Controller Discretisation: A Gap Metric Framework for Analysis and Synthesis

Michael Cantoni and Glenn Vinnicombe

Abstract— **Although techniques for directly synthesising sampled-data (SD) compensators are available in the literature, feedback controller *design* is perhaps best understood in a purely continuous-time setting. As such, a feedback controller is often designed in the continuous-time domain and then discretised for digital implementation. It is important for the discretisation step involved to yield a SD approximation which captures the essential features of the original controller from the perspective of closed-loop behaviour. In this paper, a gap metric framework is developed for studying the controller discretisation problem for linear time-invariant (LTI) plants and controllers. Importantly, knowledge of a gap metric distance between an LTI controller and a SD approximation permits explicit characterisation of the possible difference in closed-loop performance, with any LTI plant for which the LTI controller is known to work well, accounting for inter-sample behaviour. The central result of the new framework gives rise to an algorithm for computing a gap metric measure of the distance between an LTI controller and a given discretisation, and a technique for synthesising a SD approximation which is optimal with respect to this metric.**

Index Terms— **Controller Discretisation, Digital Redesign, Sampled-Data, Approximation, Gap Metric, Robustness, Periodic, Time Varying**

I. INTRODUCTION

The dynamics of many engineering systems naturally evolve in continuous time. Correspondingly, control system design is typically carried out, and perhaps most well-understood, in the continuous-time domain. This results in control laws with continuous-time dynamics. It is often the case, however, that the implementation of such control schemes will involve a digital computer, which can only process information discretely in time. Accordingly, system discretisation (in time) has been studied for many years [1], [2], [3], [4], [5].

Motivated by established gap-metric robust performance results for LTI feedback system [6], [7], [8], [9], a new framework for studying the feedback controller discretisation problem is developed in this paper. Central to the framework is the use of a gap-like metric to measure the difference between an LTI controller and a SD approximation. Indeed, knowledge of this measure of distance permits characterisation of the possible degradation of closed-loop performance with any LTI plant for which the LTI controller is known to work well, accounting for intersample behaviour.

A variant of the following LTI result [9], [10], which permits one of the systems to be a *periodic* SD controller, is established in this paper as a cornerstone of the framework described above:

Proposition 1.1: [9] Given LTI systems C and C_1, and a number β, there exists an LTI system R (dependent on C and β only) such that $\delta_\nu(C, C_1) \leq \beta \Leftrightarrow \mathscr{F}_\ell(R^{-1}, C_1) \in \mathcal{H}^\infty$ and $\|\mathscr{F}_\ell(R^{-1}, C_1)\|_\infty \leq 1$, where $\delta_\nu(C, C_1)$ denotes the ν-gap distance between C and C_1, and $\mathscr{F}_\ell(\cdot, \cdot)$ denotes the standard lower-linear fractional transformation (LFT).

A result of this kind is suggestive of an algorithm for computing the distance between an LTI controller C and a given discretisation C_{sd}, and a technique for synthesising a SD approximation; i.e. synthesise C_{sd} so that $\mathscr{F}_\ell(R^{-1}, C_{\mathrm{sd}})$ is stable and contractive (for some small

β). Indeed, this paper is by and large devoted to establishing that $\mathscr{F}_\ell(R^{-1}, C_{\mathrm{sd}})$ is stable and contractive if, and only if, C_{sd}:

(*i*) lies within a pointwise gap metric distance, closely related to the ν-gap metric, from C; and

(*ii*) stabilises any LTI plant with which C achieves a certain level of closed-loop performance.

Importantly, testing if $\mathscr{F}_\ell(R^{-1}, C_{\mathrm{sd}})$ is stable and contractive for a given C_{sd}, or synthesising a C_{sd} so that it is, can be achieved via established methods [11], [12], [13].

The paper develops along the following line. First, a generalised frequency domain framework for studying the controller discretisation problem is established (cf. Sec. II). As part of this, a motivational gap metric robust performance result is also discussed within the generalised frequency domain framework. Subsequently, a new signal-based characterisation of a pointwise gap-metric measure of distance is developed. This characterisation is inspired by the so-called "strong-necessity" of a robustness result associated with the ν-gap metric for LTI systems [8], [9], by which the distance between two LTI controllers can be expressed in terms of a stability condition being satisfied for one of the controllers in closed-loop with *any* LTI plant for which the other controller achieves a certain level of closed-loop performance (cf. Sec. III and Sec. IV). A simple numerical example is presented to demonstrate the tools developed.

II. PRELIMINARIES

A. Basic operator theoretic notation

Throughout, the symbols $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{C}_+$, $\mathbb{C}_-$, $\mathbb{T}$, $\mathbb{D}$, $j\mathbb{R}$ and $\mathbb{H}$ denote the real, integer and complex numbers, the open right-half, open left-half, unit circle and open unit disc of the complex plane, the imaginary axis, and the interval $[0, h) \subset \mathbb{R}$ for $h > 0$, respectively.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be Hilbert spaces over either $\mathbb{R}$ or $\mathbb{C}$. The inner product on $\mathcal{V}_1$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{V}_1}$, and $\| \cdot \|_{\mathcal{V}_1}$ denotes the norm induced by the inner product. Let $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \mathcal{V}_1 \to \mathcal{V}_2$ be a linear operator, where $\mathrm{dom}(\mathbf{X})$ is called the domain of $\mathbf{X}$. The graph of $\mathbf{X}$, taken with respect to the ambient space $\mathcal{V}_2 \times \mathcal{V}_1$ (resp. $\mathcal{V}_1 \times \mathcal{V}_2$), is defined to be the subspace $\mathrm{gr}(\mathbf{X}) := \left(\begin{smallmatrix} \mathbf{X} \\ I \end{smallmatrix} \right) \mathrm{dom}(\mathbf{X})$ (resp. $\mathrm{gr}(\mathbf{X}) := \left(\begin{smallmatrix} I \\ \mathbf{X} \end{smallmatrix} \right) \mathrm{dom}(\mathbf{X})$).

The Banach space $\mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ is the set of linear operators $\mathbf{X} : \mathcal{V}_1 \to \mathcal{V}_2$ for which $\mathrm{dom}(\mathbf{X}) = \mathcal{V}_1$ and the induced norm $\|\mathbf{X}\| := \sup_{u \in \mathcal{V}_1, u \neq 0} \frac{\|\mathbf{X}u\|_{\mathcal{V}_2}}{\|u\|_{\mathcal{V}_1}} < \infty$. Occasionally, the space $\mathcal{B}(\mathcal{V}_1, \mathcal{V}_1)$ may be denoted by $\mathcal{B}(\mathcal{V}_1)$ for convenience. The Hilbert adjoint of an operator $\mathbf{X} \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ is denoted by $\mathbf{X}^* \in \mathcal{B}(\mathcal{V}_2, \mathcal{V}_1)$. If $\mathbf{X}^* \mathbf{X} = I$ then $\mathbf{X}$ is called an isometry. On the other hand, if $\mathbf{X}^*$ is an isometry then $\mathbf{X}$ is called a coisometry. Two Hilbert spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ are said to be isomorphic if there exists a bijective isometry $\mathbf{Y} \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ such that $\mathcal{V}_2 = \mathbf{Y}\mathcal{V}_1$. Moreover, two operators $\mathbf{X}_1 : \mathrm{dom}(\mathbf{X}_1) \subset \mathcal{V}_1 \to \mathcal{V}_2$ and $\mathbf{X}_2 : \mathrm{dom}(\mathbf{X}_2) \subset \mathcal{V}_3 \to \mathcal{V}_4$ are said to be equivalent if $\mathrm{gr}(\mathbf{X}_1)$ is isomorphic to $\mathrm{gr}(\mathbf{X}_2)$.

B. Signal and system representations

In this section various aspects of the frequency-domain setting of [11], [14] are reviewed. A signal is simply considered to be a function mapping from some domain of definition into a Hilbert space. Of particular interest are the signal spaces $\mathcal{L}^2_{j\mathbb{R}}(\mathbb{C}^q)$ and $\mathcal{L}^2_{\mathbb{T}}(\mathcal{L}^{2,p}_{\mathbb{H}})$, of square integrable functions on $j\mathbb{R}$ and $\mathbb{T}$, that take values in $\mathbb{C}^q$ and $\mathcal{L}^2_{\mathbb{H}}(\mathbb{C}^p)$ (which denotes the space of square integrable functions on the interval $[0, h)$), respectively. For convenience, the space in which signals take their values may often be dropped. Note that $\mathcal{L}^2_{j\mathbb{R}}(\mathbb{C}^p)$ and $\mathcal{L}^2_{\mathbb{T}}(\mathcal{L}^{2,p}_{\mathbb{H}})$ are isomorphic, via the Fourier transform (by which $\mathcal{L}^2_{j\mathbb{R}}$ is isomorphic to the time-domain signal space $\mathcal{L}^2_{\mathbb{R}}$ of square integrable functions on $\mathbb{R}$), the $\mathbf{W}$-transform (which decomposes functions on $\mathbb{R}$

into segments of length h according to $\underline{f}_k(\theta) = (\mathbf{W}f)_k(\theta) := f(\theta + kh)$ for $\theta \in \mathbb{H}$) and the $\mathbf{Z}$-transform $(\mathbf{Z}\underline{f})(\varphi) := \sum_{k \in \mathbb{Z}} \varphi^k \underline{f}_k$. Also of interest are $\mathcal{H}^2_{\mathbb{C}_+}$, the subspace of all $f \in \mathcal{L}^2_{j\mathbb{R}}$ which can be continued analytically into $\mathbb{C}_+$ so that they are uniformly bounded in the usual way, and $\mathcal{H}^2_{\mathbb{D}}$, the subspace of all $f \in \mathcal{L}^2_{\mathbb{T}}$ which can similarly be continued analytically into $\mathbb{D}$. $\mathcal{H}^2_{\mathbb{C}_+}$ and $\mathcal{H}^2_{\mathbb{D}}$ are also isomorphic via the Fourier-, $\mathbf{W}$- and $\mathbf{Z}$-transforms. The intermediate time-domain signal space involved in this case is $\mathcal{L}^{2+}_{\mathbb{R}} := \{f \in \mathcal{L}^2_{\mathbb{R}} : f(t) = 0 \text{ for } t < 0\}$.

Characterising systems as multiplication operators on the spaces defined above, is adequate to address the SD approximation problem outlined in the introduction. Most would be familiar with the equivalence (via the Fourier transform isomorphism) between a time-domain representation of a finite-dimensional LTI state-space system and multiplication by a corresponding transfer function in:

(i) $\mathcal{R}^{p,m}$, defined to be the set of functions $P : j\mathbb{R} \to \mathcal{B}(\mathbb{C}^m, \mathbb{C}^p)$ (a.e.) of the form $P(\varphi) = \mathsf{C}(\varphi I - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D}$ for (almost) all $\varphi \in j\mathbb{R}$, with $\mathsf{A} \in \mathbb{R}^{n \times n}$.

It is also true, that a time-domain representation of a finite-dimensional linear *periodically time-varying* state-space system (including LTI systems) is equivalent to multiplication by a corresponding frequency-domain symbol in:

(ii) $\mathcal{D}^{p,m}$, defined to be the set of functions $P : \mathbb{T} \to \mathcal{B}(\mathcal{L}^{2,m}_{\mathbb{H}}, \mathcal{L}^{2,p}_{\mathbb{H}})$ (a.e.) of the form $P(\varphi) = \varphi\mathsf{C}(I - \varphi\mathsf{A})^{-1}\mathsf{B} + \mathsf{D}$ with $P(\varphi)$ for (almost) all $\varphi \in \mathbb{T}$, and $\mathsf{A} \in \mathbb{R}^{n \times n}$.

In particular, given a finite-dimensional LTI filter $\mathbf{F} : u \mapsto u_F$, governed by the system of differential equations $\dot{x}_F = \mathsf{A}_F x_F + \mathsf{B}_F u$ and $u_F = \mathsf{C}_F x_F$, with $\mathrm{spec}(\mathsf{A}_F) \subset \mathbb{C}_-$ (where $\mathrm{spec}(\cdot)$ denotes the spectrum of an operator), and an h-periodic SD system $\mathbf{S} : u_F \mapsto y$, governed by the difference equations $x_S(k+1) = \mathsf{A}_S x_S(k) + \mathsf{B}_S u_F(kh)$ and $y(kh + \theta) = \mathsf{C}_S x_S(k) + \mathsf{D}_S u_F(kh)$ for $\theta \in \mathbb{H}$, it follows, by considering the evolution of these equations over intervals of time of duration h, that the $\mathcal{L}^2_{\mathbb{R}}$-graph (resp. $\mathcal{L}^{2+}_{\mathbb{R}}$-graph) of the pre-filtered periodic SD system $\mathbf{C}_{\mathrm{sd}} := \mathbf{SF}$ is isomorphic (via $\mathbf{W}$ and $\mathbf{Z}$) to the $\mathcal{L}^2_{\mathbb{T}}(\mathcal{L}^2_{\mathbb{H}})$-graph (resp. $\mathcal{H}^2_{\mathbb{D}}(\mathcal{L}^2_{\mathbb{H}})$-graph) of multiplication by

$$C_{\mathrm{sd}}(\varphi) = \left(\begin{array}{cc|c} \grave{\mathsf{A}}_F & 0 & \grave{\mathsf{B}}_F \\ \mathsf{B}_S\mathsf{C}_F & \mathsf{A}_S & 0 \\ \hline \grave{\mathsf{C}}_1 & \grave{\mathsf{C}}_2 & 0 \end{array}\right) \in \mathcal{D},$$

where $\grave{\mathsf{A}}_F : x \mapsto \exp(h\mathsf{A}_F)x$, $\grave{\mathsf{B}}_F : u \mapsto \int_0^h \exp((h-\tau)\mathsf{A}_F)\mathsf{B}_F u(\tau)\,d\tau$, $\grave{\mathsf{C}}_1 : x \mapsto \mathsf{D}_S\mathsf{C}_F x$, $\grave{\mathsf{C}}_2 : x \mapsto \mathsf{C}_S x$ and $\left(\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right)$ denotes a realisation $\varphi\mathsf{C}(I - \varphi\mathsf{A})^{-1}\mathsf{B} + \mathsf{D} \in \mathcal{D}$. Similarly, multiplication by any $P \in \mathcal{R}$ is equivalent (via the Fourier transform, $\mathbf{Z}$-transform and $\mathbf{W}$-transform isomorphisms), to multiplication by a $\underline{P} \in \mathcal{D}$. Throughout, an underline is used to denote the "time-lifted" equivalent of a transfer function in $\mathcal{R}$.

Given $P \in \mathcal{R}$ (resp. $\mathcal{D}$), $\mathbf{L}_P$ denotes the Laurant operator defined by $(\mathbf{L}_P u)(\varphi) := P(\varphi)u(\varphi)$ for all $u \in \mathrm{dom}(\mathbf{L}_P) \subset \mathcal{L}^2_{j\mathbb{R}}$ (resp. $\mathcal{L}^2_{\mathbb{T}}$) and $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$). Similarly, the multiplication operator $\mathbf{M}_P$ is defined by $(\mathbf{M}_P u)(\varphi) := P(\varphi)u(\varphi)$ for all $u \in \mathrm{dom}(\mathbf{M}_P) \subset \mathcal{H}^2_{\mathbb{C}_+}$ (resp. $\mathcal{H}^2_{\mathbb{D}}$) and $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$). The space of functions $P \in \mathcal{R}$ that satisfy $\|P\|_\infty := \sup_{\varphi \in j\mathbb{R}} \|P(\varphi)\| < \infty$, is denoted by $\mathcal{R}\mathcal{L}^\infty_{j\mathbb{R}}$. The symbol $\mathcal{R}\mathcal{H}^\infty_{\mathbb{C}_+}$ denotes the Hardy space of all $P \in \mathcal{R}\mathcal{L}^\infty_{j\mathbb{R}}$ that can be continued analytically into $\mathbb{C}_+$.[1] Similarly, the space of functions $P \in \mathcal{D}$ that satisfy $\|P\|_\infty := \sup_{\varphi \in \mathbb{T}} \|P(\varphi)\| < \infty$, is denoted by $\mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$, and $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$ denotes the Hardy space of all $P \in \mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$ that can be continued analytically into $\mathbb{D}$.[2] When $P \in \mathcal{R}\mathcal{L}^\infty_{j\mathbb{R}}$ (resp. $\mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$),

$\mathbf{L}_P \in \mathcal{B}(\mathcal{L}^2_{j\mathbb{R}}, \mathcal{L}^2_{j\mathbb{R}})$ (resp. $\mathcal{B}(\mathcal{L}^2_{\mathbb{T}}, \mathcal{L}^2_{\mathbb{T}})$), $\mathrm{dom}(\mathbf{L}_P) = \mathcal{L}^2_{j\mathbb{R}}$ (resp. $\mathcal{L}^2_{\mathbb{T}}$) and $\|\mathbf{L}_P\| = \|P\|_\infty$. Moreover, when $P \in \mathcal{R}\mathcal{H}^\infty_{\mathbb{C}_+}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$), $\mathbf{M}_P \in \mathcal{B}(\mathcal{H}^2_{\mathbb{C}_+}, \mathcal{H}^2_{\mathbb{C}_+})$ (resp. $\mathcal{B}(\mathcal{H}^2_{\mathbb{D}}, \mathcal{H}^2_{\mathbb{D}})$), $\mathrm{dom}(\mathbf{M}_P) = \mathcal{H}^2_{\mathbb{C}_+}$ (resp. $\mathcal{H}^2_{\mathbb{D}}$) and $\|\mathbf{M}_P\| = \|P\|_\infty$ [15].

If a (transfer) function $P \in \mathcal{R}\mathcal{H}^\infty_{\mathbb{C}_+}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$) is such that $P(\varphi)$ is an isometry for all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), then P is called inner. On the other hand, if $P \in \mathcal{R}\mathcal{H}^\infty_{\mathbb{C}_+}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$) is such that $P(\varphi)$ is a coisometry for all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), then P is called coinner. Henceforth, P^* is used to denote the function that satisfies $P^*(\varphi) = P(\varphi)^*$ for (almost) all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$).

C. Feedback systems

In this subsection, and the next, attention is directed towards systems represented by transfer functions in $\mathcal{D}$. Recall that this includes all transfer functions in $\mathcal{R}$.

Consider the standard feedback configuration, shown in Fig. 1. When it exists (in an appropriate sense), the transfer function from (d_1, d_2) to (y_c, u_c, y_p, u_p) is denoted by $[P, C]$. Now suppose that
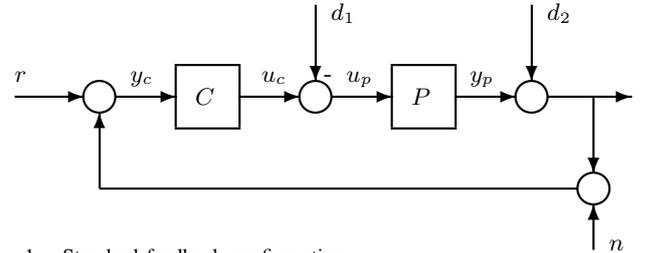

Fig. 1. Standard feedback configuration

$P \in \mathcal{D}^{m,p}$ and $C \in \mathcal{D}^{p,m}$. When $(I - PC)$ and $(I - CP)$ are invertible in $\mathcal{D}^{m,m}$ and $\mathcal{D}^{p,p}$ respectively (i.e. $[P, C]$ is well-posed), the transfer function from the signals $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ to $\begin{pmatrix} u_c \\ y_c \end{pmatrix}$ is

$$T_1(P,C) = \begin{pmatrix} C \\ I \end{pmatrix}(I - PC)^{-1}\begin{pmatrix} -P & I \end{pmatrix} \tag{1}$$

and the transfer function from $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ to $\begin{pmatrix} u_p \\ y_p \end{pmatrix}$ is $T_2(P,C) := T_1(P,C) - I$. Note that $T_1(P,C)$ and $T_2(P,C)$ capture all closed-loop transfer functions commonly employed in robustness and performance analysis [16], [9]. When $[P,C] \in \mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$), it can be shown that

$$\boldsymbol{b}(P,C) := \|T_1(P,C)\|_\infty^{-1} = \|T_2(P,C)\|_\infty^{-1} \leq 1.$$

If $[P,C] \notin \mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$) then $\boldsymbol{b}(P,C) := 0$. In the so-called $\mathcal{H}^\infty$ loop-shaping paradigm for feedback compensator design [16], [9], $\boldsymbol{b}(P,C)$ is used as a generic measure of closed-loop performance and robust stability.

Using the geometric framework of [17], the following proposition (which is used in the proof of Thm. 4.1) is readily established:

Proposition 2.1: Given $P \in \mathcal{D}^{m,p}$ and $C \in \mathcal{D}^{p,m}$, let $\mathcal{G} := \mathrm{gr}(\mathbf{L}_P)$ (resp. $\mathrm{gr}(\mathbf{M}_P)$) and $\mathcal{K} := \mathrm{gr}(\mathbf{L}_C)$ (resp. $\mathrm{gr}(\mathbf{M}_C)$), where the graphs here are taken with respect to the ambient space $\mathcal{L} := \mathcal{L}^2_{\mathbb{T}}(\mathcal{L}^{2,p}_{\mathbb{H}}) \times \mathcal{L}^2_{\mathbb{T}}(\mathcal{L}^{2,m}_{\mathbb{H}})$ (resp. $\mathcal{H}^2_{\mathbb{D}}(\mathcal{L}^{2,p}_{\mathbb{H}}) \times \mathcal{H}^2_{\mathbb{D}}(\mathcal{L}^{2,m}_{\mathbb{H}})$). Then $[P,C] \in \mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$) if, and only if, $\mathcal{K} \cap \mathcal{G} = \{0\}$ and $\mathcal{K} + \mathcal{G} = \mathcal{L}$. Furthermore, given $Q \in \mathcal{D}^{\hat{m},\hat{p}}$ and $F \in \mathcal{D}^{\hat{p},\hat{m}}$, with $\hat{p} + \hat{m} = p + m$, such that $\mathbf{L}_R\mathcal{K} = \mathrm{gr}(\mathbf{L}_F)$ (resp. $\mathbf{M}_R\mathcal{K} = \mathrm{gr}(\mathbf{M}_F)$) and $\mathbf{L}_R\mathcal{G} = \mathrm{gr}(\mathbf{L}_Q)$ (resp. $\mathbf{M}_R\mathcal{G} = \mathrm{gr}(\mathbf{M}_Q)$), for some $R, R^{-1} \in \mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$), where the graphs here are taken with respect to the partitioning $\mathcal{L}^2_{\mathbb{T}}(\mathcal{L}^{2,\hat{p}}_{\mathbb{H}}) \times \mathcal{L}^2_{\mathbb{T}}(\mathcal{L}^{2,\hat{m}}_{\mathbb{H}})$ (resp. $\mathcal{H}^2_{\mathbb{D}}(\mathcal{L}^{2,\hat{p}}_{\mathbb{H}}) \times \mathcal{H}^2_{\mathbb{D}}(\mathcal{L}^{2,\hat{m}}_{\mathbb{H}})$) of $\mathcal{L}$, the following equivalence holds:

$[P,C] \in \mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$) $\Leftrightarrow$ $[F,Q] \in \mathcal{D}\mathcal{L}^\infty_{\mathbb{T}}$ (resp. $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$).

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[1]Note that this corresponds to those $P \in \mathcal{R}$ for which a realisation $\left(\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right)$, with $\mathrm{spec}(A) \subset \mathbb{C}_-$, exists.

[2]Note that $\mathcal{D}\mathcal{H}^\infty_{\mathbb{D}}$ corresponds to those $P \in \mathcal{D}$ for which a realisation $\left(\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right)$, with $\mathrm{spec}(A) \subset \mathbb{D}$, exists.

D. A key gap metric robustness result

Important feedback system robustness results, which provide substantial motivation for using a gap metric to quantify approximation error, are now established in the frequency domain framework developed thus far. Recall that the $\mathcal{L}_{\mathbb{T}}^2$-gap between two systems $C_1, C_2 \in \mathcal{D}$ is defined to be the gap (or aperture [18]) between $\mathcal{K}_1 := \mathrm{gr}(\mathbf{L}_{C_1})$ and $\mathcal{K}_2 := \mathrm{gr}(\mathbf{L}_{C_2})$ [9], [19]:

$$\mathrm{gap}(\mathcal{K}_1, \mathcal{K}_2) := \|\mathbf{\Pi}_{\mathcal{K}_1} - \mathbf{\Pi}_{\mathcal{K}_2}\|,$$

where $\mathbf{\Pi}_{\mathcal{K}}$ denotes the orthogonal projection onto $\mathcal{K}$. Using the fact (see Appendix A) that the $\mathcal{L}_{\mathbb{T}}^2$-graph of any $C \in \mathcal{D}$ can be characterised as the range (resp. kernel) of a multiplication operator with (i) inner (resp. coinner) symbol, and (ii) left (resp. right) inverse corresponding multiplication by transfer functions in $\mathcal{DH}_{\mathbb{D}}^\infty$, it can be shown (see Appendix B) that

$$\mathrm{gap}(\mathcal{K}_1, \mathcal{K}_2) = \sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi)),$$

where for any $X, Y \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$,

$$\boldsymbol{\kappa}(X, Y) := \sup_{\substack{x \in \mathrm{gr}(X) \\ x \neq 0}} \inf_{\substack{y \in \mathrm{gr}(Y) \\ y \neq 0}} \frac{\|x - y\|_{\mathcal{V}}}{\|x\|_{\mathcal{V}}} = \sup_{\substack{x \in \mathrm{gr}(X) \\ x \neq 0}} \inf_{\substack{y \in \mathrm{gr}(Y) \\ y \neq 0}} \frac{\|x - y\|_{\mathcal{V}}}{\|y\|_{\mathcal{V}}} \quad (2)$$

and $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$. The final equality in (2) is established in [20, Appendix]. As such, it follows from the central results of [21, Sec. III], that given $[P, C_1] \in \mathcal{DL}_{\mathbb{T}}^\infty$ (resp. $\mathcal{DH}_{\mathbb{D}}^\infty$) and $[P, C_2] \in \mathcal{DL}_{\mathbb{T}}^\infty$ (resp. $\mathcal{DH}_{\mathbb{D}}^\infty$),

$$\delta(C_1, C_2) \le \|T_1(P, C_1) - T_1(P, C_2)\|_\infty \le \frac{\delta(C_1, C_2)}{\boldsymbol{b}(P, C_1) \cdot \boldsymbol{b}(P, C_2)}, \quad (3)$$

where $\delta(C_1, C_2) := \sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi))$. Furthermore, $\arcsin \boldsymbol{b}(P, C_2) \ge \arcsin \boldsymbol{b}(P, C_1) - \arcsin \delta(C_1, C_2)$, which is important from the perspective of robust stability in the face of plant uncertainty. These results clearly indicate that $\boldsymbol{\kappa}(\cdot, \cdot)$ is a sensible pointwise measure of distance, from the perspective of capturing the difference between two closed-loop systems.

III. SIGNAL-BASED CHARACTERISATION OF THE POINTWISE GAP

The ν-gap distance between two LTI controllers can be characterised in terms of a stability condition being satisfied for one of the controllers in closed-loop with *any* LTI plant for which the other controller achieves a certain level of closed-loop performance [9]. In this section, a variation of this result is established, by which one of the controllers can be periodically time-varying. This is achieved in two steps. The first step involves a new characterisation of "plant signals" that would be consistent with a specified level of closed-loop performance for a given $C \in \mathcal{R}$. This, in turn, leads to the required characterisation of the pointwise gap.

Central to the development of the first step is an inner-coinner-outer factorisation that is implicitly established in [22]. Before this can be presented, some additional notation is required. Given $H =: \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \in \mathcal{R}$ (resp. $\mathcal{D}$) and $Q \in \mathcal{R}$ (resp. $\mathcal{D}$), if $(I - H_{22}Q)^{-1} \in \mathcal{R}$ (resp. $\mathcal{D}$), the lower linear fractional transformation (LFT) $\mathscr{F}_\ell(H, Q) := H_{11} + H_{12}Q(I - H_{22}Q)^{-1}H_{21}$. Furthermore, given $\Theta =: \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \in \mathcal{R}$ (resp. $\mathcal{D}$), the Redheffer star product

$$\Theta \star H := \begin{pmatrix} \mathscr{F}_\ell(\Theta, H_{11}) & \Theta_{12}(I - \Theta_{22}H_{11})^{-1}H_{12} \\ H_{21}(I - H_{11}\Theta_{22})^{-1}\Theta_{21} & \mathscr{F}_u(H, \Theta_{22}) \end{pmatrix},$$

where $\mathscr{F}_u(H, Q) := H_{22} + H_{21}Q(I - H_{11}Q)^{-1}H_{12}$, provided the required inverses exist in $\mathcal{R}$ (resp. $\mathcal{D}$).

Now, for a given controller with transfer function $C \in \mathcal{R}$, let

$$H := \begin{pmatrix} 0 & C & C \\ 0 & I & I \\ -I & C & C \end{pmatrix}.$$

Then for a system with transfer function $P \in \mathcal{D}$, it follows that

$$\mathscr{F}_\ell(\underline{H}, P) = \begin{pmatrix} \underline{C} \\ I \end{pmatrix}(I - P\underline{C})^{-1}\begin{pmatrix} -P & I \end{pmatrix} \in \mathcal{D},$$

provided $(I - P\underline{C})$ is invertible in $\mathcal{D}$.[3] Given a $\beta > 0$, note that $[P, \underline{C}] \in \mathcal{DL}_{\mathbb{T}}^\infty$ and $\boldsymbol{b}(P, \underline{C}) > \beta \Leftrightarrow \mathscr{F}_\ell(\underline{H}, P) \in \mathcal{DL}_{\mathbb{T}}^\infty$ and $\|\mathscr{F}_\ell(\underline{H}, P)\|_\infty < \frac{1}{\beta}$. In fact, defining $H_\beta := \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\beta}I \end{pmatrix} H \begin{pmatrix} \beta I & 0 \\ 0 & \beta I \end{pmatrix}$, it follows that $[P, \underline{C}] \in \mathcal{DL}_{\mathbb{T}}^\infty$ and $\boldsymbol{b}(P, \underline{C}) > \beta \Leftrightarrow \mathscr{F}_\ell(\underline{H}_\beta, P) \in \mathcal{DL}_{\mathbb{T}}^\infty$ and $\|\mathscr{F}_\ell(\underline{H}_\beta, P)\|_\infty < 1$. For a given controller $C \in \mathcal{R}$, it can be shown by following an argument presented in [22, Sec. VI], that for $\beta < \boldsymbol{b}_{\mathrm{opt}}(C) := \sup_{P \in \mathcal{R}} \boldsymbol{b}(P, C)$,

$$H_\beta = M \star \tilde{M} \star R, \quad (4)$$

where $M \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ is inner with $M_{21}, M_{21}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$, $\tilde{M} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ is coinner with $\tilde{M}_{12}, \tilde{M}_{12}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$, and R is invertible in $\mathcal{R}$ with $R_{12}^{-1}, R_{21}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$. Details of this factorisation are in Appendix C.

Remark 3.1: The factorisation (4) can be used, as is done implicitly in [22], to characterise all $P \in \mathcal{R}$ that satisfy $\mathscr{F}_\ell(H_\beta, P) \in \mathcal{RL}_{j\mathbb{R}}^\infty$ (resp. $\mathcal{RH}_{\mathbb{C}_+}^\infty$) and $\|\mathscr{F}_\ell(H_\beta, P)\|_\infty < 1$. Indeed, $Q := \mathscr{F}_\ell(R, P) \in \mathcal{RL}_{j\mathbb{R}}^\infty$ (resp. $\mathcal{RH}_{\mathbb{C}_+}^\infty$) and $\|Q\|_\infty < 1$, for all such $P \in \mathcal{R}$. This parametrisation can be combined with the so-called "strong-necessity" LTI ν-gap robustness results described in [9], to obtain the characterisation of the ν-gap given in Prop. 1.1. A different approach, however, is needed here, since the "standard" proof (cf. [22, Lemma 15]) of the required parametrisation fails to carry through at several points. In light of this, a new signal-based framework, which may also be useful in contexts beyond the discretisation problem considered here, is established below. ♣
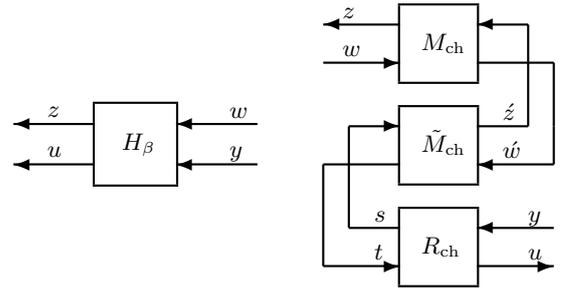
Fig. 2. Chain-Scattering representation of H_β

By virtue of the of properties M, $\tilde{M}$ and R described above, it is possible to interpret the factorisation (4) within the chain-scattering formalism [23] – see Fig. 2. Define

$$M_{\mathrm{ch}} := \begin{pmatrix} M_{12} - M_{11}M_{21}^{-1}M_{22} & M_{11}M_{21}^{-1} \\ -M_{21}^{-1}M_{22} & M_{21}^{-1} \end{pmatrix} \in \mathcal{R},$$

$$\tilde{M}_{\mathrm{ch}} := \begin{pmatrix} \tilde{M}_{12}^{-1} & -\tilde{M}_{12}^{-1}\tilde{M}_{11} \\ \tilde{M}_{22}\tilde{M}_{12}^{-1} & \tilde{M}_{21} - \tilde{M}_{22}\tilde{M}_{12}^{-1}\tilde{M}_{11} \end{pmatrix} \in \mathcal{R}$$

and

$$R_{\mathrm{ch}} := \begin{pmatrix} R_{12} - R_{11}R_{21}^{-1}R_{22} & R_{11}R_{21}^{-1} \\ -R_{21}^{-1}R_{22} & R_{21}^{-1} \end{pmatrix} \in \mathcal{R},$$

noting that $R_{\mathrm{ch}}^{-1} \in \mathcal{R}$. Since $M_{21}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ and $\tilde{M}_{12}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$, it follows that $M_{\mathrm{ch}} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ and $\tilde{M}_{\mathrm{ch}} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$. Moreover, it can be shown that $R_{\mathrm{ch}}, R_{\mathrm{ch}}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ – see Appendix C. Using the fact that M is inner and that $\tilde{M}$ is coinner, it also follows that

$$M_{\mathrm{ch}}^*(\varphi)J_1M_{\mathrm{ch}}(\varphi) = J_2 \quad \text{and} \quad \tilde{M}_{\mathrm{ch}}(\varphi)J_2\tilde{M}_{\mathrm{ch}}^*(\varphi) = J_3, \quad (5)$$

---

[3]Recall that an underline is used to denote the "time-lifted" equivalent.

for all $\varphi \in j\mathbb{R}$, where J_1, J_2 and J_3 are signature matrices of the form $\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$, partitioned conformably with M_{ch} and $\tilde{M}_{\text{ch}}$, as appropriate. To summarise, M_{ch} is J-inner, $\tilde{M}_{\text{ch}}$ is J-coinner and R_{ch} is outer. Now, given any signals $y, u \in \mathcal{L}_{j\mathbb{R}}^2$, define $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) := R_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$. Since multiplication by $\tilde{M}_{\text{ch}}$ is surjective, there exists at least one pair of signals $\acute{z}, \acute{w} \in \mathcal{L}_{j\mathbb{R}}^2$, such that $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \tilde{M}_{\text{ch}}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)$. Furthermore, the corresponding signals $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) := M_{\text{ch}}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right) \in \mathcal{L}_{j\mathbb{R}}^2$ satisfy $\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right) = H_\beta\left(\begin{smallmatrix} w \\ y \end{smallmatrix}\right)$, as illustrated in Fig. 2.
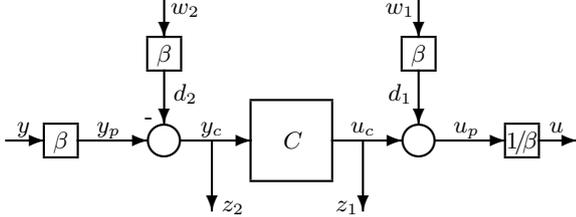

Fig. 3.   Signals in the standard feedback configuration

Fig. 3 shows the relationship between the signals w, z, u and y introduced above, and the signals in the standard feedback configuration of Fig. 1. Note, in particular, that

$$\beta\left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right) - \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right), \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) = \frac{1}{\beta}\left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right). \quad (6)$$

With this, the properties of M_{ch}, $\tilde{M}_{\text{ch}}$ and R_{ch}, give rise to the following characterisation of "plant signals" consistent with a specified level of closed-loop performance for a given $C \in \mathcal{R}$.

Theorem 3.2: Given $C \in \mathcal{R}$ and $0 < \beta < \boldsymbol{b}_{\text{opt}}(C)$, let $R \in \mathcal{R}$ (and correspondingly $R_{\text{ch}} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$) be as defined above. Then for any $\left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right) \in \mathcal{L}_{\mathbb{T}}^2$ and any $\varphi \in \mathbb{T}$, the following are equivalent:[4]

(i) $\dfrac{\left\| \left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right)(\varphi) - \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right)(\varphi) \right\|_{\mathcal{L}_{\mathbb{H}}^2}}{\left\| \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right)(\varphi) \right\|_{\mathcal{L}_{\mathbb{H}}^2}} > \beta$ for all $\left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right) \in \text{gr}(\mathbf{L}_{\underline{C}})$;

(ii) $\langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right)(\varphi), J_3 \underline{R}_{\text{ch}}\left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} < 0$, where J_3 is a matrix of the form $\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$, partitioned conformably with $\underline{R}_{\text{ch}}$.

Proof: (i)$\Rightarrow$(ii): Let $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) := \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right) := J_2 \tilde{\underline{M}}_{\text{ch}}^* J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) = \frac{1}{\beta}\left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right)$. Then $\underline{M}_{\text{ch}}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right) = \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)$, and using (5), it follows that

$$\langle \left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi), J_2\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} = \langle J_2 \tilde{\underline{M}}_{\text{ch}}^* J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi), \tilde{\underline{M}}_{\text{ch}}^* J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2}$$
$$= \langle \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi), J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2}.$$

Now note that $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \underline{M}_{\text{ch}}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)$ satisfies $\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right) = \underline{H}_\beta\left(\begin{smallmatrix} w \\ y \end{smallmatrix}\right)$. Hence, in view of (5),

$$\langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} = \langle \left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi), \underline{M}_{\text{ch}}^* J_1 \underline{M}_{\text{ch}}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2}$$
$$= \langle \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)(\varphi), J_1\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} < 0,$$

since (i) may be rewritten as $\|z(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2 < \|w(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2$ – see (6). Finally, note that

$$\langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} < 0$$
$$\Leftrightarrow \langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y_p \\ u_p \end{smallmatrix}\right)(\varphi), J_3\underline{R}_{\text{ch}}\left(\begin{smallmatrix} y_p \\ u_p \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} < 0. \quad (7)$$

(ii)$\Rightarrow$(i): Fix $z = \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right) \in \text{gr}(\mathbf{L}_{\underline{C}})$ and let $w = \frac{1}{\beta}\left[\left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right) - \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right) \right]$, $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) = \frac{1}{\beta}\left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$. Then, $\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right) = \underline{H}_\beta\left(\begin{smallmatrix} w \\ y \end{smallmatrix}\right)$ and it follows that there exists a pair of signals $\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right) \in \mathcal{L}_{\mathbb{T}}^2$ such that $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \underline{M}_{\text{ch}}\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \tilde{\underline{M}}\left(\begin{smallmatrix} \acute{w} \\ s \end{smallmatrix}\right)$ – see Figures 2 and 3. Now, since $\tilde{\underline{M}}$ is coinner it follows that $\|\acute{z}(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2 + \|t(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2 \leq \|\acute{w}(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2 +$

$\|s(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2$ and hence, in view of (5), that

$$\langle \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)(\varphi), J_1\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} = \langle \left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi), J_2\left(\begin{smallmatrix} \acute{z} \\ \acute{w} \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2}$$
$$\leq \langle \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi), J_3\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2}$$
$$= \langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} < 0,$$

where the final inequality holds because of (7) and the hypothesis that *(ii)* holds. Therefore,

$$\left\| \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right)(\varphi) \right\|_{\mathcal{L}_{\mathbb{H}}^2}^2 = \|z(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2 < \|w(\varphi)\|_{\mathcal{L}_{\mathbb{H}}^2}^2 = \frac{1}{\beta}\left\| \left(\begin{smallmatrix} u_p \\ y_p \end{smallmatrix}\right)(\varphi) - \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right)(\varphi) \right\|_{\mathcal{L}_{\mathbb{H}}^2}^2,$$

as required. $\blacksquare$

Corollary 3.3: Given $C \in \mathcal{R}$ and $0 < \beta < \boldsymbol{b}_{\text{opt}}(C)$, let $R \in \mathcal{R}$ (and correspondingly $R_{\text{ch}} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$) be as defined above. For $C_1 \in \mathcal{D}$, the following are equivalent:

(i) $\langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} \geq 0$, for all $\varphi \in \mathbb{T}$ and all $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) \in \text{gr}(\mathbf{L}_{C_1})$;

(ii) $\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\underline{C}(\varphi), C_1(\varphi)) \leq \beta$.

Proof: For any fixed $\varphi \in \mathbb{T}$, it follows by Theorem 3.2, that $\langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)(\varphi), J_3\underline{R}_{\text{ch}}\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} \geq 0$ is equivalent to the existence of a $\left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right) \in \text{gr}(\mathbf{L}_{\underline{C}})$ satisfying

$$\frac{\left\| \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)(\varphi) - \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right)(\varphi) \right\|_{\mathcal{L}_{\mathbb{H}}^2}}{\left\| \left(\begin{smallmatrix} u_c \\ y_c \end{smallmatrix}\right)(\varphi) \right\|_{\mathcal{L}_{\mathbb{H}}^2}} \leq \beta.$$

Thus, the result holds by the definition of $\boldsymbol{\kappa}(\cdot, \cdot)$ – see (2). $\blacksquare$

IV.   SD APPROXIMATION IN THE POINTWISE GAP

This section serves to present the main result of the paper. This result leads directly to a procedure for computing the gap between a given LTI controller and a particular discretisation, and a procedure for synthesising an optimal SD approximation. Note that the main result does take into account the "standard" requirement of closed-loop stability (i.e. all closed-loop transfer functions in $\mathcal{H}^\infty$).

Theorem 4.1: Given $C \in \mathcal{R}$, a pre-filtered, periodic SD controller with corresponding transfer function $C_{\text{sd}} \in \mathcal{D}$,[5] and $0 < \beta < \boldsymbol{b}_{\text{opt}}(C)$, let $R \in \mathcal{R}$ (and correspondingly $R_{\text{ch}} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$) be as defined in Sec. III and Appendix C. Then for any $P \in \mathcal{R}$, that satisfies $[P, C] \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ and $\boldsymbol{b}(P, C) > \beta$, the following are equivalent:[6]

(i) $\mathscr{F}_\ell(\underline{R}^{-1}, C_{\text{sd}}) \in \mathcal{DH}_{\mathbb{D}}^\infty$ and $\|\mathscr{F}_\ell(\underline{R}^{-1}, C_{\text{sd}})\|_\infty \leq 1$;

(ii) $\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\underline{C}(\varphi), C_{\text{sd}}(\varphi)) \leq \beta$ and $[\underline{P}, C_{\text{sd}}] \in \mathcal{DH}_{\mathbb{D}}^\infty$.

Proof: (i)$\Rightarrow$(ii): Let $F_{\text{sd}} := \mathscr{F}_\ell(\underline{R}^{-1}, C_{\text{sd}}) \in \mathcal{DH}_{\mathbb{D}}^\infty$, and given any $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) \in \text{gr}(\mathbf{L}_{C_{\text{sd}}})$, let $s := \left(\begin{smallmatrix} I & 0 \end{smallmatrix}\right)\underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{L}_{\mathbb{T}}^2$. Now, since $\|F_{\text{sd}}\|_\infty \leq 1$,

$$\langle \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi), J_3\underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} = \langle \left(\begin{smallmatrix} I \\ F_{\text{sd}} \end{smallmatrix}\right)s(\varphi), J_3\left(\begin{smallmatrix} I \\ F_{\text{sd}} \end{smallmatrix}\right)s(\varphi) \rangle_{\mathcal{L}_{\mathbb{H}}^2} \geq 0,$$

for all $\varphi \in \mathbb{T}$, where J_3 is a signature matrix partitioned conformably with $\underline{R}_{\text{ch}}$. In view of this, Corol. 3.3 implies that $\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\underline{C}(\varphi), C_{\text{sd}}(\varphi)) \leq \beta$. So it remains to show that $[\underline{P}, C_{\text{sd}}] \in \mathcal{DH}_{\mathbb{D}}^\infty$. To see this, first note that $\underline{Q} := \mathscr{F}_\ell(\underline{R}, \underline{P}) \in \mathcal{DH}_{\mathbb{D}}^\infty$ and $\|\underline{Q}\|_\infty < 1$, since $\boldsymbol{b}(P, C) > \beta$ (cf. Rem. 3.1). Then using a standard small gain argument (cf. [24]), observe that $[\underline{Q}, F_{\text{sd}}] \in \mathcal{DH}_{\mathbb{D}}^\infty$. Finally, since $\mathbf{M}_{\underline{R}_{\text{ch}}}\text{gr}(\mathbf{C}_{C_{\text{sd}}}) = \text{gr}(\mathbf{M}_{F_{\text{sd}}})$ and $\mathbf{M}_{\underline{R}_{\text{ch}}}\text{gr}(\mathbf{M}_{\underline{P}}) = \text{gr}(\mathbf{M}_{Q})$ (cf. Fig. 4), it follows by Prop. 2.1, that $[\underline{P}, C_{\text{sd}}] \in \mathcal{DH}_{\mathbb{D}}^\infty$.

(ii)$\Rightarrow$(i): Corol. 3.3 guarantees that $\mathbf{M}_{F_{\text{sd}}}$ is contractive on $\text{dom}(\mathbf{M}_{F_{\text{sd}}}) \subset \mathcal{H}_{\mathbb{D}}^2$. To see this, consult Fig. 4 and note that for any $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) \in \text{gr}(\mathbf{M}_{F_{\text{sd}}})$ there exists a $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) \in \text{gr}(\mathbf{M}_{C_{\text{sd}}}) \subset \text{gr}(\mathbf{L}_{C_{\text{sd}}})$, such that $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = \underline{R}_{\text{ch}}\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$. Now the part of *(ii)* which ensures $[\underline{P}, C_{\text{sd}}] \in \mathcal{DH}_{\mathbb{D}}^\infty$, is used to show that $\text{dom}(\mathbf{M}_{F_{\text{sd}}}) = \mathcal{H}_{\mathbb{D}}^2$, and hence, that $F_{\text{sd}} \in \mathcal{DH}_{\mathbb{D}}^\infty$ with $\|F_{\text{sd}}\|_\infty \leq 1$. Suppose $\text{dom}(\mathbf{M}_{F_{\text{sd}}}) \neq \mathcal{H}_{\mathbb{D}}^2$ and

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[4]Recall that the underline denotes "time-lifted" equivalent in $\mathcal{D}$.

[5]See the end of Section II-B for details regarding the construction of this from a time-domain representation.

[6]Again, recall that an underline denotes the "time-lifted" equivalent.

define $Q := \mathscr{F}_\ell(\underline{R}, \underline{P})$, noting that $\underline{Q} \in \mathcal{DH}_\mathbb{D}^\infty$ with $\|\underline{Q}\|_\infty < 1$, since $\boldsymbol{b}(\underline{P}, \underline{C}) > \beta$ (see Rem. 3.1). Moreover, since $\mathrm{gr}(\mathbf{M}_{C_{\mathrm{sd}}}) = \mathbf{M}_{\underline{R}_{\mathrm{ch}}^{-1}} \mathrm{gr}(\mathbf{M}_{F_{\mathrm{sd}}})$ and $\mathrm{gr}(\mathbf{M}_{\underline{P}}) = \mathbf{M}_{\underline{R}_{\mathrm{ch}}^{-1}} \mathrm{gr}(\mathbf{M}_{\underline{Q}})$, it follows by Prop. 2.1, that $[F_{\mathrm{sd}}, \underline{Q}] \in \mathcal{DH}_\mathbb{D}^\infty$. Accordingly, by the Large Gain Theorem [25], $\sup_{x \in \mathrm{dom}(\mathbf{M}_{F_{\mathrm{sd}}})} \|\underline{Q}F_{\mathrm{sd}}x\|_{\mathcal{H}_\mathbb{D}^2}/\|x\|_{\mathcal{H}_\mathbb{D}^2} \geq 1$. This, however, contradicts $\|\underline{Q}\|_\infty < 1$ and $\mathbf{M}_{F_{\mathrm{sd}}}$ being contractive on its domain. So $\mathrm{dom}(\mathbf{M}_{F_{\mathrm{sd}}}) = \mathcal{H}_\mathbb{D}^2$ must hold, as required. ∎
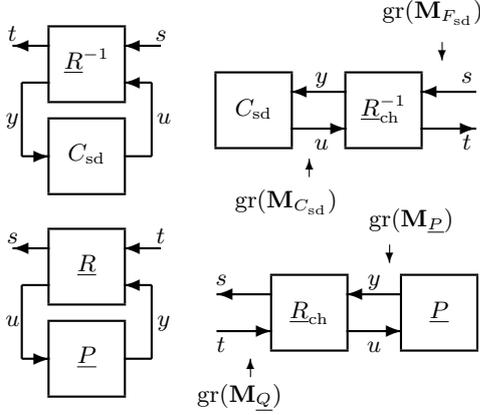

Fig. 4. $\mathrm{gr}(\mathbf{M}_{C_{\mathrm{sd}}})$, $\mathrm{gr}(\mathbf{M}_{F_{\mathrm{sd}}})$, $\mathrm{gr}(\mathbf{M}_{\underline{P}})$ and $\mathrm{gr}(\mathbf{M}_{\underline{Q}})$ relationships

Theorem 4.1 provides the required characterisation of the pointwise gap metric distance between a given LTI controller and a SD approximation, accounting for closed-loop stability with any LTI plant for which the LTI controller achieves a certain level of closed-loop performance. Indeed, it gives rise to the following two procedures for analysis and synthesis.

Procedure 1: **(Analysis)** Given a strictly proper LTI controller C and a pre-filtered periodic SD controller C_{sd}, find the smallest $\beta < \boldsymbol{b}_{\mathrm{opt}}(C)$ such that $\mathscr{F}_\ell(\underline{R}^{-1}, C_{\mathrm{sd}}) \in \mathcal{DH}_\mathbb{D}^\infty$ and $\|\mathscr{F}_\ell(\underline{R}^{-1}, C_{\mathrm{sd}})\|_\infty \leq 1$, where R is defined in (16) of Appendix C. By Theorem 4.1, this smallest value for β bounds the gap between C and C_{sd}. Finding the smallest β can be achieved using established $\mathcal{H}^\infty$ SD analysis methods [11], [12], [13].

Procedure 2: **(Synthesis)** Given a strictly proper LTI controller C, a fixed sample rate and an anti-aliasing filter F, find the smallest $\beta < \boldsymbol{b}_{\mathrm{opt}}(C)$, such that there exists a SD controller $C_{\mathrm{sd}} = \mathscr{H}C_{\mathrm{d}}\mathscr{S}F$ satisfying $\mathscr{F}_\ell(\underline{R}^{-1}, C_{\mathrm{sd}}) \in \mathcal{DH}_\mathbb{D}^\infty$ and $\|\mathscr{F}_\ell(\underline{R}^{-1}, C_{\mathrm{sd}})\|_\infty \leq 1$, where $\mathscr{S}$ is an ideal sampler, $\mathscr{H}$ is a zero-order hold and R is defined in (16) of Appendix C. The value for β, and the corresponding optimal C_{sd}, can be obtained using established $\mathcal{H}^\infty$ SD synthesis methods [11], [12], [13]. Note that the state-space dimension of C_{d} is at most that of C plus that of F. Finally, observe that the SD approximation obtained would only be useful if the smallest β achieved is less than $\boldsymbol{b}(P, C)$. If not, the fixed pre-filter and/or sampling period should be re-designed.

V. Concluding Example

Consider the LTI plant $P(s) = \frac{3}{s}$ and an LTI controller $C(s) = \frac{-5.25(s+1)^2}{s(s+1.5)(s+3.5)}$, for which $\boldsymbol{b}(P, C) = 0.312$. Figure 5 shows the frequency response of the discrete-time component of three SD approximations of C, for a fixed anti-aliasing filter $F = \frac{10}{s+10}$ and sampling frequency of 60rad/sec. The discretisations shown are the result of: (*i*) Proc. 2; (*ii*) Taking the Cayley transform ($z = \frac{1+sh/2}{1-sh/2}$ with h the sampling period) of $C(s)$; (*iii*) Taking the Cayley transform of $\hat{C}(s) = \frac{-5.25(s+1)^2(0.1s+1)}{s(s+1.5)(s+3.5)}$ (i.e. C with F factored out). The resulting closed-loop characteristics are summarised in Figure 6 and Table I, in which $\Delta := T_1(\underline{P}, \underline{C}) - T_1(\underline{P}, C_{\mathrm{sd}})$.

It can be seen that the discretisations obtained via approaches (*i*) and (*iii*) are very similar. Of note though, is the slightly lower gain (cf. Fig. 5) obtained via (*i*). This seems to result in significantly less degradation of $\boldsymbol{b}(P, C)$, at the cost of a slightly greater difference between nominal closed-loop performance, and that achieved with the SD controller. It would be interesting to investigate the use of appropriate weights (perhaps related to P) to direct the gap approximation procedure, in order to maintain a handle on this phenomenon. The example also demonstrates that F should be factored out of C (if possible), before discretising via a bilinear transform. The anti-aliasing filter is automatically taken into account in approach (*i*).

The closed-loop characteristics for the discretisations obtained when the sampling frequency is increased to 500 rad/sec (with the same fixed F), are summarised in Table II. Note that near nominal closed-loop characteristics are recovered via approaches (*i*) and (*iii*), whereas the discretisation obtained via (*ii*) is still relatively poor. In fact, for this sampling frequency, the closed-loop step responses for the discretisations obtained via (*i*) and (*iii*) are identical to the LTI controlled case (see Figure 6).
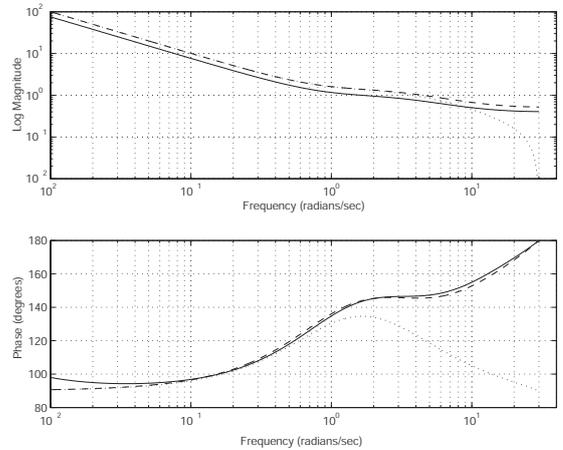
Fig. 5. C_{d} Frequency responses – samp. rate 60 rad/sec: (solid) gap approximation; (dot) Cayley transform without factoring out the aa-filter; (dash) Cayley transform first factoring out the aa-filter

Approach	$\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\underline{C}, C_{\mathrm{sd}})$	$\boldsymbol{b}(\underline{P}, C_{\mathrm{sd}})$	$\|\Delta\|_\infty$
(*i*)	0.295	0.276	2.06
(*ii*)	0.474	0.085	9.34
(*iii*)	0.306	0.221	1.72

TABLE I

Closed-Loop characteristics – samp. rate 60 rad/sec

Approach	$\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(\underline{C}, C_{\mathrm{sd}})$	$\boldsymbol{b}(\underline{P}, C_{\mathrm{sd}})$	$\|\Delta\|_\infty$
(*i*)	0.040	0.312	0.306
(*ii*)	0.316	0.156	3.67
(*iii*)	0.042	0.302	0.157

TABLE II

Closed-Loop characteristics – samp. rate 500 rad/sec

Appendix A
Normalised coprime factorisations and graph symbols

Given a realisation $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ of a function $C \in \mathcal{R}^{p,m}$ (resp. $\mathcal{D}^{p,m}$), that is stabilisable and detectable,[7] there exists (by definition) an

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[7] Such realisations can be constructed from any realisation in the usual way via a Kalman Decomposition – see [24, Sec. 3.3] for $C \in \mathcal{R}^{p,m}$ and [26, Sec. 2.4.1] for $C \in \mathcal{D}^{p,m}$.
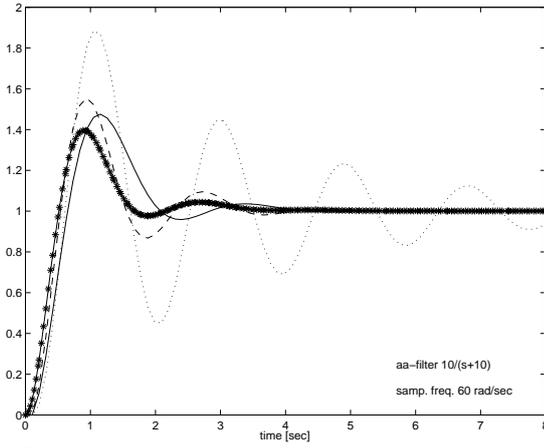
Fig. 6. Closed-Loop step responses: (solid) gap approximation; (dot) Cayley transform without factoring out the aa-filter; (dash) Cayley transform first factoring out the aa-filter; (star) Continuous-time LTI control

$\mathsf{F} \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^m)$ (resp. $\mathcal{B}(\mathbb{R}^n, \mathcal{L}_{\mathbb{H}}^{2,m})$) and an $\mathsf{L} \in \mathcal{B}(\mathbb{R}^p, \mathbb{R}^n)$ (resp. $\mathcal{B}(\mathcal{L}_{\mathbb{H}}^{2,p}, \mathbb{R}^n)$), such that $\mathrm{spec}(\mathsf{A} + \mathsf{BF}) \subset \mathbb{C}_-$ (resp. $\mathrm{spec}(\mathsf{A} + \mathsf{BF}) \subset \mathbb{D}$) and $\mathrm{spec}(\mathsf{A} + \mathsf{LC}) \subset \mathbb{C}_-$ (resp. $\mathrm{spec}(\mathsf{A} + \mathsf{LC}) \subset \mathbb{D}$). Defining

$$\begin{pmatrix} D & -Y \\ N & X \end{pmatrix} := \left(\begin{array}{c|cc} \mathsf{A} + \mathsf{BF} & \mathsf{BV} & -\mathsf{LS}^{-1} \\ \hline \mathsf{F} & \mathsf{V} & 0 \\ \mathsf{C} + \mathsf{DF} & \mathsf{DV} & \mathsf{S}^{-1} \end{array} \right) \in \mathcal{RH}_{\mathbb{C}_+}^\infty \quad (8)$$
$$(\text{resp. } \mathcal{DH}_{\mathbb{D}}^\infty)$$

and

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{N} & \tilde{D} \end{pmatrix} := \left(\begin{array}{c|cc} \mathsf{A} + \mathsf{LC} & -(\mathsf{B} + \mathsf{LD}) & \mathsf{L} \\ \hline \mathsf{V}^{-1}\mathsf{F} & \mathsf{V}^{-1} & 0 \\ \mathsf{SC} & -\mathsf{SD} & \mathsf{S} \end{array} \right) \in \mathcal{RH}_{\mathbb{C}_+}^\infty \quad (9)$$
$$(\text{resp. } \mathcal{DH}_{\mathbb{D}}^\infty),$$

for invertible $\mathsf{S} \in \mathcal{B}(\mathbb{R}^p, \mathbb{R}^p)$ (resp. $\mathsf{S} \in \mathcal{B}(\mathcal{L}_{\mathbb{T}}^{2,p}, \mathcal{L}_{\mathbb{T}}^{2,p})$) and $\mathsf{V} \in \mathcal{B}(\mathbb{R}^m, \mathbb{R}^m)$ (resp. $\mathsf{V} \in \mathcal{B}(\mathcal{L}_{\mathbb{H}}^{2,m}, \mathcal{L}_{\mathbb{H}}^{2,m})$), it follows that $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ are coprime factorisations, in that

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{N} & \tilde{D} \end{pmatrix} \begin{pmatrix} D & -Y \\ N & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Moreover, an appropriate choice of $\mathsf{F}, \mathsf{L}, \mathsf{V}, \mathsf{S}$ yields normalised factors, in the sense that $\binom{N}{D}$ is inner and $\begin{pmatrix} -\tilde{D} & \tilde{N} \end{pmatrix}$ is coinner. In particular, let $\mathsf{R} := (I + \mathsf{D}^*\mathsf{D})$ and $\tilde{\mathsf{R}} := (I + \mathsf{DD}^*)$, and define $\mathsf{F} := -\mathsf{R}^{-1}(\mathsf{B}^*\mathsf{X} + \mathsf{D}^*\mathsf{C})$ (resp. $\mathsf{F} := -(\mathsf{R} + \mathsf{B}^*\mathsf{XB})^{-1}(\mathsf{B}^*\mathsf{XA} + \mathsf{D}^*\mathsf{C})$) and $\mathsf{L} := -(\mathsf{BD}^* + \mathsf{YC}^*)\tilde{\mathsf{R}}^{-1}$ (resp. $\mathsf{L} := -(\mathsf{BD}^* + \mathsf{AYC}^*)(\tilde{\mathsf{R}} + \mathsf{CYC}^*)$), where $0 \leq \mathsf{X} = \mathsf{X}^* \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^n)$ is the stabilising solution to the continuous-time algebraic Riccati equation

$$(\mathsf{A} - \mathsf{BR}^{-1}\mathsf{D}^*\mathsf{C})^*\mathsf{X} + \mathsf{X}(\mathsf{A} - \mathsf{BR}^{-1}\mathsf{D}^*\mathsf{C})$$
$$- \mathsf{XBR}^{-1}\mathsf{B}^*\mathsf{X} + \mathsf{C}^*\tilde{\mathsf{R}}^{-1}\mathsf{C} = 0,$$

which satisfies $\mathrm{spec}(\mathsf{A} + \mathsf{BF}) \subset \mathbb{C}_-$ by definition (resp. the stabilising solution to the discrete-time algebraic Riccati equation[8]

$$\mathsf{X} = (\mathsf{A} - \mathsf{BR}^{-1}\mathsf{D}^*\mathsf{C})^*\mathsf{X}(I + \mathsf{BR}^{-1}\mathsf{B}^*\mathsf{X})^{-1}(\mathsf{A} - \mathsf{BR}^{-1}\mathsf{D}^*\mathsf{C})$$
$$+ \mathsf{C}^*\tilde{\mathsf{R}}^{-1}\mathsf{C},$$

which satisfies $\mathrm{spec}(\mathsf{A} + \mathsf{BF}) \subset \mathbb{D}$ by definition), and $0 \leq \mathsf{Y} = \mathsf{Y}^* \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^n)$ is the solution to the continuous-time algebraic Riccati

equation

$$(\mathsf{A} - \mathsf{BD}^*\tilde{\mathsf{R}}^{-1}\mathsf{C})\mathsf{Y} + \mathsf{Y}(\mathsf{A} - \mathsf{BD}^*\tilde{\mathsf{R}}^{-1}\mathsf{C})^*$$
$$- \mathsf{YC}^*\tilde{\mathsf{R}}^{-1}\mathsf{CY} + \mathsf{BR}^{-1}\mathsf{B}^* = 0,$$

that satisfies $\mathrm{spec}(\mathsf{A} + \mathsf{LC}) \subset \mathbb{C}_-$ (resp. the solution to the discrete-time algebraic Riccati equation

$$\mathsf{Y} = (\mathsf{A} - \mathsf{BD}^*\tilde{\mathsf{R}}^{-1}\mathsf{C})\mathsf{Y}(I + \mathsf{C}^*\tilde{\mathsf{R}}^{-1}\mathsf{CY})^{-1}(\mathsf{A} - \mathsf{BD}^*\tilde{\mathsf{R}}^{-1}\mathsf{C})^*$$
$$+ \mathsf{BR}^{-1}\mathsf{B}^*,$$

that satisfies $\mathrm{spec}(\mathsf{A} + \mathsf{BF}) \subset \mathbb{D}$).[9] Finally, setting $\mathsf{V} := \mathsf{R}^{-\frac{1}{2}}$ (resp. $\mathsf{V} := (\mathsf{R} + \mathsf{B}^*\mathsf{XB})^{-\frac{1}{2}}$) and $\mathsf{S} := \tilde{\mathsf{R}}^{-\frac{1}{2}}$ (resp. $\mathsf{S} := (\tilde{\mathsf{R}} + \mathsf{CYC}^*)^{-\frac{1}{2}}$) it follows that $\binom{N}{D}$ and $\begin{pmatrix} -\tilde{D} & \tilde{N} \end{pmatrix}$, defined in (8) and (9), are inner and coinner, respectively – See [24, Corol. 13.29] for $C \in \mathcal{R}^{p,m}$ and [26, Lemma 5.4] for $C \in \mathcal{D}^{p,m}$.

Given coprime factorisations $C = ND^{-1} = \tilde{D}^{-1}\tilde{N} \in \mathcal{R}^{p,m}$ (resp. $\mathcal{D}^{p,m}$), defining

$$K := \begin{pmatrix} N \\ D \end{pmatrix} \in \mathcal{RH}_{\mathbb{C}_+}^\infty \text{ (resp. } \mathcal{DH}_{\mathbb{D}}^\infty)$$

and

$$\tilde{K} := \begin{pmatrix} -\tilde{D} & \tilde{N} \end{pmatrix} \in \mathcal{RH}_{\mathbb{C}_+}^\infty \text{ (resp. } \mathcal{DH}_{\mathbb{D}}^\infty),$$

it follows that $\mathrm{gr}(\mathbf{M}_C) = \mathrm{ran}(\mathbf{M}_K) = \ker(\mathbf{M}_{\tilde{K}}) \subset \mathcal{H}_{\mathbb{C}_+}^2$ (resp. $\mathcal{H}_{\mathbb{D}}^2$) and $\mathrm{gr}(\mathbf{L}_C) = \mathrm{ran}(\mathbf{L}_K) = \ker(\mathbf{L}_{\tilde{K}}) \subset \mathcal{L}_{j\mathbb{R}}^2$ (resp. $\mathcal{L}_{\mathbb{T}}^2$). Similarly, for almost all[10] frequencies $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), $\mathrm{ran}(K(\varphi)) = \ker(\tilde{K}(\varphi)) \subset \mathbb{C}^{m+p}$ (resp. $\mathcal{L}_{\mathbb{H}}^{2,m+p}$). The functions K and $\tilde{K}$ are called right and left graph symbols, which are said to be normalised if $K^*K = I$ and $\tilde{K}\tilde{K}^* = I$. When K and $\tilde{K}$ are normalised,

$$\begin{pmatrix} K^* \\ \tilde{K} \end{pmatrix} \begin{pmatrix} K & \tilde{K}^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (10)$$

Correspondingly, $\mathrm{ran}(\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}) = \mathbb{C}^{m+p}$ (resp. $\mathcal{L}_{\mathbb{H}}^{2,m+p}$) at any $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$). Moreover, for almost all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), $\ker(\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}) = \ker(K(\varphi)^*) \cap \ker(\tilde{K}(\varphi)) = \mathrm{ran}(K(\varphi))^\perp \cap \mathrm{ran}(K(\varphi)) = \{0\}$, where $\perp$ denotes the orthogonal complement of a Hilbert space.[11] So $\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}$ is bijective for almost all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), and hence, in light of (10), it follows that

$$\begin{pmatrix} K(\varphi) & \tilde{K}^*(\varphi) \end{pmatrix} \begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix} = I. \quad (11)$$

In fact this is true for all $\varphi \in j\mathbb{R}$ (resp. $\mathbb{T}$), since K and $\tilde{K}$ are continuous on $j\mathbb{R}$ (resp. $\mathbb{T}$).[12]

APPENDIX B
EQUIVALENCE OF THE POINTWISE AND $\mathcal{L}_{\mathbb{T}}^2$-GAPS

Given Hilbert space $\mathcal{V}$ and bounded operator $\mathbf{X} \in \mathcal{B}(\mathcal{V}, \mathcal{V})$, let $\boldsymbol{\mu}(\mathbf{X}) = \sup_{\|u\|_{\mathcal{V}_1} = 1} \|\mathbf{X}u\|_{\mathcal{V}_2}$ denote the induced norm, and

$$\boldsymbol{\tau}(\mathbf{X}) := \inf_{\|u\|_{\mathcal{V}_1} = 1} \|\mathbf{X}u\|_{\mathcal{V}_2}$$

denote the minimum modulus.

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[8]Note that this is a standard *finite-dimensional* Riccati equation, since $(\mathsf{A} - \mathsf{BR}^{-1}\mathsf{D}^*\mathsf{C}) \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^n)$, $\mathsf{BR}^{-1}\mathsf{B}^* \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathsf{C}^*\tilde{\mathsf{R}}^{-1}\mathsf{C} \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^n)$.

[9]The required solutions X and Y exist since $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ is a stabilisable and detectable realisation. See [24, Corol. 13.8] for $C \in \mathcal{R}^{p,m}$ and [26, Prop. 2.20] for $C \in \mathcal{D}^{p,m}$.

[10]Except for possibly finitely many points at which $D(\varphi)$ (or $\tilde{D}(\varphi)$) may not be boundedly invertible; i.e. at poles of C on $j\mathbb{R}$ (resp. $\mathbb{T}$).

[11]Note that $\mathrm{cl}(\mathrm{ran}(K(\varphi))) = \mathrm{ran}(K(\varphi))$ since $K(\varphi)$ is left-invertible.

[12]In the $C \in \mathcal{R}$ case, such delicate arguments are not required, since at each frequency $\begin{pmatrix} K^*(\varphi) \\ \tilde{K}(\varphi) \end{pmatrix}$ is square and finite-dimensional, which combined with (10) is enough to imply that it is unitary.

Lemma 1: If $\left(\begin{smallmatrix}\mathbf{A}\\\mathbf{B}\end{smallmatrix}\right) \in \mathcal{B}(\mathcal{V}, \mathcal{V} \times \mathcal{V})$ is an isometry, then $\boldsymbol{\mu}(\mathbf{B})^2 = 1 - \boldsymbol{\tau}(\mathbf{A})^2$. Similarly, if $\left(\tilde{\mathbf{A}} \quad \tilde{\mathbf{B}}\right) \in \mathcal{B}(\mathcal{V} \times \mathcal{V}, \mathcal{V})$ is a coisometry and $\boldsymbol{\tau}(\tilde{\mathbf{A}}) = \boldsymbol{\tau}(\tilde{\mathbf{A}}^*)$, then $\boldsymbol{\mu}(\tilde{\mathbf{B}})^2 = 1 - \boldsymbol{\tau}(\tilde{\mathbf{A}})^2$.

Proof: If $\left(\begin{smallmatrix}\mathbf{A}\\\mathbf{B}\end{smallmatrix}\right)$ is an isometry, it follows that for any $u \in \mathcal{V}$,

$$\langle u, u \rangle_{\mathcal{V}} = \langle \mathbf{B}u, \mathbf{B}u \rangle_{\mathcal{V}} + \langle \mathbf{A}u, \mathbf{A}u \rangle_{\mathcal{V}}, \quad (12)$$

by which it is immediate that $\boldsymbol{\mu}(\mathbf{B})^2 = 1 - \boldsymbol{\tau}(\mathbf{A})^2$. The coisometry case follows similarly, on noting that $\left(\begin{smallmatrix}\tilde{\mathbf{A}}^*\\\tilde{\mathbf{B}}^*\end{smallmatrix}\right)$ is an isometry and $\boldsymbol{\mu}(\tilde{\mathbf{B}}^*) = \boldsymbol{\mu}(\tilde{\mathbf{B}})$. ∎

Given C_1 and $C_2 \in \mathcal{D}$, with normalised right and left graph symbols K_1 and K_2, and $\tilde{K}_1$ and $\tilde{K}_2$, respectively, define

$$\begin{aligned}
\boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi)) &:= \boldsymbol{\mu}(\tilde{K}_1 K_2(\varphi)) \\
&= \boldsymbol{\mu}(\tilde{K}_2 K_1(\varphi)) \\
&=: \boldsymbol{\kappa}(C_2(\varphi), C_1(\varphi)). \quad (13)
\end{aligned}$$

The second equality follows by Lemma 1, since the operator $K_1^*\left(K_2 \quad \tilde{K}_2^*\right)(\varphi)$ is a coisometry and $\left(\begin{smallmatrix}K_1^*\\\tilde{K}_1\end{smallmatrix}\right) K_2(\varphi)$ is an isometry (see (10) and (11)) for all $\varphi \in \mathbb{T}$,[13] and therefore,

$$\begin{aligned}
\boldsymbol{\mu}(\tilde{K}_2 K_1(\varphi)) &= \boldsymbol{\mu}(K_1^* \tilde{K}_2^*(\varphi)) \\
&= \sqrt{1 - \boldsymbol{\tau}(K_1^* K_2(\varphi))^2} \\
&= \boldsymbol{\mu}(\tilde{K}_1 K_2(\varphi)). \quad (14)
\end{aligned}$$

Moreover, since the $K_i(\varphi)$ and $\tilde{K}_i(\varphi)$ are isometries and coisometries, respectively,

$$\begin{aligned}
\boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi)) &:= \boldsymbol{\mu}(\tilde{K}_1 K_2(\varphi)) \\
&= \boldsymbol{\mu}(\tilde{K}_1^* \tilde{K}_1 K_2 K_2^*(\varphi)) \\
&= \|\mathbf{\Pi}_{\mathrm{gr}(C_1(\varphi))^\perp} \mathbf{\Pi}_{\mathrm{gr}(C_2(\varphi))}\| \\
&= \sup_{\substack{x \in \mathrm{gr}(C_1(\varphi)) \\ x \neq 0}} \inf_{\substack{y \in \mathrm{gr}(C_2(\varphi)) \\ y \neq 0}} \frac{\|x - y\|_{\mathcal{L}_{\mathbb{H}}^2}}{\|x\|_{\mathcal{L}_{\mathbb{H}}^2}}.
\end{aligned}$$

It can be shown that $\boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi)) = 0$ if, and only if, $C_1(\varphi) = C_2(\varphi)$, and that $\boldsymbol{\kappa}(C_1(\varphi), C_3(\varphi)) \leq \boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi)) + \boldsymbol{\kappa}(C_2(\varphi), C_3(\varphi))$ for any other $C_3 \in \mathcal{D}$ [8], [9]. As such, with (2), it follows that $\boldsymbol{\kappa}(\cdot, \cdot)$ is a metric pointwise in frequency.

The relationship between the pointwise metric just described and the $\mathcal{L}_{\mathbb{T}}^2$-gap can now be established. In particular, recall that the $\mathcal{L}_{\mathbb{T}}^2$-gap between two systems with transfer functions C_1 and $C_2 \in \mathcal{D}$ is defined to be the gap (or aperture – see [18] for example) between $\mathcal{K}_1 := \mathrm{gr}(\mathbf{L}_{C_1})$ and $\mathcal{K}_2 := \mathrm{gr}(\mathbf{L}_{C_2})$ [9], [19]:

$$\begin{aligned}
\mathrm{gap}(\mathcal{K}_1, \mathcal{K}_2) &:= \|\mathbf{\Pi}_{\mathcal{K}_1} - \mathbf{\Pi}_{\mathcal{K}_2}\| \\
&= \max_{i,j \in \{1,2\}} \vec{\mathrm{gap}}(\mathcal{K}_i, \mathcal{K}_j), \quad (15)
\end{aligned}$$

where $\mathbf{\Pi}_{\mathcal{K}_i}$ denotes the orthogonal projection onto $\mathcal{K}_i$ and

$$\vec{\mathrm{gap}}(\mathcal{K}_i, \mathcal{K}_j) := \|\mathbf{\Pi}_{\mathcal{K}_j^\perp} \mathbf{\Pi}_{\mathcal{K}_i}\| = \sup_{x \in \mathcal{K}_i, x \neq 0} \inf_{y \in \mathcal{K}_j} \frac{\|x - y\|_{\mathcal{L}_{\mathbb{T}}^2}}{\|x\|_{\mathcal{L}_{\mathbb{T}}^2}}.$$

Now, since $K_i K_i^* K_i q = K_i q$ for any $q \in \mathcal{L}_{\mathbb{T}}^2$ and $K_i K_i^* + \tilde{K}_i^* \tilde{K}_i = I$ see (10) and (11)) it follows that the projections required to calculate the directed gaps can be expressed in terms of the normalised right and left graph symbols K_i and $\tilde{K}_i$, respectively,

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[13]To have $\boldsymbol{\tau}(K_1^* K_2(\varphi)) = \boldsymbol{\tau}(K_2^* K_1(\varphi))$, it is sufficient for the D-terms of C_1 and C_2 to be compact operators. In view of the realizations in Appendix A, this implies $K_2^* K_1(\varphi)$ is Fredholm with index 0 (as a compact perturbation of an invertible operator), and therefore, it is injective iff it is surjective, whereby the minimum modulus equality holds. The compactness property is satisfied in the case of $\mathbf{W}$-transformed transfer functions from $\mathcal{R}$, and the transfer fuction of a sampled-data controller shown in Section II-B.

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of C_i. Indeed, $\mathbf{\Pi}_{\mathcal{K}_i} = \mathbf{L}_{K_i K_i^*}$ and $\mathbf{\Pi}_{\mathcal{K}_i^\perp} = \mathbf{L}_{\tilde{K}_i^* \tilde{K}_i}$. In light of this, it follows that

$$\begin{aligned}
\vec{\mathrm{gap}}(\mathcal{K}_i, \mathcal{K}_j) &= \|\tilde{K}_j^* \tilde{K}_j K_i K_i^*\|_\infty \\
&= \|\tilde{K}_j K_i\|_\infty \\
&= \|\tilde{K}_i K_j\|_\infty \\
&= \|\tilde{K}_i^* \tilde{K}_i K_j K_j^*\|_\infty \\
&= \vec{\mathrm{gap}}(\mathcal{K}_j, \mathcal{K}_i),
\end{aligned}$$

where the third equality holds because of (14). Hence, by the definition of $\boldsymbol{\kappa}(\cdot, \cdot)$, and in view of (15), the following relationship holds:

$$\sup_{\varphi \in \mathbb{T}} \boldsymbol{\kappa}(C_1(\varphi), C_2(\varphi)) = \|\tilde{K}_1 K_2\|_\infty = \mathrm{gap}(\mathcal{K}_1, \mathcal{K}_2).$$

APPENDIX C
INNER-COINNER-OUTER FACTORISATION

Details of the factorisation (4) are given here in the notation of [22]. Suppose $C \in \mathcal{R}$ has a stabilisable and detectable realisation $\left(\begin{smallmatrix}\mathsf{A} & \mathsf{B}\\\mathsf{C} & \mathsf{D}\end{smallmatrix}\right)$. Furthermore, let $\mathsf{X} = \mathsf{X}^* \geq 0$ be the stabilising solution to the generalised control Riccati equation (GCARE)

$$\mathsf{A}^* \mathsf{X} + \mathsf{X}\mathsf{A} - \mathsf{X}\mathsf{B}\mathsf{B}^*\mathsf{X} + \mathsf{C}^*\mathsf{C} = 0,$$

and $\mathsf{Z} = \mathsf{Z}^* \geq 0$ be the stabilising solution to generalised filtering Riccati equation (GFARE)[14]

$$\mathsf{A}\mathsf{Z} + \mathsf{Z}\mathsf{A}^* - \mathsf{Z}\mathsf{C}^*\mathsf{C}\mathsf{Z} + \mathsf{B}\mathsf{B}^* = 0.$$

Let $\gamma := \frac{1}{\beta} \geq \frac{1}{\boldsymbol{b}_{\mathrm{opt}}(C)} = \sqrt{1 + \mathrm{rad}(\mathsf{X}\mathsf{Z})}$ and define:

(i) $M := \left(\begin{array}{c|cc|c} \mathsf{A} + \mathsf{B}\mathsf{F}_\infty & \left(0 \quad \frac{1}{\gamma}\mathsf{B}\right) & \frac{\sqrt{\gamma^2-1}}{\gamma}\mathsf{B} \\ \hline \begin{pmatrix}\mathsf{C}\\\mathsf{F}_\infty\end{pmatrix} & \begin{pmatrix}0 & 0\\0 & \frac{1}{\gamma}I\end{pmatrix} & \begin{pmatrix}0\\\frac{\sqrt{\gamma^2-1}}{\gamma}I\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}I & 0\\0 & \frac{\sqrt{\gamma^2-1}}{\gamma}I\end{pmatrix} & \begin{pmatrix}0\\-\frac{1}{\gamma}I\end{pmatrix} \end{array}\right)$, where $\mathsf{F}_\infty := -\mathsf{B}^*\mathsf{X}$;

(ii) $\tilde{M} := \left(\begin{array}{c|cc|c} \mathsf{A}_{\mathrm{tmp}} + \gamma \mathsf{L}_{\mathrm{tmp}}\mathsf{C} & \left(-\mathsf{L}_{\mathrm{tmp}} \quad \frac{1}{\sqrt{\gamma^2-1}}\mathsf{B}\right) & \frac{\gamma}{\sqrt{\gamma^2-1}}\mathsf{Y}_{\mathrm{tmp}}\mathsf{F}_\infty^* \\ \hline \frac{-\gamma}{\sqrt{\gamma^2-1}} & (0 \quad 0) & (-I \quad 0) \\ \gamma\mathsf{C} & I & 0 \end{array}\right)$,

where $\mathsf{A}_{\mathrm{tmp}} := \mathsf{A} - \frac{1}{\gamma^2-1}\mathsf{B}\mathsf{F}_\infty$, $\mathsf{L}_{\mathrm{tmp}} := -\gamma\mathsf{Y}_{\mathrm{tmp}}\mathsf{C}^*$, $\mathsf{Y}_{\mathrm{tmp}} = \mathsf{Y}_\infty(I - \mathsf{X}\mathsf{Y}_\infty)^{-1} \geq 0$, $\mathsf{Y}_\infty := \frac{1}{\gamma^2-1}\mathsf{Z}$; and

(iii)

$$R := \left(\begin{array}{c|cc} \mathsf{A}_R & -\mathsf{L}_{\mathrm{tmp}} & \mathsf{B}_{2R} \\ \hline \frac{-\gamma}{\sqrt{\gamma^2-1}}\mathsf{F}_\infty & 0 & \frac{1}{\sqrt{\gamma^2-1}}I \\ \gamma\mathsf{C} & I & 0 \end{array}\right) \in \mathcal{R}, \quad (16)$$

where $\mathsf{A}_R := \mathsf{A}_{\mathrm{tmp}} + \frac{\gamma^2}{\gamma^2-1}\mathsf{Y}_{\mathrm{tmp}}\mathsf{F}_\infty^*\mathsf{F}_\infty$ and $\mathsf{B}_{2R} := \frac{\gamma}{\gamma^2-1}\mathsf{B} - \frac{\gamma}{\gamma^2-1}\mathsf{Y}_{\mathrm{tmp}}\mathsf{F}_\infty^*$.

That, $H_\beta = M \star \tilde{M} \star R$ follows directly by substitution. It is now shown that:

(i) $M \in \mathcal{R}\mathcal{H}_{\mathbb{C}_+}^\infty$ is inner, with $M_{12}^{-1} \in \mathcal{R}\mathcal{H}_{\mathbb{C}_+}^\infty$;
(ii) $\tilde{M} \in \mathcal{R}\mathcal{H}_{\mathbb{C}_+}^\infty$ is coinner, with $\tilde{M}_{21}^{-1} \in \mathcal{R}\mathcal{H}_{\mathbb{C}_+}^\infty$; and
(iii) $R_{12}^{-1}, R_{21}^{-1}, R_{\mathrm{ch}}, R_{\mathrm{ch}}^{-1} \in \mathcal{R}\mathcal{H}_{\mathbb{C}_+}^\infty$.

Since $\mathsf{X} \geq 0$ is the stabilising solution to GCARE, it follows that

$$\mathrm{spec}(\mathsf{A} - \mathsf{B}\mathsf{B}^*\mathsf{X}) = \mathrm{spec}(\mathsf{A} + \mathsf{B}\mathsf{F}_\infty) \subset \mathbb{C}_-$$

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[14]The required stabilising solutions X and Z exist since $(\mathsf{A}, \mathsf{B}, \mathsf{C})$ is stabilisable and detectable [24, Corol. 13.8]

and hence, that $M \in \mathcal{RH}_{\mathbb{C}_+}^\infty$. Defining $\mathsf{A}_{\mathsf{F}_\infty} := \mathsf{A} + \mathsf{B}\mathsf{F}_\infty$ and rearranging GCARE to obtain

$$\mathsf{A}_{\mathsf{F}_\infty}^* \mathsf{X} + \mathsf{X}\mathsf{A}_{\mathsf{F}_\infty} + \mathsf{C}^*\mathsf{C} + \mathsf{F}_\infty^*\mathsf{F}_\infty = 0, \tag{17}$$

it also follows that X is the Observability Gramian of M. Furthermore,

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma}I \end{pmatrix} \quad \begin{pmatrix} I & 0 \\ 0 & \frac{\sqrt{\gamma^2-1}}{\gamma}I \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{\sqrt{\gamma^2-1}}{\gamma}I \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{1}{\gamma}I \end{pmatrix} \right) \begin{pmatrix} \begin{pmatrix} \mathsf{C} \\ \mathsf{F}_\infty \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\gamma}\mathsf{B}^* \end{pmatrix} \\ \frac{\sqrt{\gamma^2-1}}{\gamma}\mathsf{B}^* \end{pmatrix}\mathsf{X} = 0,$$

and therefore, by [24, Lemma 13.29], M is inner. Finally, note that

$$M_{21}^{-1} = \left(\begin{array}{c|c} \mathsf{A} + \mathsf{B}\mathsf{F}_\infty & \bullet \\ \hline \bullet & \bullet \end{array} \right),$$

and hence, that $M_{21}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$.

Now consider

$$W := \left(\begin{array}{c|cc} \mathsf{A}_{\mathrm{tmp}}^* + \gamma\mathsf{C}^*\mathsf{L}_{\mathrm{tmp}}^* & \frac{-\gamma}{\sqrt{\gamma^2-1}}\mathsf{F}_\infty^* & \gamma\mathsf{C}^* \\ \hline \begin{pmatrix} -\mathsf{L}_{\mathrm{tmp}}^* \\ \frac{1}{\sqrt{\gamma^2-1}}\mathsf{B}^* \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} -I \\ 0 \end{pmatrix} \\ \frac{\gamma}{\sqrt{\gamma^2-1}}\mathsf{F}_\infty\mathsf{Y}_{\mathrm{tmp}} & I & 0 \end{array} \right)$$

and note that W is inner if, and only if, $\tilde{M}$ is co-inner, since $W \in \mathcal{RH}_{\mathbb{C}_+}^\infty \Leftrightarrow \tilde{M} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ and $W(\bar{\varphi})^* W(\bar{\varphi}) = \tilde{M}(\varphi)\tilde{M}^*(\varphi)$ for any $\varphi \in j\mathbb{R}$, where the bar denotes complex conjugate. Now define $\mathsf{Y}_\infty = \frac{1}{\gamma^2-1}\mathsf{Z}$ and rearrange GFARE to obtain

$$\mathsf{A}\mathsf{Y}_\infty + \mathsf{Y}_\infty\mathsf{A}^* + \frac{1}{\gamma^2-1}\mathsf{B}\mathsf{B}^* - (\gamma^2-1)\mathsf{Y}_\infty\mathsf{C}^*\mathsf{C}\mathsf{Y}_\infty = 0. \tag{18}$$

Applying the similarity transformation $\begin{pmatrix} I & -\mathsf{X} \\ 0 & I \end{pmatrix}$ to the Hamiltonian

$$\begin{pmatrix} \mathsf{A}^* & -(\gamma^2-1)\mathsf{C}^*\mathsf{C} \\ \frac{-1}{\gamma^2-1}\mathsf{B}\mathsf{B}^* & -\mathsf{A} \end{pmatrix},$$

associated with the algebraic Riccati equation (18), yields the Hamiltonian

$$\begin{pmatrix} \mathsf{A}_{\mathrm{tmp}}^* & \frac{\gamma^2}{\gamma^2-1}\mathsf{F}_\infty^*\mathsf{F}_\infty - \gamma^2\mathsf{C}^*\mathsf{C} \\ \frac{-1}{\gamma^2-1}\mathsf{B}\mathsf{B}^* & -\mathsf{A}_{\mathrm{tmp}} \end{pmatrix},$$

which may be associated with the algebraic Riccati equation

$$\mathsf{A}_{\mathrm{tmp}}\mathsf{Y}_{\mathrm{tmp}} + \mathsf{Y}_{\mathrm{tmp}}\mathsf{A}_{\mathrm{tmp}}^* + \frac{1}{\gamma^2-1}\mathsf{B}\mathsf{B}^*$$
$$+ \mathsf{Y}_{\mathrm{tmp}}(\frac{\gamma^2}{\gamma^2-1}\mathsf{F}_\infty^*\mathsf{F}_\infty - \gamma^2\mathsf{C}^*\mathsf{C})\mathsf{Y}_{\mathrm{tmp}} = 0. \tag{19}$$

Since $\gamma > \sqrt{1 + \boldsymbol{\rho}(\mathsf{X}\mathsf{Z})}$, $\mathsf{Y}_{\mathrm{tmp}} = \mathsf{Y}_\infty(I - \mathsf{X}\mathsf{Y}_\infty)^{-1} \geq 0$ is thus a solution of (19) and hence (cf. [24, Thm. 13.5]),

$$\mathrm{spec}(\mathsf{A}_{\mathrm{tmp}}^* + \frac{\gamma^2}{\gamma^2-1}\mathsf{F}_\infty^*\mathsf{F}_\infty\mathsf{Y}_{\mathrm{tmp}} - \gamma^2\mathsf{C}^*\mathsf{C}\mathsf{Y}_{\mathrm{tmp}}) \subset \mathbb{C}_-. \tag{20}$$

From this one can deduce that $(\frac{\gamma}{\sqrt{\gamma^2-1}}\mathsf{F}_\infty\mathsf{Y}_{\mathrm{tmp}}, \mathsf{A}_{\mathrm{tmp}\infty}^*)$ is detectable, where $\mathsf{A}_{\mathrm{tmp}\infty}^* := \mathsf{A}_{\mathrm{tmp}}^* + \gamma\mathsf{C}^*\mathsf{L}_{\mathrm{tmp}}^* = \mathsf{A}_{\mathrm{tmp}}^* - \gamma^2\mathsf{C}^*\mathsf{C}\mathsf{Y}_{\mathrm{tmp}}$, and thus, that $\left(\begin{pmatrix} \mathsf{B}_{1\mathrm{tmp}\infty}^* \\ \frac{\gamma}{\sqrt{\gamma^2-1}}\mathsf{F}_\infty\mathsf{Y}_{\mathrm{tmp}} \end{pmatrix}, \mathsf{A}_{\mathrm{tmp}\infty}^* \right)$ is detectable, where $\mathsf{B}_{1\mathrm{tmp}\infty} := \begin{pmatrix} -\mathsf{L}_{\mathrm{tmp}} & \frac{1}{\sqrt{\gamma^2-1}}\mathsf{B} \end{pmatrix}$. Indeed, rearranging (19) to become

$$\mathsf{A}_{\mathrm{tmp}\infty}\mathsf{Y}_{\mathrm{tmp}} + \mathsf{Y}_{\mathrm{tmp}}\mathsf{A}_{\mathrm{tmp}\infty}^*$$
$$+ \mathsf{B}_{1\mathrm{tmp}\infty}\mathsf{B}_{1\mathrm{tmp}\infty}^* + \frac{\gamma^2}{\gamma^2-1}\mathsf{Y}_{\mathrm{tmp}}\mathsf{F}_\infty^*\mathsf{F}_\infty\mathsf{Y}_{\mathrm{tmp}} = 0,$$

it follows that $\mathsf{Y}_{\mathrm{tmp}} \geq 0$ is the Observability Gramian of W and hence that $W \in \mathcal{RH}_{\mathbb{C}_+}^\infty$ – see [24, Lemma 3.19]. In fact,

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix} & I \\ 0 \end{pmatrix} \begin{pmatrix} \mathsf{B}_{1\mathrm{tmp}\infty}^* \\ \frac{\gamma}{\sqrt{\gamma^2-1}}\mathsf{F}_\infty\mathsf{Y}_{\mathrm{tmp}} \end{pmatrix} + \begin{pmatrix} \frac{-\gamma}{\sqrt{\gamma^2-1}}\mathsf{F}_\infty \\ \gamma\mathsf{C} \end{pmatrix}\mathsf{Y}_{\mathrm{tmp}} = 0.$$

So by [24, Lemma 13.29], it also follows that W is inner, and thus, that $\tilde{M}$ is coinner, as required. Furthermore,

$$\tilde{M}_{12}^{-1} = \left(\begin{array}{c|c} \mathsf{A}_{\mathrm{tmp}} - \gamma^2\mathsf{Y}_{\mathrm{tmp}}\mathsf{C}^*\mathsf{C} + \frac{\gamma^2}{\gamma^2-1}\mathsf{Y}_{\mathrm{tmp}}\mathsf{F}_\infty^*\mathsf{F}_\infty & \bullet \\ \hline \bullet & \bullet \end{array} \right),$$

which in view of (20), implies that $\tilde{M}_{12}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$.

Finally, observe that

$$R_{21}^{-1} = \left(\begin{array}{c|c} \mathsf{A}_{\mathrm{tmp}} + \frac{\gamma^2}{\gamma^2-1}\mathsf{Y}_{\mathrm{tmp}}\mathsf{F}_\infty^*\mathsf{F}_\infty - \gamma^2\mathsf{Y}_{\mathrm{tmp}}\mathsf{C}^*\mathsf{C} & \bullet \\ \hline \bullet & \bullet \end{array} \right),$$

$$R_{\mathrm{ch}}^{-1} = \left(\begin{array}{c|cc} \mathsf{A}_{\mathrm{tmp}} + \frac{\gamma^2}{\gamma^2-1}\mathsf{Y}_{\mathrm{tmp}}\mathsf{F}_\infty^*\mathsf{F}_\infty - \gamma^2\mathsf{Y}_{\mathrm{tmp}}\mathsf{C}^*\mathsf{C} & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} \right),$$

$$R_{12}^{-1} = \left(\begin{array}{c|c} \mathsf{A} + \mathsf{B}\mathsf{F}_\infty & \bullet \\ \hline \bullet & \bullet \end{array} \right)$$

and

$$R_{\mathrm{ch}} = \left(\begin{array}{c|cc} \mathsf{A} + \mathsf{B}\mathsf{F}_\infty & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} \right).$$

Hence, in view of (17) and (20), it follows that $R_{12}^{-1}, R_{21}^{-1}, R_{\mathrm{ch}}, R_{\mathrm{ch}}^{-1} \in \mathcal{RH}_{\mathbb{C}_+}^\infty$, as claimed.

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