ⊢_{IL} ◇χ → ◇ψ, hence we will have ◇ψ ∈ x, as required. To prove (iv)n+1, suppose □ψ ∈ z'n+1 − z'n. Then there is a χ ∈ z'n such that ⊢_{IL} χ → □ψ, and by the monotonicity rule, we get ⊢_{IL} ◇χ → □ψ. By the induction hypothesis, (i)n holds, so χ ∈ z'n implies ◇χ ∈ x, and hence ◇□ψ ∈ x. Applying the adjoint axiom Ad3, we can conclude that ψ ∈ x, and we are done. Thus we have established that x R∗ x'.

The proof of the equality R∗3(U φ) = U φ is the mirror image of the proof for ◇. The verification that R∗ possesses any relational properties prescribed by axiom schema Γ in given in [15] for the modal logics IT ⊕ φ, ITB ⊕ φ, IS4 ⊕ φ and IS5 ⊕ φ. The remaining combinations, and the lifting to the tense language with properties of R∗, are straightforward.
Claim 6 ("Truth Lemma"): \( \llbracket \varphi \rrbracket_{T^t} = U_{\varphi} \) for all \( \varphi \in \mathcal{L}^t \).

Hence \( \varphi \) is a theorem of \( \mathbf{IL} = \mathbf{IK}^t \oplus \Gamma \) iff \( \llbracket \varphi \rrbracket_{T^t} = X_{\text{fp}} \).

Proof of Claim 6: The base case for atomic propositions \( p \in AP \) holds by definition of the valuation \( u_* \) in \( \mathcal{M}_* \). In Claim 5, we have established the inductive step for \( \otimes \) and \( \diamond \) with \( \llbracket \otimes \varphi \rrbracket_{T^t} = R^{-\gamma}_{\varphi}(U_{\varphi}) = U_{\otimes \varphi} \) and \( \llbracket \diamond \varphi \rrbracket_{T^t} = R^t_{\varphi}(U_{\varphi}) = U_{\diamond \varphi} \). We depart from Fischer Servi’s completeness proof in [15] on the topology/pre-order, so we give the case for \( \Box \). First, from the semantic definition plus the induction hypothesis, we have \( \llbracket \Box \varphi \rrbracket_{T^t} = \text{int}_{\tau_t} \left( R^{-\gamma}_{\varphi}(\llbracket \varphi \rrbracket_{T^t}) \right) = \text{int}_{\tau_t} \left( R_{\varphi}(U_{\varphi}) \right) \). Hence \( \llbracket \Box \varphi \rrbracket_{T^t} \subseteq \text{int}_{\tau_t} \left( U_{\Box \varphi} \right) = U_{\Box \varphi} \). For the other direction, first observe that for any formulae \( \psi, \psi' \in \mathcal{L}^t \), we have \( U_{\psi} \subseteq U_{\psi'} \) iff \( \vdash_{\mathbf{IL}} \psi \to \psi' \), since \( \mathbf{IL} \) is the intersection of all its prime theories. We thus have the following chain of equivalences: \( U_{\psi} \subseteq R_{\varphi}(U_{\psi}) \) iff \( R^t_{\varphi}(U_{\psi}) \subseteq U_{\psi} \) iff \( U_{\Box \psi} \subseteq U_{\psi} \) iff \( \vdash_{\mathbf{IL}} \Box \psi \to \psi \). Since \( \mathbf{IL} \) is the intersection of all its prime theories, we can conclude that \( U_{\Box \varphi} \subseteq R_{\varphi}(U_{\varphi}) \), and hence \( U_{\Box \varphi} \subseteq \text{int}_{\tau_t} \left( R_{\varphi}(U_{\varphi}) \right) = \llbracket \Box \varphi \rrbracket_{T^t} \). By a mirror argument, we get \( \llbracket \Box \varphi \rrbracket_{T^t} = \text{int}_{\tau_t} \left( R_t^t(U_{\varphi}) \right) = \text{int}_{\tau_t} \left( U_{\Box \varphi} \right) = U_{\Box \varphi} \).

7 An extended Gödel translation, and topological bisimulations

The Fischer Servi extension of the Gödel translation [13, 15, 14], extends to a function \( \text{GT} : \mathcal{L}^t \to \mathcal{L}_B^t \) defined by the clauses in Assertion (4) plus:

\[
\begin{align*}
\text{GT}(\otimes \varphi) & := \otimes \text{GT}(\varphi) & \text{GT}(\Box \varphi) & := \Box \text{GT}(\varphi) \\
\text{GT}(\Box \varphi) & := \Box \text{GT}(\varphi) & \text{GT}(\Box \varphi) & := \Box \text{GT}(\varphi)
\end{align*}
\]

(13)

Directly from the semantic clauses in Definitions 4.3 and 4.4, we can see that for all models \( \mathcal{M} = (\mathcal{F}, v) \) over an l.s.c. topological frame \( \mathcal{F} \), if \( \mathcal{M}' = (\mathcal{F}, v') \) is the variant open model with valuation \( v'(p) := \text{int}_{\tau}(v(p)) \), then for all formulae \( \varphi \in \mathcal{L}^t \),

\[
\llbracket \varphi \rrbracket_{T^t} = \llbracket \text{GT}(\varphi) \rrbracket_{T^t}
\]

(14)

Consequently, we have semantic faithfulness: \( \varphi \in \mathbb{K}^t \mathcal{T} \) iff \( \text{GT}(\varphi) \in \mathbb{K}^t \mathcal{LST} \); and the openness property: \( \text{GT}(\varphi) \leftrightarrow \Box \text{GT}(\varphi) \in \mathbb{K}^t \mathcal{LST} \).

Proposition 7.1 [Faithfulness of extended Gödel translation]

Let \( \Gamma \) be any finite set of \( \mathcal{L}^t \) from the list in (10) above. Then for all \( \varphi \in \mathcal{L}^t \),

(1) \( \varphi \) is a theorem of \( \mathbf{IK}^t \oplus \Gamma \) iff \( \text{GT}(\varphi) \) is a theorem of \( \mathbf{K}^t \mathcal{LSC} \oplus \Gamma \).
\(2\text{.) } \text{GT}(\varphi) \leftrightarrow \Box \text{GT}(\varphi) \text{ is a theorem of } K^4 \text{LSC}.\)

**Proof.** Since we already have semantic faithfulness explicitly in Assertion (14), we can use soundness and completeness to prove the deductive faithfulness of the translation. \(\dagger\)

The result can also be derived from a much more general result for (an equivalent) Gödel translation given in [35], Theorem 8, on the faithful embedding of modal logics \(L = \text{IntK} \oplus \Gamma_1\) (which includes \(\text{IK} \oplus \Gamma = \text{IntK} \oplus \text{F1} \oplus \text{F2} \oplus \Gamma\) into bi-modal logics in the interval between \((\text{S4}_4 \oplus \text{K}^4) \oplus \text{GT}(\Gamma_1)\) and \((\text{Grz}_4 \oplus \text{K}^4) \oplus \text{GT}(\Gamma_1) \oplus \text{mix}, \) where \(\text{Grz}_4 = \text{S4}_4 \oplus \Box(\Box \varphi \to \Box \varphi) \to \varphi\) and \(\text{mix} = (\Box \Box \varphi \leftrightarrow \Box \varphi) \land (\Box \Box \varphi \leftrightarrow \Box \varphi).\) We have restricted the schema in \(\Gamma\) to those from a “safe” list of relational properties that don’t require translating, since the schema characterize the same relations in the intuitionistic and classical semantics.

The Gödel translation is a syntactic function \(\text{GT} : \mathcal{L}^4 \to \mathcal{L}^4,\) which naturally expresses a semantic relationship between the canonical model spaces \(X_{ip}\) and \(Y_{im}.\) Define a set-valued map \(G : X_{ip} \sim Y_{im}\) by:

\[
G(x) := \{ y \in Y_{im} \mid \text{GT}(x) \subseteq y \}
\]

Note that the image \(\text{GT}(x)\) of an intuitionistic prime theory \(x \in X_{ip}\) will in general have many classical maximal consistent extensions \(y \in Y_{im}.\)

**Theorem 7.2** [Topological properties of Gödel map and canonical models]

Let \(\mathcal{M}_* = (X_{ip}, T_*, R_*, u_*)\) and \(\mathcal{M}_\square = (Y_{im}, S_{im}, Q_{im}, v_{im})\) be the canonical models for the intuitionistic logic \(\text{IL} := \text{IK} \oplus \Gamma\) and the classical logic \(\text{IL} := K^4 \text{LSC} \oplus \Gamma\), respectively, from the proof of Theorem 6.1. Let \(\mathcal{M}_\square^G = (Y_{im}, S_{im}, Q_{im}, v_{im})\) be the open and l.s.c. model obtained from \(\mathcal{M}_\square\) by taking \(S_{im}^\square\) to be the proper sub-topology of \(S_{im}\) which has as a basis the open sets \(\{ V_{\Box \text{GT}(\varphi)} \mid \varphi \in \mathcal{L}^4 \}\), and by taking the valuation \(v_{im}^G(p) := \text{int}_{S_{im}^G}(v_{im}(p)) = V_{\Box p}\).

Then the maps \(G : X_{ip} \sim Y_{im}\) and \(G^{-1} : Y_{im} \sim X_{ip}\) have the following properties:

1. both \(G\) and \(G^{-1}\) are l.s.c. with respect to \(T\) and \(S_{im}^G ;\)

2. they satisfy the relational equalities: \(R_\square \bullet G = G \bullet Q_{im}\) and \(Q_{im} \bullet G^{-1} = G^{-1} \bullet R_\square ;\)

3. \(G^3(u_*(p)) \subseteq v_*(p)\) and \(G^{-3}(v_*(p)) \subseteq u_*(p)\) for all atomic propositions \(p \in AP.\)

Hence \(G\) is a tense topo-bisimulation between the two open and l.s.c. models \(\mathcal{M}_*\) and \(\mathcal{M}_\square^G.\)

Furthermore:

\(M_{\square}^G\) will be an l.s.c. model, as \(Q_{im}^G\) and \(Q^{-1}_{im}\) will still be l.s.c. w.r.t. the sub-topology \(S_{im}^G ;\) using equation (12) and \(\text{GT}(\varphi) \leftrightarrow \Box \text{GT}(\varphi),\) we have \(Q_{im}^G(V_{\Box \text{GT}(\varphi)}) = V_{\Box \text{GT}(\varphi)}\) and \(Q^{-1}_{im}(V_{\Box \text{GT}(\varphi)}) = V_{\Box \text{GT}(\varphi)}.\)
(4.) both maps $G$ and $G^{-1}$ are total and surjective; and

(5.) the topologies $T_\ast$ on $X_{ip}$, and $S^{\ast}_{\Pi}$ on $Y_{im}$, are both compact, with the compactness of the first 
derivable from the second using $G$ and $G^{-1}$.

**Proof.** For Part (1.), we need only look at the basic opens in $T_\ast$ and $S^{\ast}_{\Pi}$. Using the equivalence
\[
\square GT(\varphi) \leftrightarrow GT(\varphi)
\]
from Proposition 7.1, it is readily established that for all $\varphi \in \mathcal{L}^k$:
\[
G^{-3}(V_{\square GT(\varphi)}) = U_\varphi \quad \text{and} \quad G^3(U_\varphi) = V_{\square GT(\varphi)} \quad (15)
\]

For Part (2.), we begin with a claim that for all formulae $\varphi \in \mathcal{L}^k$:
\[
\begin{align*}
(R_\ast \bullet G)^{-3}(V_{\square GT(\varphi)}) &= U_{\varphi} \quad \Rightarrow \quad (G \bullet Q_\Pi)^{-3}(V_{\square GT(\varphi)}) \\
(Q_\Pi \bullet G^{-1})^3(V_{\square GT(\varphi)}) &= U_{\varphi} \quad \Rightarrow \quad (G^{-1} \bullet R_\ast)^3(V_{\square GT(\varphi)}) \\
(R_\ast \bullet G)^3(U_\varphi) &= V_{\square GT(\varphi)} \quad \Rightarrow \quad (G \bullet Q_\Pi)^3(U_\varphi) \\
(Q_\Pi \bullet G^{-1})^{-3}(U_\varphi) &= V_{\square GT(\varphi)} \quad \Rightarrow \quad (G^{-1} \bullet R_\ast)^{-3}(U_\varphi)
\end{align*}
\]  

These equations are simply verified by appeal to Assertion (15) just above for $G$, together with results established in the completeness proof for Theorem 6.1: Assertion (12) for $Q_\Pi$ in the proof of Claim 2, and Claim 5 for $R_\ast$. To complete the proof of Part (2.), define two relations between the canonical model spaces, $H_1 : X_{ip} \hookrightarrow Y_{im}$ and $H_2 : Y_{im} \hookrightarrow X_{ip}$, as follows:

\[
x \ H_1 \ y \iff (\forall \varphi \in \mathcal{L}^k)[ \quad \varphi \in x \Rightarrow \square GT(\varphi) \in y \quad \land \quad ]
\]

\[
y \ H_2 \ x \iff (\forall \varphi \in \mathcal{L}^k)[ \quad \varphi \in x \Rightarrow \square GT(\varphi) \in y \quad \land \quad ]
\]

Using Assertions (16) plus the definitions of the maps, it is straightforward albeit tedious to verify that $(R_\ast \bullet G) = H_1 = (G \bullet Q_\Pi)$ and $(Q_\Pi \bullet G^{-1}) = H_2 = (G^{-1} \bullet R_\ast)$.

For the first two equalities, we start with the inclusions $(R_\ast \bullet G) \subseteq H_1$ and $(G \bullet Q_\Pi) \subseteq H_1$. For the first, suppose that $x(R_\ast \bullet G) y$, and let $\varphi \in \mathcal{L}^k$ be arbitrary. Then appealing to Assertions (16) for the $(R_\ast \bullet G)^3$ and $(R_\ast \bullet G)^{-3}$ operators, plus the occasional adjoint axiom, we proceed through the four clauses of the definition of the $H_1$ relation. (i) If $\varphi \in x$ then $x \in U_\varphi$, hence by (16),
y \in V_{\square \text{GT}(\varphi)}$, and so $\square \text{GT}(\varphi) \in y$.  (ii) If $\square \varphi \in x$ then $x \in U_{\square \varphi}$, hence by (16), $y \in V_{\square \text{GT}(\Diamond \square \varphi)}$, and hence $\square \text{GT}(\Diamond \square \varphi) \in y$, and thus by axiom Ad4, we have $\square \text{GT}(\varphi) \in y$.  (iii) If $\square \text{GT}(\varphi) \in y$ then $\Diamond \varphi \in x$, by (16).  (iv) If $\square \text{GT}(\Diamond \varphi) \in y$, then by (16), we have $\Diamond \Diamond \varphi \in x$, and hence by axiom Ad3, we have $\varphi \in x$.  Thus we have $xH_1 y$, as required.  The proof of the second inclusion repeats verbatim that for the first, with the appeal to Assertions (16) this time for the $(G \cdot Q_{\square})^{-3}$ and $(G \bullet Q_{\square})^{-3}$ operators.

For the first converse, suppose $xH_1 y$, and suppose, for a contradiction, that not $x(R_* \bullet G)y$.  So for every $x' \in X_{ip}$ such that $\text{GT}(x') \subseteq y$, we have $x' \notin R_*(x)$.  So fix $x' \in G^{-1}(y)$; this set is not empty, as we will establish in Part (4.) below (without circularity).  Define $X_{ip}^{m} := \{x_0 \in X_{ip} \mid (\forall x \in X_{ip}) \ x_0 \notin x \}$ to be the (proper) subset of prime IL theories that are $\subseteq$-maximal.  Then there exists $x_0 \in X_{ip}^{m}$ such that $x' \subseteq x_0$ and $\text{GT}(x_0) \subseteq y$, and hence $x_0 \notin R_*(x)$.  We consider in turn the four clauses in the definition of $R_*$, and for each derive a contradiction with $xH_1 y$.  For clause (i), there exists a formula $\psi$ such that $\psi \in x_0$ and $\Diamond \psi \notin x$.  The first of these implies $\square \text{GT}(\psi) \in y$, and hence by the third clause of $H_1$, we have $\Diamond \psi \in x$; contradiction.  For clause (ii), we have $\psi \in x$ and $\Diamond \psi \notin x_0$.  The second of these implies $\square \text{GT}(\Diamond \psi) \notin y$, and hence by the contrapositive of the first clause of $H_1$, we get a contradiction from $\psi \notin x$.  For clause (iii), we have $\square \psi \in x$ and $\psi \notin x_0$.  Now the first of these implies $\square \text{GT}(\psi) \in y$, by the second clause of $H_1$, and then we can conclude $\psi \in x_0$, giving a contradiction.  For clause (iv), we have $\Diamond \psi \in x_0$ and $\psi \notin x$.  The first of these implies $\square \text{GT}(\Diamond \psi) \in y$, and hence $\psi \in x$, by the fourth clause of $H_1$, which contradicts $\psi \notin x$.  Hence we have established that $H_1 \subseteq (R_* \bullet G)$.

For the second converse, $H_1 \subseteq (G \bullet Q_{\square})$, suppose not $x(G \bullet Q_{\square})y$.  So for every $y_2 \in Q_{\square}^{-1}(y)$, there exists a formula $\varphi_2 \in x$ such that $\text{GT}(\varphi_2) \notin y_2$, and hence $\neg \text{GT}(\varphi_2) \in y_2$, since these are classical maximal consistent theories.  Then $\Diamond \neg \text{GT}(\varphi_2) \in y$, by the definition of $Q_{\square}$, and hence $\neg \Diamond \text{GT}(\varphi_2) \in y$.  Thus $\Diamond \text{GT}(\varphi_2) \notin y$, and hence $\square \text{GT}(\Diamond \varphi_2) \notin y$.  This last fact combined with $\varphi_2 \in x$ yields the conclusion that not $xH_1 y$.

The four inclusions for $(Q_{\square} \bullet G^{-1}) = H_2 = (G^{-1} \bullet R_*)$ are proved by “mirror” arguments, uniformly inter-changing $\Diamond$ and $\square$ with $\Diamond$ and $\square$.

Part (3.) transcribes as the inclusions $G^2(U_p) \subseteq V_{\square p}$ and $G^{-3}(V_{\square p}) \subseteq U_p$, which are immediate since we have established equalities in the proof of Part (1.).  The conjunction of Parts (1.), (2.) and (3.) is the definition of $G$ being a tense topo-bisimulation.

For Part (4.), the totality of $G$ (equivalently, the surjectivity of $G^{-1}$) is immediate, since every prime theory $x \in X_{ip}$ is $\text{IL}$-consistent, hence the image $\text{GT}(x) \subseteq L_{\square}$ is $\text{L}_{\square}$-consistent, and so has a maximal $\text{L}_{\square}$-consistent superset $y \supseteq \text{GT}(x)$ with $y \in Y_{\text{im}}$, by Lindenbaum’s Lemma.  For the surjectivity of $G$ (equivalently, the totality of $G^{-1}$), define the (proper) subset of the language.
\( \mathcal{L}^\exists, \mathcal{G}^\exists := \{ \psi \in \mathcal{L}^\exists_i \mid (\exists \varphi \in \mathcal{L}^\exists) \left[ \mathcal{F}^{\mathcal{L}^\exists_i} \psi \leftrightarrow \text{GT}(\varphi) \right] \}, \) consisting of all formulae \( \mathcal{L}^\exists_i \)-equivalent to the image under the GT translation of some \( \exists \)-free formula in \( \mathcal{L}^\exists_i \). Now for any maximal \( \mathcal{L}^\exists \)-consistent theory \( y \in Y^\exists \), define the subset \( y^\exists := y \cap \mathcal{G}^\exists \). Let \( X^\exists_{ip} \) be the subset of prime \( \mathcal{II} \) theories that are \( \subseteq \)-maximal, defined above in the proof of Part (2). Then every \( x_0 \in X^\exists_{ip} \) is a maximal \( \mathcal{II} \)-consistent theory, and is also a classical \( \mathcal{L}^\exists \)-consistent theory that is maximal within the \( \exists \)-free language \( \mathcal{L}^\exists \). So by the deductive faithfulness of the Gödel translation, for every \( y \in Y^\exists \), there is a maximal \( x_0 \in X^\exists_{ip} \) such that \( \text{GT}(x_0) = y^\exists \), and hence \( \text{GT}(x_0) \subseteq y \). Hence \( G \) is surjective.

For Part (5.), the compactness of the topology \( \mathcal{S}^\exists_i \) on \( Y^\exists \) is immediate, because \( \mathcal{S}^\exists_i \) is a sub-topology of a compact topology \( \mathcal{S}_{\mathcal{I}} \). (Any cover drawn from open sets in \( \mathcal{S}^\exists_i \) will also be a cover in \( \mathcal{S}_{\mathcal{I}} \), so it has a finite sub-cover.) To show that the topology \( \mathcal{T}_x \) on \( X_{ip} \) is compact, let \( \mathcal{C} \) be any family of open sets in \( \mathcal{T}_x \) whose union is all of \( X_{ip} \). We may assume each of the sets in \( \mathcal{C} \) is basic open sets, so there exists a set \( C_0 \subseteq \mathcal{L}^\exists \) of formulae such that \( \mathcal{C} = \{ U_\varphi \mid \varphi \in C_0 \} \). Now by the surjectivity of \( G \) and distribution over unions of existential pre/post-image operators, we have

\[
Y^\exists = G^3(x_{ip}) = \bigcup_{\varphi \in C_0} G^2(U_\varphi) = \bigcup_{\varphi \in C_0} V^{\text{GT}(\varphi)}.
\]

Since the topology \( \mathcal{S}^\exists_i \) on \( Y^\exists \) is compact, there is a finite subset \( F \subseteq C_0 \) such that \( Y^\exists = \bigcup_{\varphi \in F} V^{\text{GT}(\varphi)} \). But then by the totality of \( G \), we have \( X_{ip} = G^{-3}(Y^\exists) = \bigcup_{\varphi \in F} G^{-2}(V^{\text{GT}(\varphi)}) = \bigcup_{\varphi \in F} U_\varphi \). Hence the family of opens \( \mathcal{C} \) has a finite subcover, and we are done.

...
**PROOF.** Let $\mathcal{M}_i = (X_{ip}, T_i, R_i, u_*)$ be the canonical model for the logic $\mathbf{IK}^i$, and for $i \in \{1, 2\}$, define functions $B_i : X_i \rightarrow X_{ip}$, on states $w \in X_i$, by:

$$B_i(w) := \{ \varphi \in \mathcal{L}^i \mid w \in \llbracket \varphi \rrbracket_i^{\lambda_i} \}$$

It is readily established that, for every $w \in X_i$, the set of formulae $B_i(w)$ is non-empty and $\mathbb{K}^i$-semantically-prime, and hence by Theorem 6.1 it is a prime theory of $\mathbf{IK}^i$, so $B_i(w) \in X_{ip}$ and the function is well-defined. Moreover, the inverse $(B_i)^{-1} : X_{ip} \rightarrow X_i$ satisfies $(B_i)^{-1}(x) = \llbracket x \rrbracket_i^{\lambda_i}$; i.e. $(B_i)^{-1}(x)$ is the realization set in the model $\mathcal{M}_i$ for $x \subseteq \mathcal{L}^i$ as a set of formulae. Note that $B_i$ is total on $X_i$; equivalently, $(B_i)^{-1}$ is surjective onto $X_i$.

The bulk of this proof will consist in establishing that $B_i$ is a tense topo-bisimulation between the models $\mathcal{M}_i$ and $\mathcal{M}_*$. Supposing this is established, we then define $B : X_1 \sim X_2$ by $B := B_1 \bullet (B_2)^{-1}$. Then by composition, $B$ will be a tense topo-bisimulation between the models $\mathcal{M}_1$ and $\mathcal{M}_2$, and $B$ will be both total on $X_1$ and surjective on $X_2$. From the definition of the maps $B_1$ and $B$, it is immediate that:

$$w B z \text{ iff } B_1(w) = B_2(z) \text{ iff } (\forall \varphi \in \mathcal{L}^i) [w \in \llbracket \varphi \rrbracket_i^{\lambda_i+1} \iff z \in \llbracket \varphi \rrbracket_i^{\lambda_i+2}]$$

for all states $w \in X_1$ and $z \in X_2$. (The only if direction also comes from Theorem 5.2.)

So it remains to show that $B_i$ is a tense topo-bisimulation between the models $\mathcal{M}_i$ and $\mathcal{M}_*$. Evaluating pre- and post-images, directly from the definitions, we get:

$$(B_i)^{-3}(U_{\varphi}) = \llbracket \varphi \rrbracket_i^{\lambda_i} \quad \text{and} \quad (B_i)^{3}(\llbracket \varphi \rrbracket_i^{\lambda_i}) = U_{\varphi} \quad (17)$$

for all formulae $\varphi \in \mathcal{L}^i$. By assumption, the topology $T_i$ is manifest in $\mathcal{M}_i$, and we know of the canonical model that $T_*$ is manifest in $\mathcal{M}_*$. So the family of open sets $O_i(\mathcal{M}_i) = \{ \llbracket \varphi \rrbracket_i^{\lambda_i} \mid \varphi \in \mathcal{L}^i \}$ constitute a basis for the topology $T_i$. Hence we can conclude from Assertions (17) that the two maps $B_i : X_i \rightarrow X_{ip}$ and $(B_i)^{-1} : X_{ip} \rightarrow X_i$ are l.s.c. with respect to $T_*$ and $T_i$.

To establish the relational equalities $R_i \bullet B_i = B_i \bullet R_*$ and $(B_i)^{-1} \bullet R_* = R_i \bullet (B_i)^{-1}$, we follow a similar strategy as in the proof of Part (2.) of Theorem 7.2 above. Using Assertions (17) together with the diamond clauses in Definition 4.3, and Claim 5 in the proof of Theorem 6.1, that $R_i^{-3}(U_{\varphi}) = U_{\varphi}$ and $R_i^{3}(U_{\varphi}) = U_{\varphi}$, it is readily verified that for all formulae $\varphi \in \mathcal{L}^i$:

$$
\begin{align*}
R_i \bullet B_i)^{-3}(U_{\varphi}) &= \llbracket \varphi \rrbracket_i^{\lambda_i} \quad \text{and} \quad (B_i \bullet R_*)^{-3}(U_{\varphi}) \quad (18) \\
(R \bullet (B_i)^{-1})^{-3}(U_{\varphi}) &= \llbracket \varphi \rrbracket_i^{\lambda_i} \quad \text{and} \quad ((B_i)^{-1} \bullet R_i)^{-3}(U_{\varphi}) \\
(R_i \bullet B_i)^{3}(\llbracket \varphi \rrbracket_i^{\lambda_i}) &= U_{\varphi} \quad \text{and} \quad (B_i \bullet R_*)^{3}(\llbracket \varphi \rrbracket_i^{\lambda_i}) \\
(R \bullet (B_i)^{-1})^{3}(\llbracket \varphi \rrbracket_i^{\lambda_i}) &= U_{\varphi} \quad \text{and} \quad ((B_i)^{-1} \bullet R_i)^{3}(\llbracket \varphi \rrbracket_i^{\lambda_i})
\end{align*}
$$
Now we define two relations $J_1 : X_i \sim X_{ip}$ and $J_2 : X_{ip} \sim X_i$ as follows:

\[
J_1 w x \iff (\forall \varphi \in L^i) (\varphi \in x \Rightarrow w \in \left[\Diamond \varphi \right]^{M_i}) \land (w \in \left[\varphi \right]^{M_i} \Rightarrow \varphi \in x)
\]

\[
J_2 x w \iff (\forall \varphi \in L^i) (\varphi \in x \Rightarrow w \in \left[\Diamond \varphi \right]^{M_i}) \land (w \in \left[\varphi \right]^{M_i} \Rightarrow \varphi \in x)
\]

The two equalities $B_i \cdot R_s = J_1$ and $R_s \cdot (B_i)^{-1} = J_2$, as well as the two inclusions $R_i \cdot B_i \subseteq J_1$ and $(B_i)^{-1} \cdot R_i \subseteq J_2$, are easy consequences of Assertions (18) together with the definition of the canonical relation $R_s$ in $M_s$. The remaining two inclusions take more effort, and (of course) use the conditions characterizing the class $\mathcal{D}_0$.

For the first converse inclusion, that $J_1 \subseteq R_i \cdot B_i$, suppose that $w J_1 x$. We need to find an $R_i$-successor $w^* \in R_i(w)$ that realizes $x$ as a theory in $M_i$. The realization set satisfies $\left\langle x \right\rangle^{M_i} = (B_i)^{-1}(x)$, and hence $w^* \in R_i(w) \cap \left\langle x \right\rangle^{M_i}$ would witness that $w (R_i \cdot B_i) x$, as required.

Since $R_i$ has realization saturation in $M_i$, we want to establish the hypothesis of that property holds of the state $w \in X_i$ and the set of formulae in $A = x \subseteq L^i$; since $x \in X_{ip}$ is a prime theory of $IK^t$, it is $\text{ITT}$-semantically-prime by Theorem 6.1. So we need to show that for every finite subset $\left\{ \varphi_1, \ldots, \varphi_n \right\} \subseteq x$, there is a $w_1 \in R_i(w)$ such that $w_1 \in \left[\varphi_k \right]^{M_i}$ for each $k \in \{1, \ldots, n\}$, and for every finite subset $\left\{ \psi_1, \ldots, \psi_m \right\} \subseteq \partial x$, there is a $w_2 \in R_i(w)$ such that $w_2 \in bd_{R_i}(\left[\psi_j \right]^{M_i})$ for each $j \in \{1, \ldots, m\}$.

Now if $\left\{ \varphi_1, \ldots, \varphi_n \right\} \subseteq x$, then the conjunction $\varphi_0 \in x$, where $\varphi_0 := \bigwedge_{1 \leq k \leq n} \varphi_k$, since $x$ is $\text{IK}^t$-deductively closed. Then $w J_1 x$ and $\varphi_0 \in x$ imply that $w \in \left[\Diamond \varphi_0 \right]^{M_i}$, and hence there exists a $w_1 \in R_i(w)$ such that $w_1 \in \left[\varphi_k \right]^{M_i}$ for each $k \in \{1, \ldots, n\}$.

Suppose $\left\{ \psi_1, \ldots, \psi_m \right\} \subseteq \partial x$. Then $\psi_0 \notin x$, where $\psi_0 := \bigvee_{1 \leq j \leq m} (\psi_j \lor \neg \psi_j)$, since $x$ has the disjunction property. Then $\Diamond \Box \psi_0 \notin x$, since $x$ is $\text{IK}^t$-deductively closed (applying axiom $\text{Ad4}$). Then $w J_1 x$ and $\Diamond \Box \psi_0 \notin x$ imply that $w \notin \left[\Diamond \Box \psi_0 \right]^{M_i}$, and hence $w \in cl_{R_i}(R_i^{-3} (X - \left[\psi_0 \right]^{M_i}))$. Now set:

\[
D := X - \left[\psi_0 \right]^{M_i} = \bigcap_{1 \leq j \leq m} (X - \left[\psi_j \right]^{M_i}) \cap (X - \left[\neg \psi_j \right]^{M_i}) = \bigcap_{1 \leq j \leq m} bd_{R_i}(\left[\psi_j \right]^{M_i})
\]

with the last equality by Assertion (5). Since the relation $R_i$ is boundary-closed in $M_i$, we can conclude that the set $R_i^{-3}(D)$ is closed in $T_i$, so $cl_{R_i}(R_i^{-3}(D)) = R_i^{-3}(D)$, and thus $w \in R_i^{-3}(D)$. Hence there exists a $w_2 \in R_i(w)$ such that $w_2 \in bd_{R_i}(\left[\psi_j \right]^{M_i})$ for each $j \in \{1, \ldots, m\}$.

The final inclusion $J_2 \subseteq (B_i)^{-1} \cdot R_i$ is equivalent to $(J_2)^{-1} \subseteq R_i^{-1} \cdot B_i$, and this is proved by the symmetric “mirror” argument, obtained by uniformly replacing $J_1$ with $(J_2)^{-1}$, replacing $\Diamond$ and $\Box$ with $\Diamond$ and $\Box$, and replacing $R$ and $R_i$ with the converse relations $R^{-1}$ and $R_i^{-1}$. The bisimulation conditions for $p \in AP$ are that $(B_i)^{-3}(\left[p \right]^{M_i}) \subseteq U_p$ and $(B_i)^{-3}(U_p) \subseteq \left[p \right]^{M_i}$.
These are trivially satisfied as by Assertion (17), we already have the equalities for all formulae. Hence $B_1$ is a tense topo-bisimulation between the models $\mathcal{M}_t$ and $\mathcal{M}_s$, and we are done. \hfill \dagger

The remaining question is whether $\mathcal{M}_s$ itself is in the Hennessy-Milner class $\mathcal{D}_0$; at this stage, we have to leave the question open (although we believe the answer is no).

**Proposition 7.4** The canonical model $\mathcal{M}_* = (X_{ip}, T_s, R_s, u_s)$ for the logic $\mathbf{IK}^1$ is such that $T_s$ is manifest in $\mathcal{M}_s$, and $R_s$ and $R_s^{-1}$ both have realization saturation in $\mathcal{M}_s$. However, it is an open question whether the relations $R_s$ and $R_s^{-1}$ are boundary-closed in $\mathcal{M}_s$.

**PROOF.** We have already used the fact that the topology $T_s$ is manifest in the model $\mathcal{M}_s$ (in the proof of Theorem 7.3 above), after first noting it in Claim 4 of the proof of Theorem 6.1. To see that $R_s$ has realization saturation in $\mathcal{M}_s$, fix $x \in X_{ip}$ and an $\mathbb{I}^1_\mathbb{T}$-semantically-prime set of formulae $\mathcal{A} \subseteq \mathcal{L}^1$. Then by Theorem 6.1, $\mathcal{A}$ is a prime theory of $\mathbf{IK}^1$, and hence there is a (unique) $y \in X_{ip}$ such that $y = \mathcal{A}$. The conclusion of the realization saturation property is that $R_s(x) \cap \{y\}^{m_*} \neq \emptyset$; since the realization set reduces to $\{y\}^{m_*} = \{y\}$, the desired conclusion is equivalent to $x R_s y$.

Suppose that for every finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq y$, there is an $x_1 \in R_s(x)$ such that $x_1 \in U_{\varphi_k}$ for each $k \in \{1, \ldots, n\}$, and for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \partial y$, there is an $x_2 \in R_s(x)$ such that $x_2 \in bd_{T_s}(U_{\psi_j})$ for each $j \in \{1, \ldots, m\}$. Note that $bd_{T_s}(U_\psi) = \{z \in X_{ip} \mid \psi \notin z \text{ and } \neg \psi \notin z\} = \{z \in X_{ip} \mid \psi \in \partial z\}$. To prove that $x R_s y$, we go through the four clauses of the definition of $R_s$: (i) $\varphi \in y \Rightarrow \diamond \varphi \in x$; (ii) $\square \varphi \in x \Rightarrow \varphi \in y$; (iii) $\varphi \in x \Rightarrow \varphi \in y$; and (iv) $\varphi \in y \Rightarrow \varphi \in x$. For clause (i), suppose $\varphi \in y$. Then there exists $x_1 \in R_s(x) \cap U_{\varphi}$, and thus $x \in R_s^{-2}(U_{\varphi}) = U_{\varphi}$, and so $\diamond \varphi \in x$. For clause (ii), suppose $\varphi \notin y$. We split into two cases: Case I: $\neg \varphi \in y$, and Case II: $\neg \varphi \notin y$. For Case I, $\neg \varphi \in y$ implies there exists $x_1 \in R_s(x) \cap U_{\neg \varphi}$, and hence $x \in R_s^{-2}(U_{\neg \varphi}) = U_{\neg \varphi}$, so $\ looph \varphi \notin x$. For Case II, $\neg \varphi \notin y$ means $\varphi \notin \partial y$ and hence there exists $x_2 \in R_s(x) \cap bd_{T_s}(U_{\varphi})$. Now we have $x R_s x_2$ and $\varphi \notin x_2$, and these imply $\looph \varphi \notin x$ by clause (ii) of $R_s$ for $x_2$. For clause (iii), suppose $\varphi \notin y$. We again split into two cases; here: Case I: $\neg \varphi \in y$, and Case II: $\neg \varphi \notin y$. For Case I, $\neg \varphi \in y$ implies there exists $x_1 \in R_s(x) \cap U_{\neg \varphi}$, and hence $x \in R_s^{-2}(U_{\neg \varphi}) = U_{\neg \varphi}$, so $\looph \neg \varphi \in x$. Since $x$ is deductively closed, we then get $\neg \varphi \in x$ (by axiom Ad3), and hence $\varphi \notin x$. For Case II, $\neg \varphi \notin y$ means $\varphi \notin \partial y$ and hence there exists $x_2 \in R_s(x) \cap bd_{T_s}(U_{\varphi})$. Now we have $x R_s x_2$ and $\varphi \notin x_2$, and these imply $\varphi \notin x$ by clause (iii) of $R_s$ for $x_2$. For clause (iv), suppose $\varphi \in y$. Then there exists $x_1 \in R_s(x) \cap U_{\varphi}$, and thus $x \in R_s^{-2}(U_{\varphi}) = U_{\varphi}$, and so $\looph \varphi \in x$. Then since $x$ is deductively closed, we get $\neg \varphi \in x$ (by axiom Ad3), and hence $\varphi \in x$. So now we have $x R_s y$, and thus $R_s$ has realization saturation in $\mathcal{M}_s$. The proof for $R_s^{-1}$ is symmetric. \hfill \dagger
By definition, \( R_* \) will fail to be boundary-closed in \( \mathcal{M}_* \) iff there exists a finite set of formulae \( \{ \psi_1, \ldots, \psi_m \} \subseteq \mathcal{L}^i \) and a state \( x \in cl_{\tau}(R^{\mathfrak{3}}_*(D)) - R^{\mathfrak{3}}_*(D) \), where \( D := \bigcap_{1 \leq j \leq m} bd_{\tau}(U_{\psi_j}) \). Set \( \psi_0 := \bigvee_{1 \leq j \leq m}(\psi_j \lor \neg\psi_j) \). As in the proof of Theorem 7.3 above, we have:

\[
cl_{\tau}(R^{\mathfrak{3}}_*(D)) = \{ x \in X_{ip} \mid \boxdot \psi_0 \not\models x \} \quad \text{and} \quad D = \{ z \in X_{ip} \mid \psi_0 \not\models z \}
\]

Via the Gödel translation \( GT : \mathcal{L}^k \to \mathcal{L}^q \), we can move over to the classical topological semantics; in particular, for any open l.s.c. model \( \mathcal{M} \), we have \( \models \varphi^i_{\mathcal{M}} = \models \varphi^{GT}(\varphi) \) for all \( \varphi \in \mathcal{L}^k \), from Assertion (14). Define an explicit boundary operator in the language \( \mathcal{L}^k \) by:

\[
\nabla \xi := \diamond \xi \land \neg \Box \xi \quad \text{for formulae } \xi \in \mathcal{L}_k^i.
\]

(One may note that \( \diamond \nabla \xi \leftrightarrow \nabla \xi \land \neg \Box \nabla \xi \), and hence \( \nabla \nabla \xi \leftrightarrow \nabla \xi \), as well as \( \neg(\nabla \xi \land (\Box \xi \lor \neg \Box \xi)) \), are all theorems of classical \( \mathbf{S4} \).)

Then observe that the boundary-closed property is classically expressible. For every model \( \mathcal{M}_i \in \mathfrak{D}_0 \) and every finite set of formulae \( \{ \psi_1, \ldots, \psi_m \} \subseteq \mathcal{L}^i \), we have in the classical semantics that:

\[
\mathcal{M}_i \models \diamond \nabla \xi_0 \iff \nabla \xi_0 \quad \text{and} \quad (\xi_0 \iff \neg GT(\psi_0)) \in \kappa^k_{\mathbf{LST}}
\]

where:

\[
\xi_0 := \bigwedge_{1 \leq j \leq m} \nabla GT(\psi_j) \quad \text{and} \quad [\xi_0]^{\mathcal{M}_*} = D \quad \text{and} \quad \psi_0 := \bigvee_{1 \leq j \leq m}(\psi_j \lor \neg\psi_j)
\]

or equivalently:

\[
\mathcal{M}_i \models GT(\boxdot \psi_0) \iff \boxdot GT(\psi_0)
\]

Now as constructed in the proof of Theorem 7.3 above, we have a total and functional tense topo-bisimulation \( B_i : X_i \to X_{ip} \) between each \( \mathcal{M}_i \in \mathfrak{D}_0 \) and the canonical model \( \mathcal{M}_* \). Set \( X_0 := \bigcup_{\mathcal{M}_i \in \mathfrak{D}_0} ran(B_i) = \bigcup_{\mathcal{M}_i \in \mathfrak{D}_0} B_i^2(X_i) \), which is an open set in \( X_{ip} \) w.r.t. \( \tau_* \) since each map \( B_i^{-1} \) is l.s.c. Then by Part (2) of Theorem 5.2 for the classical multi-modal logic, we can conclude that for every prime theory \( x \in X_0 \), and for every set of formulae \( \{ \psi_1, \ldots, \psi_m \} \subseteq \mathcal{L}^i \), we have \( x \in [GT(\boxdot \psi_0) \iff \boxdot GT(\psi_0)]^{\mathcal{M}_*} \), and conversely, if there exists a set of formulae \( \{ \psi_1, \ldots, \psi_m \} \subseteq \mathcal{L}^i \) such that \( x \in [GT(\boxdot \psi_0) \land \neg GT(\boxdot \psi_0)]^{\mathcal{M}_*} \), then \( x \in (X_{ip} - X_0) \). Also note that \( x \in (X_{ip} - X_0) \) iff \( x \not\in \mathcal{A} \) for every set \( \mathcal{A} \subseteq \mathcal{L}^i \) that is realized as a (intuitionistic) theory in some model \( \mathcal{M}_i \in \mathfrak{D}_0 \).

Thus we can conclude that \( R_* \) will fail to be boundary-closed in \( \mathcal{M}_* \) iff there is a prime theory \( x \in (X_{ip} - X_0) \) and formulae \( \{ \psi_1, \ldots, \psi_m \} \subseteq \mathcal{L}^i \) such that \( x \in [GT(\boxdot \psi_0) \land \neg GT(\boxdot \psi_0)]^{\mathcal{M}_*} \).

By Theorem 7.2, we also have a total and surjective tense topo-bisimulation \( G : X_{ip} \sim Y_{\mathfrak{M}} \) between \( \mathcal{M}_* \) and the open and l.s.c. model \( \mathcal{M}_{\mathfrak{M}} = (Y_{\mathfrak{M}}, S_\mathfrak{M}, Q_\mathfrak{M}, v_\mathfrak{M}) \) derived from the canonical model \( \mathcal{M}_* \) for the classical logic \( \mathbf{K^3LSC} \). Set \( Y_0 := G^3(X_0) \), so \( Y_0 \) is open in \( Y_{\mathfrak{M}} \) w.r.t.
the topology $S_\alpha^n$, since $G^{-1}$ is l.s.c. Then again by Part (2.) of Theorem 5.2 for the classical semantics, for every maximal $\mathbf{K}^4\mathbf{LSC}$-consistent theory $y \in Y_\alpha$, and for every set of formulae $\{\psi_1, \ldots, \psi_m\} \subseteq \mathcal{L}$, we have $y \in \llbracket\Diamond\top\top\top\Diamond(\psi_0) \leftrightarrow \Box\Diamond(\psi_0)\rrbracket^{\alpha^n}$, and conversely, if there exists a set of formulae $\{\psi_1, \ldots, \psi_m\} \subseteq \mathcal{L}$ such that $y \in \llbracket\Box\Diamond(\psi_0) \land \neg\Diamond(\Box(\psi_0))\rrbracket^{\alpha^n}$, then $y \in (Y_\alpha - Y_\beta)$. Now $y \in \llbracket\Box\Diamond(\psi_0) \land \neg\Diamond(\Box(\psi_0))\rrbracket^{\alpha^n}$ iff $y \in \llbracket\Diamond\Diamond\Diamond\Diamond(\psi_0) \land \neg\Diamond(\psi_0)\rrbracket^{\alpha^n}$ iff the set $Q_{\alpha}^{-3}(\llbracket\psi_0\rrbracket)^{\alpha^n}$ is not closed in the topology $S_\alpha^n$ on $Y_{\alpha^n}$.

So we now have the equivalence: the relation $R_\alpha$ will fail to be boundary-closed in the model $\mathcal{M}_\alpha$ iff the relation $Q_{\alpha}$ fails to be boundary-closed in the model $\mathcal{M}_{\alpha^n}$.

# # NOTE TO CO-AUTHORS FROM JEN: # # I think it must be possible to do a classical consistency argument to prove the “slightly shrunk” (by the topology and the atomic valuation) almost-canonical model $\mathcal{M}_{\alpha^n}$ fails to have the boundary-closed property. Any input, please!

8 Discussion: decidability and open problems

We conclude the paper with a brief discussion of decidability and open problems. There are some positive results on extensions $\mathbf{IK} \oplus \Gamma$ for some subsets $\Gamma$ of the schemes:

$$\{\mathbf{T}\mathbf{D}\mathbf{S}, \mathbf{B}\mathbf{D}\mathbf{S}, \mathbf{D}\mathbf{S}, 4\mathbf{D}\mathbf{S}, 5\mathbf{D}\mathbf{S}, \mathbf{D}\mathbf{S}, 5\mathbf{D}\mathbf{S}\}$$

Simpson [31] proves the finite model property over bi-relational frames for extensions where $\Gamma \subseteq \{\mathbf{T}\mathbf{D}\mathbf{S}, \mathbf{B}\mathbf{D}\mathbf{S}, \mathbf{D}\mathbf{S}\}$, and Grefe [19] proves the result independently for $\mathbf{IK}$. Earlier work via algebraic models proves the finite model property over bi-relational frames for $\mathbf{IS5} := \mathbf{IK} \oplus \mathbf{T}\mathbf{D}\mathbf{S} \oplus 5\mathbf{D}\mathbf{S}$ (also known as $\mathbf{MIPQ}$). As documented in Simpson’s thesis [31], §3.3, Bull claimed a proof of the finite algebraic model property for $\mathbf{MIPQ}$ in a 1965 paper; the mistake was then corrected by Fischer Servi in 1978 and independently by Ono in 1977, the latter using a prime filter construction to obtain a finite bi-relational model from a finite algebraic model. Wolter and Zakharyaschev also have finite model property results for a number of extensions of the weaker logic $\mathbf{IntK}$; see [35], Theorem 17 and Example 19.

Decidability and the finite model property for other extensions remain open questions; in particular, for $\mathbf{IS4} := \mathbf{IK} \oplus \mathbf{T}\mathbf{D}\mathbf{S} \oplus 4\mathbf{D}\mathbf{S}$. Ewald in [12] claimed the finite model property for tense logics $\mathbf{IK}^\dagger \oplus \Gamma$ for $\Gamma$ included in $\{\mathbf{T}\mathbf{D}\mathbf{S}, \mathbf{B}\mathbf{D}\mathbf{S}, \mathbf{D}\mathbf{S}\}$, but a fatal flaw was identified by Stirling and detailed by Simpson in [31], §8.2. Ewald also claimed that decidability could be established using Rabin’s monadic second-order theory $\mathbf{S2S}$ of the binary tree with two successors. Simpson disputes this, claiming that the confluence relationships between the intuitionistic preorder $\leq$ and the modal accessibility relation $R$ required by the $\mathbf{Zig}(\leq, R)$ and $\mathbf{Zig}(\leq, R^{-1})$
frame conditions present a major obstacle to applying the techniques of Rabin to these decision problems. We agree with Simpson’s analysis, particularly for transitive accessibility relations \(R\), and concur that the bi-relational confluence properties present a serious challenge to establishing decidability. The corresponding classical bi-modal or tri-modal logics are likewise proving to be stubbornly resistant to the resolution of decidability: the question is still open for K\(\mathbf{LSC}\) and K\(\mathbf{LSC}^\dagger\), and the bi-modal K\(\mathbf{LSC}_0 := (\mathsf{S4} \boxplus \mathsf{K}\Box) \oplus (\Diamond \Box \varphi \rightarrow \Box \Diamond \varphi)\), where only the forwards relation \(R\) need be l.s.c. A further open problem is whether topological completeness in Euclidean space is possible for intuitionistic modal/tense logics, and topological modal/tense logics, extending the results of McKinsey and Tarski for intuitionistic propositional logic and for classical \(\mathsf{S4}\).

In the course of this paper, we have made a number of original contributions. We have given a complete topological semantics for the intuitionistic modal and tense logics of Fischer Servi [15] and Ewald [12], and generalized the known bi-relational frame conditions to lower semi-continuity properties of the relation with respect to the topology. By exhibiting a non-Alexandrov topology on the canonical model over the space of prime theories of the logic, we properly extend the semantics of the logics beyond the existing classes of bi-relational models. We then use this canonical model to exhibit a topological bisimulation with lower semi-continuity that is maximal in perserving intuitionistic semantics.

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