Computing the distance between time-periodic dynamical systems

Shanon Vuglar and Michael Cantoni

***Abstract*— A recent generalization of Vinnicombe's ν-gap metric accommodates linear time-varying (LTV) dynamics for a class of systems with graphs that admit normalized coprime representations. Such graph representations exist for systems generated by stabilizable and detectable LTV state-space models. In general, construction of normalized coprime representations is computationally challenging. In this paper, a numerical method for constructing normalized graph representations is provided for periodic state-space models. The construction involves the periodic solutions of corresponding periodic differential Riccati equations. Based on this, a bisection algorithm for computing the ν-gap metric is also provided.**

***Index Terms*— Robustness analysis; numerical method**

I. Introduction

The ν-gap metric [1] for linear time-invariant (LTI) systems is a measure of the distance that is relevant within the context of analyzing the robustness of feedback interconnections. A generalization of the ν-gap metric for a class of LTV systems is proposed in [2]; see [6] for a recent clarification. Specifically, the generalization is defined for LTV systems that admit normalized strong left and right graph representations, which are also required to generate forward Hankel operators that are compact.

In [2], [3], existence of the required normalized coprime representations is established for LTV state-space models that are stabilizable and detectable. The approach is constructive. It involves the solutions of time-varying differential Riccati equations over doubly-infinite time with boundary conditions at $+\infty$ and $-\infty$, respectively. In general, obtaining these solutions is intractable. To make progress from this perspective, attention is restricted to periodic systems below.

The main contribution of this paper is a tractable method for computing the ν-gap metric distance between stabilizable and detectable, periodic linear state-space models, via the construction of normalized coprime representations. The method involves computation of the stabilizing periodic solutions to periodic differential Riccati equations, which circumvents the issue of dealing directly with boundary conditions at $+\infty$ and $-\infty$. Several numerical methods exist for computing the periodic solutions; e.g., [4]. Finally, a method is given for calculating the ν-gap metric. This involves the additional verification of a family of Fredholm index conditions. For the system class considered, this is equivalent to an eigenvalue condition on a Monodromy matrix.

This work is supported by the Australian Research Council (DP130104510)

S. Vuglar and M. Cantoni are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Australia. Email: shanonvuglar@vuglar.com; cantoni@unimelb.edu.au.

II. Preliminaries

Let $\boldsymbol{L}_2^n$ denote the Hilbert space of square Lebesgue integrable functions $w : \mathbb{R} \to \mathbb{R}^n$, with inner-product $\langle w, v\rangle_{\boldsymbol{L}_2} = \int_{-\infty}^{\infty} w(t)^T v(t)\,\mathrm{d}t$ and norm $\|\cdot\|_2$. For any interval $\mathbb{I} \subset \mathbb{R}$, let $\boldsymbol{L}_2^n(\mathbb{I})$ denote the subspace of $w \in \boldsymbol{L}_2^n$ such that $w(t) = 0$ for all $t \in \mathbb{R} \setminus \mathbb{I}$. For convenience, the spatial dimension n is often omitted, and compatible dimensions is implicit throughout.

For $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \boldsymbol{L}_2 \to \boldsymbol{L}_2$, the image is denoted by $\mathrm{img}(\mathbf{X}) := \{w \in \boldsymbol{L}_2 : w = \mathbf{X}v; v \in \mathrm{dom}(\mathbf{X})\}$, and the kernel by $\ker(\mathbf{X}) := \{v \in \mathrm{dom}(\mathbf{X}) : \mathbf{X}v = 0\}$. The graph of $\mathbf{X}$ is denoted by

$$\mathcal{G}(\mathbf{X}) := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \boldsymbol{L}_2 \times \boldsymbol{L}_2 \quad \middle| \quad y = \mathbf{X}u, \quad u \in \mathrm{dom}(\mathbf{X}) \right\}.$$

The Banach space of operators $\mathbf{X} : \boldsymbol{L}_2 \to \boldsymbol{L}_2$ for which there exists $c > 0$ such that $\|\mathbf{X}u\|_2 \le c\|u\|_2$ for all $u \in \boldsymbol{L}_2$ is denoted by $\mathscr{L}$. For $X \in \mathscr{L}$, $\gamma(\mathbf{X}) := \sup_{\|u\|_2=1} \|\mathbf{X}u\|_2$ and $\mu(\mathbf{X}) := \inf_{\|u\|_2=1} \|\mathbf{X}u\|_2$. The adjoint of $\mathbf{X} \in \mathscr{L}$ is denoted by $\mathbf{X}^* \in \mathscr{L}$. It is said that $\mathbf{K} \in \mathscr{L}$ is *compact* if for every bounded sequence $\{x_n\} \subset \boldsymbol{L}_2$, the sequence $\{\mathbf{K}x_n\}$ admits a convergent subsequence in $\boldsymbol{L}_2$ [5]. The operator $\mathbf{X} \in \mathscr{L}$ is said to be Fredholm if the dimensions of $\ker(\mathbf{X})$ and $\mathrm{coker}(\mathbf{X})$ are both finite, where $\mathrm{coker}(\cdot)$ denotes the quotient space $\boldsymbol{L}_2/\,\mathrm{img}(\mathbf{X}) := \{[w] : w \in \boldsymbol{L}_2\}$ and $[w] := w + \mathrm{img}(\mathbf{X})$ denotes the equivalence class of w defined with respect to the equivalence relation: $w_1 \sim w_2$ if $w_1 - w_2 \in \mathrm{img}(\mathbf{X})$. If $\mathbf{X} \in \mathscr{L}$ is Fredholm, then the Fredholm index is given by $\mathrm{ind}(\mathbf{X}) := \dim\ker(\mathbf{X}) - \dim\mathrm{coker}(\mathbf{X})$.

Let $\mathbb{R}_{>\tau} := (\tau, +\infty)$ and $\mathbb{R}_{<\tau} := (-\infty, \tau)$ for $\tau \in \mathbb{R}$. The orthogonal projection $\boldsymbol{L}_2(\mathbb{R}_{>\tau})$ is denoted by $\mathbf{P}_\tau \in \mathscr{L}$. With $\mathbf{Q}_\tau = \mathbf{I} - \mathbf{P}_\tau$, where $\mathbf{I}$ is the identity on $\boldsymbol{L}_2$, it follows that $\mathrm{img}(\mathbf{Q}_\tau) = \boldsymbol{L}_2(\mathbb{R}_{<\tau})$. For $\tau \in \mathbb{R}$ and $\mathbf{X} \in \mathscr{L}$ the Wiener-Hopf and Hankel operators are given by $\mathbf{T}_\tau(\mathbf{X}) := \mathbf{P}_\tau \mathbf{X}|_{\boldsymbol{L}_2(\mathbb{R}_{>\tau})}$ and $\mathbf{H}_\tau(\mathbf{X}) := \mathbf{P}_\tau \mathbf{X}|_{\boldsymbol{L}_2(\mathbb{R}_{<\tau})}$, respectively.

It is said that $\mathbf{M} : \mathrm{dom}(\mathbf{M}) \subset \boldsymbol{L}_2 \to \boldsymbol{L}_2$ is causal if $\mathbf{P}_\tau \mathbf{M}|_{\boldsymbol{L}_2(\mathbb{R}_{>\tau}) \cap \mathrm{dom}(\mathbf{M})} = \mathbf{M}|_{\boldsymbol{L}_2(\mathbb{R}_{>\tau}) \cap \mathrm{dom}(\mathbf{M})}$ for all $\tau \in \mathbb{R}$. The set of all such linear maps is denoted by $\mathscr{C}$. A system is a linear map in the set

$$\mathscr{C}_+ := \{\mathbf{M} : \mathrm{dom}(\mathbf{M}) \subset \boldsymbol{L}_{2+} \to \boldsymbol{L}_{2+} \mid \mathbf{M} \text{ is causal}\},$$

where $\boldsymbol{L}_{2+} = \bigcup_{\tau \in \mathbb{R}} \mathbf{P}_\tau \boldsymbol{L}_2$. It is said that $\mathbf{G} \in \mathscr{L} \cap \mathscr{C}$ is a right representation of $\mathcal{G}(\mathbf{M})$ if $\mathcal{G} = \mathrm{img}(\mathbf{G})$. Similarly, $\tilde{\mathbf{G}} \in \mathscr{L} \cap \mathscr{C}$ is a left representation if $\mathcal{G} = \ker(\tilde{\mathbf{G}})$. If a right (left) representation $\mathbf{G}$ ($\tilde{\mathbf{G}}$) is left (right) invertible in $\mathscr{L} \cap \mathscr{C}$, then it is a strong right (left) representation. Equivalently, such representations are called coprime representations. If a strong right (left) representation $\mathbf{G}$ ($\tilde{\mathbf{G}}$) satisfies $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ ($\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$), then it is said to be normalized.

Definition 1: $\mathscr{C}_s \subset \mathscr{C}_+$ denotes the set of systems $\mathbf{M} \in \mathscr{C}_+$ for which there exist

$$\mathbf{G} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \in \mathscr{L} \cap \mathscr{C}, \qquad \tilde{\mathbf{G}} = \begin{bmatrix} -\tilde{\mathbf{V}} & \tilde{\mathbf{U}} \end{bmatrix} \in \mathscr{L} \cap \mathscr{C},$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Y} & \mathbf{X} \end{bmatrix} \in \mathscr{L} \cap \mathscr{C}, \quad \tilde{\mathbf{Z}} = \begin{bmatrix} -\tilde{\mathbf{X}} \\ \tilde{\mathbf{Y}} \end{bmatrix} \in \mathscr{L} \cap \mathscr{C},$$

with the following properties:

(a) $\begin{bmatrix} \mathbf{Z} \\ \tilde{\mathbf{G}} \end{bmatrix} \begin{bmatrix} \mathbf{G} & -\tilde{\mathbf{Z}} \end{bmatrix} = \begin{bmatrix} \mathbf{G} & -\tilde{\mathbf{Z}} \end{bmatrix} \begin{bmatrix} \mathbf{Z} \\ \tilde{\mathbf{G}} \end{bmatrix} = \mathbf{I}$;

(b) $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$;

(c) $\mathrm{img}(\mathbf{G}) = \ker(\tilde{\mathbf{G}})$ and $\mathcal{G}(\mathbf{M}) \cap \boldsymbol{L}_2(\mathbb{R}_{>\tau}) = \mathrm{img}(\mathbf{T}_\tau(\mathbf{G})) = \ker(\mathbf{T}_\tau(\tilde{\mathbf{G}}))$ for every $\tau \in \mathbb{R}$; and

(d) $\mathbf{H}_\tau(\mathbf{G})$ and $\mathbf{H}_\tau(\tilde{\mathbf{G}})$ are compact for every $\tau \in \mathbb{R}$.

Definition 2: The ν-gap between $\mathbf{M}_1 \in \mathscr{C}_s$ and $\mathbf{M}_2 \in \mathscr{C}_s$ is defined by

$$\delta_\nu(\mathbf{M}_1, \mathbf{M}_2) := \begin{cases} \gamma(\tilde{\mathbf{G}}_2 \mathbf{G}_1) & \text{if } \mu(\mathbf{G}_1^*\mathbf{G}_2) > 0 \text{ and } \mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2) \text{ is Fredholm and } \mathrm{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) = 0 \text{ for all } \tau \in \mathbb{R} \\ 1 & \text{otherwise} \end{cases},$$

where $\mathbf{G}_k$ (resp., $\tilde{\mathbf{G}}_k$) denotes any normalized strong right (resp. left) graph representation, with the additional properties described in Definition 1, for $\mathbf{M}_k$, $k \in 1, 2$.

Remark 1: Definition 2 corresponds to [2, Definition 4.2], however, the norm-coercivity constraint $\mu(\mathbf{G}_1^*\mathbf{G}_2) > 0$ is missing there. Further details can be found in [6].

III. Normalized Coprime Representations For Time-Periodic Systems

Consider the time-periodic state-space model

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] := \left\{ \begin{array}{l} \dot{x}(t) = A(t)x(t) + B(t)u(t); \\ y(t) = C(t)x(t). \end{array} \right. \quad (1)$$

Here, $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are continuous periodic matrix-valued functions over $\mathbb{R}$, where $A(t+T) = A(t)$; $B(t+T) = B(t)$; and $C(t+T) = C(t)$ for some $T \in \mathbb{R}$, and $\dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}$ denotes the derivative. For $t \in \mathbb{R}$, the latent signal $x(t) \in \mathbb{R}^n$, the input signal $u(t) \in \mathbb{R}^m$, and the the output signal $y(t) \in \mathbb{R}^p$, where $n, m, p \in \mathbb{N}$. Before considering normalized coprime representations for this class of time-periodic systems, it is instructive to recall several results for more general LTV systems from [2] and [3]. Ultimately, these results confirm existence of graph representations with all properties specified in Definition 1, except the normalization property (b), which is treated later in this section.

Associated with $A(\cdot)$ is a fundamental matrix $X_A : \mathbb{R} \to \mathbb{R}^{n\times n}$ which is defined to be the solution of

$$\dot{X}_A(t) = A(t)X_A(t); \quad X_A(0) = I.$$

Since $A(\cdot)$ is continuous and bounded, $X_A(\cdot)$ exists and is invertible [7, Sec II.2]. If there exists $P_A = P_A^2 \in \mathbb{R}^{n\times n}$, $\varrho > 0$, and $\varsigma > 0$ such that

$$\begin{aligned} |X_A(t)P_A X_A(s)^{-1}| &\le \varrho e^{-\varsigma(t-s)} \quad \forall\, t \ge s \quad \text{and} \\ |X_A(t)(I-P_A)X_A(s)^{-1}| &\le \varrho e^{-\varsigma(s-t)} \quad \forall\, s \ge t, \end{aligned} \quad (2)$$

then $A(\cdot)$ is said to admit an *exponential dichotomy* P_A. If $P_A = I$, then it is said that $A(\cdot)$ defines an *exponentially stable* evolution.

If $A(\cdot)$ admits an exponential dichotomy P_A, then (1) generates a bounded operator $\mathbf{M} = (u \in \boldsymbol{L}_2^m \mapsto y \in \boldsymbol{L}_2^p)$ defined by the integral equation [8, Theorem 1.2.3]

$$y(t) = \int_{-\infty}^{\infty} C(t)\kappa_A(t,s)B(s)u(s)\,\mathrm{d}s \quad \forall t \in \mathbb{R}, \quad (3)$$

where

$$\kappa_A(t,s) := \begin{cases} X_A(t)P_A X_A(s)^{-1} & t \ge s \\ -X_A(t)(I-P_A)X_A(s)^{-1} & s > t \end{cases}. \quad (4)$$

In general, the integral operator (3) is not causal. However, if $P_A = I$, then $\mathbf{M} \in \mathscr{L} \cap \mathscr{C}$.

Associated with the matrix valued function A is a multiplication operator $\mathbf{A} \in \mathscr{L}$ defined by $(x \in \boldsymbol{L}_2 \mapsto (t \in \mathbb{R} \mapsto (\mathbf{A}x)(t) := A(t)x(t)) \in \boldsymbol{L}_2)$. Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \to \boldsymbol{L}_2$ denote the differential operator defined by

$$(t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}^n) \mapsto (t \in \mathbb{R} \mapsto (\mathcal{D}x)(t) := \dot{x}(t) \in \mathbb{R}^n),$$

where

$$\mathrm{dom}(\mathcal{D}) = \left\{ x \in \boldsymbol{L}_2 \;\middle|\; \begin{array}{l} x \text{ locally absolutely continuous} \\ \dot{x} \in \boldsymbol{L}_2 \end{array} \right\}.$$

Then, on $\mathrm{dom}(\mathcal{D})$ the operator $(\mathcal{D} - \mathbf{A})$ has a bounded inverse if, and only if, A admits an exponential dichotomy [9, Theorem 1.1]. In this case, $(\mathcal{D} - \mathbf{A})^{-1} \in \mathscr{L}$ is given by

$$(t \in \mathbb{R} \mapsto z(t) \in \mathbb{R}^n) \mapsto \left(t \in \mathbb{R} \mapsto ((\mathcal{D}-\mathbf{A})^{-1}z)(t) := \int_{-\infty}^{\infty} \kappa_A(t,s)z(s)ds \in \mathbb{R}^n \right),$$

with κ_A as defined in (4) [9, Theorem 1.1]. Hence,

$$\mathbf{M} = \mathbf{C}(\mathcal{D} - \mathbf{A})^{-1}\mathbf{B},$$

where $\mathbf{B}$ and $\mathbf{C}$ denote the multiplication operators associated with the matrix valued functions B and C. Moreover $\mathrm{dom}(\mathbf{M}) = \boldsymbol{L}_2$, and

$$\mathcal{G}(\mathbf{M}) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \boldsymbol{L}_2 \times \boldsymbol{L}_2 \;\middle|\; \begin{array}{l} \exists\, x \in \boldsymbol{L}_2 \text{ for} \\ \text{which (1) is satisfied} \end{array} \right\}.$$

If A defines an exponentially stable evolution (i.e., $P_A = I$), then the inverse $(\mathcal{D} - \mathbf{A})^{-1}$ is causal, and thus, $\mathbf{M} \in \mathscr{L} \cap \mathscr{C}$.

Bounded causal solutions of (1) are of interest. From above, if A admits an exponential dichotomy, then associated with M is a bounded integral operator $\mathbf{M}$. If $P_A = I$, then $\mathbf{M}$ is causal, but in general, $\mathbf{M}$ is not causal. If A does not admit an exponential dichotomy, then it is not possible to construct $\mathbf{M}$ as above. Nonetheless, whenever the state-space model is stabilizable and detectable, in the sense described next, it is possible to construct a causal operator $\mathbf{M}_+ \in \mathscr{C}_+$ such that $\mathcal{G}(\mathbf{M}_+) \subset \boldsymbol{L}_{2+}$ is precisely the set of all causal solutions of (1) in the subspace $\boldsymbol{L}_{2+}$ of $\boldsymbol{L}_2$ signals with support that is (non-uniformly) bounded below.

The system model (1) is said to be *stabilizable* if, and only if, there exists a continuous and periodic matrix valued function $F(t+T) = F(t)$ such that $A + BF$ defines an exponentially stable evolution. The system model (1) is said to be *detectable* if, and only if, there exists a continuous and periodic matrix valued function $L(t+T) = L(t)$ such that $A + LC$ defines an exponentially stable evolution. For the remainder of this paper, attention is restricted to the class of models (1) that are stabilizable and detectable.

Define the state-space models

$$G = \begin{bmatrix} U \\ V \end{bmatrix} := \left[\begin{array}{c|c} A+BF & B \\ \hline C & 0 \\ F & I \end{array}\right], \tag{5}$$

$$\tilde{G} = \begin{bmatrix} -\tilde{V} & \tilde{U} \end{bmatrix} := \left[\begin{array}{c|cc} A+LC & -L & B \\ \hline C & -I & 0 \end{array}\right], \tag{6}$$

$$Z = \begin{bmatrix} Y & X \end{bmatrix} := \left[\begin{array}{c|cc} A+LC & -L & B \\ \hline -F & 0 & I \end{array}\right], \quad \text{and} \tag{7}$$

$$\tilde{Z} = \begin{bmatrix} -\tilde{X} \\ \tilde{Y} \end{bmatrix} := \left[\begin{array}{c|c} A+BF & L \\ \hline F & 0 \\ C & -I \end{array}\right], \tag{8}$$

where the matrix valued functions F and L are such that $(A + BF)$ and $(A + LC)$ define exponentially stable evolutions; i.e. $P_{A+BF} = P_{A+LC} = I$. Let $\mathbf{F}$ and $\mathbf{L}$ denote the multiplication operators associated with F and L respectively.

Since $(A+BF)$ and $(A+LC)$ define exponentially stable evolutions, the inverses $(\mathcal{D} - (\mathbf{A} + \mathbf{BF}))^{-1} \in \mathscr{L} \cap \mathscr{C}$ and $(\mathcal{D} - (\mathbf{A} + \mathbf{LC}))^{-1} \in \mathscr{L} \cap \mathscr{C}$ exist. Hence, the following are well defined:

$$\begin{aligned}
\mathbf{V} &:= \mathbf{I} + \mathbf{F}\,(\mathcal{D} - (\mathbf{A} + \mathbf{BF}))^{-1}\,\mathbf{B}; \\
\mathbf{U} &:= \mathbf{C}\,(\mathcal{D} - (\mathbf{A} + \mathbf{BF}))^{-1}\mathbf{B}; \\
\mathbf{X} &:= \mathbf{I} - \mathbf{F}\,(\mathcal{D} - (\mathbf{A} + \mathbf{LC}))^{-1}\mathbf{B}; \\
\mathbf{Y} &:= \mathbf{F}\,(\mathcal{D} - (\mathbf{A} + \mathbf{LC}))^{-1}\mathbf{L}; \\
\tilde{\mathbf{U}} &:= \mathbf{C}\,(\mathcal{D} - (\mathbf{A} + \mathbf{LC}))^{-1}\mathbf{B}; \\
-\tilde{\mathbf{V}} &:= -\mathbf{I} - \mathbf{C}\,(\mathcal{D} - (\mathbf{A} + \mathbf{LC}))^{-1}\mathbf{L}; \\
\tilde{\mathbf{Y}} &:= \mathbf{F}\,(\mathcal{D} - (\mathbf{A} + \mathbf{BF}))^{-1}\mathbf{L}; \quad \text{and} \\
-\tilde{\mathbf{X}} &:= -\mathbf{I} + \mathbf{C}\,(\mathcal{D} - (\mathbf{A} + \mathbf{BF}))^{-1}\mathbf{L}.
\end{aligned}$$

These operators, and the corresponding Wiener-Hopf operators $\mathbf{T}_\tau(\mathbf{G})$ etc., are all causal. Also note that

$$\begin{aligned}
\mathbf{ZG} &= \mathbf{XV} + \mathbf{YU} \\
&= \mathbf{I} + \mathbf{F}[(\mathcal{D} - (\mathbf{A} + \mathbf{BF}))^{-1} - (\mathcal{D} - (\mathbf{A} + \mathbf{LC}))^{-1} \\
&\quad + (\mathcal{D} - (\mathbf{A} + \mathbf{LC}))^{-1}(\mathbf{LC} - \mathbf{BF}) \\
&\quad \times (\mathcal{D} - (\mathbf{A} + \mathbf{BF}))^{-1}]\mathbf{B} = \mathbf{I},
\end{aligned}$$

and $\mathbf{Z}|_{\boldsymbol{L}_{2+}}\mathbf{G}|_{\boldsymbol{L}_{2+}} = \mathbf{I}|_{\boldsymbol{L}_{2+}}$. Similarly, $\tilde{\mathbf{G}}\tilde{\mathbf{Z}} = \mathbf{I}$, and $\tilde{\mathbf{G}}|_{\boldsymbol{L}_{2+}}\tilde{\mathbf{Z}}|_{\boldsymbol{L}_{2+}} = \mathbf{I}|_{\boldsymbol{L}_{2+}}$.

Now, define the causal operator $\mathbf{M}_+ \in \mathscr{C}_+$, with $\operatorname{dom}(\mathbf{M}_+) := \operatorname{img}(\mathbf{V}|_{\boldsymbol{L}_{2+}}) \subset \mathbf{L}_{2+}$, by the following:

$$\mathbf{M}_+|_{\operatorname{dom}(\mathbf{M}_+)\cap \boldsymbol{L}_2(\mathbb{R}_{>\tau})} := \mathbf{T}_\tau(\mathbf{U})\mathbf{T}_\tau(\mathbf{V})^{-1} \text{ for } \tau \in \mathbb{R}. \tag{9}$$

It is important to note that $\mathbf{T}_\tau(\mathbf{V})^{-1}$ is an operator that is only defined on $\operatorname{img}(\mathbf{V}|_{\boldsymbol{L}_2(\mathbb{R}_{>\tau})})$. Also, note that $\mathbf{T}_\tau(\mathbf{V})^{-1}$ is causal because V is causal with non-zero instantaneous gain [2, Lemma 2.3]. Hence, $\mathbf{M}_+$ is causal.

The graph $\mathcal{G}(\mathbf{M}_+)$ is precisely the subspace of all causal input-output pairs associated with a solution of (1) in $\boldsymbol{L}_{2+}$. To see this, suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ arises from a causal solution of (1). That is, there exists $\tau \in \mathbb{R}$, such that $u \in \boldsymbol{L}_2(\mathbb{R}_{>\tau})$ and $y \in \boldsymbol{L}_2(\mathbb{R}_{>\tau})$, and there exists $x : \mathbb{R} \to \mathbb{R}^n$ with $x(\tau) = 0$, such that (1) is satisfied. It follows that $x(s) = 0$ for $s < \tau$ by the variation of constants formula and integrating backwards from τ. Also, by [3, Lemma 3.6], $x \in \boldsymbol{L}_2(\mathbb{R}_{>\tau})$ because it is assumed that the state-space model is detectable. Let $q = \mathbf{Z}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. Then (by substitution) $z = x$ is the unique forward solution to the following state-space model associated with Z:

$$\begin{aligned}
\dot{z} &= (A + LC)z + Bu - Ly \\
&= (A + LC)z + Bu - LCx, \\
q &= -Fz + u,
\end{aligned}$$

with initial condition $z(\tau) = 0$. Hence, $q = u - Fx$. Let $\eta = x$. Then q, η, u and y satisfy the following state-space model associated with G:

$$\begin{aligned}
\dot{\eta} &= (A + BF)\eta + Bq, \\
u &= F\eta + q, \\
y &= C\eta,
\end{aligned} \tag{10}$$

with initial condition $\eta(\tau) = 0$. Hence, $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \operatorname{img}\mathbf{T}_\tau(\mathbf{G})$ and $y = \mathbf{M}_+u$; i.e. $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{G}(\mathbf{M}_+)$. Conversely, suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{G}(\mathbf{M}_+)$. Then there exists $\tau \in \mathbb{R}$, such that $u \in \boldsymbol{L}_2(\mathbb{R}_{>\tau})$ and $y \in \boldsymbol{L}_2(\mathbb{R}_{>\tau})$, and $y = \mathbf{T}_\tau(\mathbf{U})\mathbf{T}_\tau(\mathbf{V})^{-1}u$. Let $q = \mathbf{T}_\tau(\mathbf{V})^{-1}u$. There exists $\eta \in \boldsymbol{L}_2(\mathbb{R}_{>\tau})$ with $\eta(\tau) = 0$ such that (10) is satisfied. Let $x = \eta$. Then u, y, and x satisfy (1); i.e. $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ is a causal solution solution of (1) with forward support.

It is now established (by construction) that normalized coprime representations exist for the class of periodic LTV systems under consideration. It is then shown that the constructed coprime factors belong to the class $\mathscr{C}_s$.

Lemma 1: Consider the periodic state-stace model M defined in (1), and suppose that this is stabilizable and detectable. Then the periodic Riccati equation (PRE)

$$-\dot{P} = A^TP + PA + C^TC - PBB^TP \tag{11}$$

and the dual PRE

$$\dot{Q} = AQ + QA^T - QC^TCQ + BB^T \tag{12}$$

have symmetric periodic stabilizing solutions P and Q, respectively. Furthermore, $A - BB^TP$ and $A - QC^TC$ define exponentially stable evolutions.

Proof: See [10, Theorem 6.3]. ■

Theorem 1: Consider the periodic state-space model M defined in (1), and suppose that this is stabilizable and detectable. Let $F = -B^TP$, where P is the symmetric periodic stabilizing solution of the PRE (11), and let $L = -QC^T$, where Q is the symmetric periodic stabilizing solution of the dual PRE (12). Let $G, \tilde{G}, Z$ and $\tilde{Z}$ be the state space models defined in (5)-(8) with F and L as constructed here. Then the corresponding input-output operators $\mathbf{G}, \tilde{\mathbf{G}}, \mathbf{Z}, \tilde{\mathbf{Z}} \in \mathscr{L}$ are causal and $\mathrm{img}(\mathbf{G}|_{\boldsymbol{L}_{2+}}) = \ker(\tilde{\mathbf{G}}|_{\boldsymbol{L}_{2+}}) = \mathcal{G}(\mathbf{M}_+)$, where $\mathbf{M}_+$ is the causal operator defined in (9). Furthermore, $\mathbf{Z}\mathbf{G} = \tilde{\mathbf{G}}\tilde{\mathbf{Z}} = \mathbf{I}$, $\tilde{\mathbf{G}}\mathbf{G} = 0$, $\mathbf{Z}\tilde{\mathbf{Z}} = 0$, and $\mathbf{G}\mathbf{Z} + \tilde{\mathbf{Z}}\tilde{\mathbf{G}} = \mathbf{I}$ hold. Moreover, $\mathbf{G}^*\mathbf{G} = \tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$.

Proof: First, $\mathrm{img}(\mathbf{G}|_{L_{2+}}) = \mathcal{G}(\mathbf{M}_+)$ is established. The strong graph property follows from the existence of a causal bounded left inverse, which has already been established. Then the normalization property is established. Corresponding properties are then established for $\tilde{\mathbf{G}}|_{L_{2+}}$.

Suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathrm{img}(\mathbf{G}|_{L_{2+}})$. Then by the causality of $\mathbf{G}$ there exists $\tau \in \mathbb{R}$ and $q \in L_2^m(\mathbb{R}_{>\tau})$, such that $u \in L_2^m(\mathbb{R}_{>\tau})$, $y \in L_2^p(\mathbb{R}_{>\tau})$, $u = \mathbf{T}_\tau(\mathbf{V})q$, and $y = \mathbf{T}_\tau(\mathbf{U})q$. Hence, $y = \mathbf{T}_\tau(\mathbf{U})\mathbf{T}_\tau(\mathbf{V})^{-1}u$ and $\mathrm{img}(\mathbf{G}|_{L_{2+}}) \subset \mathcal{G}(\mathbf{M}_+)$. Conversely, suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{G}(\mathbf{M}_+)$. Then there exists $\tau \in \mathbb{R}$, such that $y = \mathbf{T}_\tau(\mathbf{U})\mathbf{T}_\tau(\mathbf{V})^{-1}u$. Let $q = \mathbf{T}_\tau(\mathbf{V})^{-1}u$. Then $q \in L_2^m(\mathbb{R}_{>\tau})$ and $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = \mathbf{G}q$; $\mathcal{G}(\mathbf{M}_+) \subset \mathrm{img}(\mathbf{G}|_{L_{2+}})$; and $\mathrm{img}(\mathbf{G}|_{L_{2+}}) = \mathcal{G}(\mathbf{M}_+)$. The strong graph representation property follows from $\mathbf{Z}|_{L_{2+}}\mathbf{G}|_{L_{2+}} = \mathbf{I}|_{L_{2+}}$, as established earlier in this section.

It is now shown that $\mathbf{G}$ is normalized. Let $\mathbf{P}$ and $\dot{\mathbf{P}}$ denote the multiplication operators associated with the periodic matrix-valued functions P and $\dot{P}$, respectively. For notational convenience, let $\mathbf{\Xi} = (\mathcal{D} - (\mathbf{A} + \mathbf{B}\mathbf{F}))$. Then,

$$\begin{aligned}
\mathbf{G} &= \begin{bmatrix} \mathbf{C}\mathbf{\Xi}^{-1}\mathbf{B} \\ \mathbf{I} + \mathbf{F}\mathbf{\Xi}^{-1}\mathbf{B} \end{bmatrix}; \\
\mathbf{G}^* &= \begin{bmatrix} \mathbf{B}^*\mathbf{\Xi}^{-*}\mathbf{C}^* & \mathbf{I} + \mathbf{B}^*\mathbf{\Xi}^{-*}\mathbf{F}^* \end{bmatrix}; \quad \text{and} \\
\mathbf{G}^*\mathbf{G} &= \mathbf{B}^*\mathbf{\Xi}^{-*}\mathbf{C}^*\mathbf{C}\mathbf{\Xi}^{-1}\mathbf{B} \\
&\qquad + \left(\mathbf{I} + \mathbf{B}^*\mathbf{\Xi}^{-*}\mathbf{F}^*\right)\left(\mathbf{I} + \mathbf{F}\mathbf{\Xi}^{-1}\mathbf{B}\right) \\
&= \mathbf{I} + \mathbf{B}^*\mathbf{\Xi}^{-*}\mathbf{C}^*\mathbf{C}\mathbf{\Xi}^{-1}\mathbf{B} - \mathbf{B}^*\mathbf{P}\mathbf{\Xi}^{-1}\mathbf{B} \\
&\qquad - \mathbf{B}^*\mathbf{\Xi}^{-*}\mathbf{P}\mathbf{B}^* + \mathbf{B}^*\mathbf{\Xi}^{-*}\mathbf{P}\mathbf{B}\mathbf{B}^*\mathbf{P}\mathbf{\Xi}^{-1}\mathbf{B} \\
&= \mathbf{I} + \mathbf{B}^*\mathbf{\Xi}^{-*}\Big(\mathbf{C}^*\mathbf{C} - (\mathcal{D} - (\mathbf{A} + \mathbf{B}\mathbf{F}))^*\mathbf{P} \\
&\qquad - \mathbf{P}\left(\mathcal{D} - (\mathbf{A} + \mathbf{B}\mathbf{F})\right) + \mathbf{P}\mathbf{B}\mathbf{B}^*\mathbf{P}\Big)\mathbf{\Xi}^{-1}\mathbf{B} \\
&= \mathbf{I} + \mathbf{B}^*\mathbf{\Xi}^{-*}\Big(\mathcal{D}\mathbf{P} - \mathbf{P}\mathcal{D} \\
&\qquad + \mathbf{C}^*\mathbf{C} + \mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^*\mathbf{P}\Big)\mathbf{\Xi}^{-1}\mathbf{B} \\
&= \mathbf{I} + \mathbf{B}^*\mathbf{\Xi}^{-*}\Big(\mathcal{D}\mathbf{P} - \mathbf{P}\mathcal{D} - \dot{\mathbf{P}}\Big)\mathbf{\Xi}^{-1}\mathbf{B}.
\end{aligned}$$

But for all $x \in \mathrm{dom}(\mathcal{D})$,

$$\left(\mathcal{D}\mathbf{P} - \mathbf{P}\mathcal{D} - \dot{\mathbf{P}}\right)(x) = \mathcal{D}(\mathbf{P}x) - \mathbf{P}(\dot{x}) - \dot{\mathbf{P}}x = 0.$$

Hence, $\mathbf{G}^*\mathbf{G} = I$; $\mathbf{G}$ is normalized.

Now consider $\tilde{\mathbf{G}}|_{L_{2+}}$. Suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \ker(\tilde{\mathbf{G}}|_{L_{2+}})$. Then there exists $\tau \in \mathbb{R}$ such that $u \in L_2^m(\mathbb{R}_{>\tau})$ and $y \in L_2^p(\mathbb{R}_{>\tau})$. Also, there exists $\xi \in L_2^n(\mathbb{R}_{>\tau})$ with $\xi(\tau) = 0$ such that

$$\begin{aligned}
\dot{\xi} &= (A + LC)\xi + Bu - Ly; \\
0 &= C\xi - y.
\end{aligned} \tag{13}$$

Equivalenty,

$$\begin{aligned}
\dot{\xi} &= A\xi + Bu; \\
y &= C\xi.
\end{aligned}$$

Hence, $x = \xi$ satisfies (1) and $\ker(\tilde{\mathbf{G}}|_{L_{2+}}) \subset \mathcal{G}(\mathbf{M}_+)$. Conversely, suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{G}(\mathbf{M}_+)$. Then there exists $\tau \in \mathbb{R}$ such that $u \in L_2^m(\mathbb{R}_{>\tau})$ and $y \in L_2^p(\mathbb{R}_{>\tau})$, and there exists $x \in L_2^n(\mathbb{R}_{>\tau})$ with $x(\tau) = 0$ such that (1) is satisfied. Then $\xi = x$ satisfies (13) and $\mathcal{G}(\mathbf{M}_+) \subset \ker(\tilde{\mathbf{G}}|_{L_{2+}})$. Hence, $\mathcal{G}(\mathbf{M}_+) = \ker(\tilde{\mathbf{G}}|_{L_{2+}})$. The strong graph property follows from $\tilde{\mathbf{G}}|_{L_{2+}}\tilde{\mathbf{Z}}|_{L_{2+}} = \mathbf{I}|_{L_{2\pm}}$, as was established earlier in this section. The identities $\mathbf{Z}\tilde{\mathbf{Z}} = 0$ and $\mathbf{G}\mathbf{Z} + \tilde{\mathbf{Z}}\tilde{\mathbf{G}} = \mathbf{I}$ follow similarly.

Finally, it remains to show that $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$. Let $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ denote the multiplication operators associated with the periodic matrix-valued functions Q and $\dot{Q}$, and let $\mathbf{\Upsilon} = (\mathcal{D} - (\mathbf{A} + \mathbf{L}\mathbf{C}))$. Then,

$$\begin{aligned}
\tilde{\mathbf{G}} &= \begin{bmatrix} -\mathbf{I} - \mathbf{C}\mathbf{\Upsilon}^{-1}\mathbf{L} & \mathbf{C}\mathbf{\Upsilon}^{-1}\mathbf{B} \end{bmatrix}; \\
\tilde{\mathbf{G}}^* &= \begin{bmatrix} -\mathbf{I} - \mathbf{L}^*\mathbf{\Upsilon}^{-1}\mathbf{C} \\ \mathbf{B}^*\mathbf{\Upsilon}^{-*}\mathbf{C}^* \end{bmatrix}; \quad \text{and} \\
\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* &= \mathbf{C}\mathbf{\Upsilon}^{-1}\mathbf{B}\mathbf{B}^*\mathbf{\Upsilon}^{-*}\mathbf{C}^* \\
&\qquad + (\mathbf{I} + \mathbf{C}\mathbf{\Upsilon}^{-1}\mathbf{L})(\mathbf{I} + \mathbf{L}^*\mathbf{\Upsilon}^{-1}\mathbf{C}) \\
&= \mathbf{I} + \mathbf{C}\mathbf{\Upsilon}^{-1}\Big(\mathbf{B}\mathbf{B}^* - \mathbf{Q}\mathbf{\Upsilon}^* - \mathbf{\Upsilon}\mathbf{Q} \\
&\qquad + \mathbf{Q}\mathbf{C}^*\mathbf{C}\mathbf{Q}\Big)\mathbf{\Upsilon}^{-*}\mathbf{C}^* \\
&= \mathbf{I} + \mathbf{C}\mathbf{\Upsilon}^{-1}\Big(\mathbf{Q}\mathcal{D} + \dot{\mathbf{Q}} - \mathcal{D}\mathbf{Q}\Big)\mathbf{\Upsilon}^{-*}\mathbf{C}^* = \mathbf{I}
\end{aligned}$$

This concludes the proof. ■

Finally, the Wiener-Hopf and Hankel properties of graph representations associated with Definition 1 of the class of periodic systems under consideration are now verified.

Theorem 2: The normalized strong graph representations $\mathbf{G}, \tilde{\mathbf{G}} \in \mathscr{L} \cap \mathscr{C}$ in Theorem 1 are such that the following additional properties hold:

(a) $\mathcal{G}(\mathbf{M}_+) \cap \boldsymbol{L}_2(\mathbb{R}_{>\tau}) = \mathrm{img}(\mathbf{T}_\tau(\mathbf{G})) = \ker(\mathbf{T}_\tau(\tilde{\mathbf{G}}))$ for every $\tau \in \mathbb{R}$; and
(b) $\mathbf{H}_\tau(\mathbf{G})$ and $\mathbf{H}_\tau(\tilde{\mathbf{G}})$ are compact for every $\tau \in \mathbb{R}$.

Therefore, system $\mathbf{M}_+$ defined in (9) is an element of $\mathscr{C}_s$.

Proof: Theorem 1 establishes the existence of normalized strong graph representations for the system $\mathbf{M}_+ \in \mathscr{C}_+$ defined in (9) for the periodic, stabilizable and detectable, state-space model (1). As such, to establish $\mathbf{M}_+ \in \mathscr{C}_s$, it remains to show (a) and (b).

(a) Given $\tau \in \mathbb{R}$, suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathrm{img}(\mathbf{T}_\tau(\mathbf{G}))$. Then $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \boldsymbol{L}_2(\mathbb{R}_{>\tau})$. Also, since $\mathrm{img}(\mathbf{T}_\tau(\mathbf{G})) \subset \mathrm{img}(\mathbf{G}|_{L_{2+}}) = \mathcal{G}(\mathbf{M}_+)$, $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{G}(\mathbf{M}_+)$. Hence, $\mathrm{img}(\mathbf{T}_\tau(\mathbf{G})) \subset \mathcal{G}(\mathbf{M}_+) \cap \boldsymbol{L}_2(\mathbb{R}_{>\tau})$. Now suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{G}(\mathbf{M}_+) \cap \boldsymbol{L}_2(\mathbb{R}_{>\tau})$. Then $y = \mathbf{T}_\tau(\mathbf{U})\mathbf{T}_\tau(\mathbf{V})^{-1}u$. Let $q = (\mathbf{T}_\tau(\mathbf{V}))^{-1}u$. Then $q \in L_2^m(\mathbb{R}_{>\tau})$ and $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = \mathbf{T}_\tau(\mathbf{G})q$; $\mathrm{img}(\mathbf{T}_\tau(\mathbf{G})) \subset \mathcal{G}(\mathbf{M}_+) \cap \boldsymbol{L}_2(\mathbb{R}_{>\tau})$. Hence, $\mathcal{G}(\mathbf{M}_+) \cap \boldsymbol{L}_2(\mathbb{R}_{>\tau}) = \mathrm{img}(\mathbf{T}_\tau(\mathbf{G}))$. Similarly, it can be shown that $\mathcal{G}(\mathbf{M}_+) \cap \boldsymbol{L}_2(\mathbb{R}_{>\tau}) = \ker(\mathbf{T}_\tau(\tilde{\mathbf{G}}))$.

(b) Since $A+BF$ defines an exponentially stable evolution, $\mathbf{H}_\tau(\mathbf{G})$ can be decomposed into the composition of two compact operators:

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} := \begin{bmatrix} C(t) \\ F(t) \end{bmatrix} \Phi_{A+BF}(t,\tau)x_\tau \quad \forall t > \tau; \text{ and}$$

$$x_\tau := \int_{-\infty}^{\tau} \Phi_{A+BF}(\tau,s)B(s)q(s)\,\mathrm{d}s.$$

Hence $\mathbf{H}_\tau(\mathbf{G})$ is compact for every $\tau \in \mathbb{R}$ [5, Sec 2.16] . In the same way, it can be shown that $\mathbf{H}_\tau(\tilde{\mathbf{G}})$ is compact for every $\tau \in \mathbb{R}$. ■

IV. Calculating the gap metric

In this section, a method is devised for computing the ν-gap distance between two periodic, stabilizable and detectable, state-space models M_1 and M_2; i.e., between the corresponding input-output operators $\mathbf{M}_{1+}$ and $\mathbf{M}_{2+}$ defined in line with (9). Throughout, $\mathbf{G}_k$ and $\tilde{\mathbf{G}}_k$ denote the corresponding strong right and left graph representations, with the additional properties specified in Definition 1.

Computation of the ν-gap can be approached in three steps: (i) obtain the normalized graph representations $\tilde{\mathbf{G}}_2$ and $\mathbf{G}_1$; (ii) test the conditions $\mu(\mathbf{G}_1^*\mathbf{G}_2) > 0$ and $\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)$ is Fredholm with $\mathrm{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) = 0 \ \forall\ \tau \in \mathbb{R}$; and, (iii) if these conditions hold, calculate $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1)$.

Normalized graph representations exist and can be constructed as detailed in Section III: Let

$$G_k = \left[\begin{array}{c|c} A_k + B_kF_k & B_k \\ \hline C_k & 0 \\ F_k & I \end{array}\right],$$

and $\mathbf{G}_k$ be the associated causal integral operator ($k \in 1,2$).

Now consider the state-space model

$$A = \begin{bmatrix} A_2 + B_2F_2 & 0 \\ F_1^*F_2 + C_1^*C_2 & -(A_1+B_1F_1)^* \end{bmatrix};$$

$$B = \begin{bmatrix} B_2 \\ F_1^* \end{bmatrix};$$

$$C = \begin{bmatrix} F_2 & -B_1^* \end{bmatrix};$$

$$D = I,$$

which is a state-space realization for $\mathbf{G}_1^*\mathbf{G}_2$. Recall that $A_1+B_1F_1$ and $A_2+B_2F_2$ both give rise to exponentially stable evolutions. Hence, A admits an exponential dichotomy, whereby $\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)$ is Fredholm for all $\tau \in \mathbb{R}$ [8, Theorem II.5.2]. Actually,

$$P_A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

with I being the same dimension as $A_2 \in \mathbb{R}^{n_2 \times n_2}$. Hence, $\mathrm{rank}\, P_A = n_2$. Furthermore, since $D^{-1} = I$ is uniformly bounded, if $A - BD^{-1}C$ also admits an exponential dichotomy, then $(\mathbf{G}_1^*\mathbf{G}_2)^{-1}$ is bounded, whereby $\mu(\mathbf{G}_1^*\mathbf{G}_2) > 0$, and the Fredholm index satisfies [8, Theorem II.5.2]

$$\mathrm{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) = \mathrm{rank}\, P_A - \mathrm{rank}\, P_{A-BD^{-1}C} \quad \forall \tau \in \mathbb{R}.$$

Note that

$$A - BD^{-1}C = \begin{bmatrix} A_2 & B_2B_1^* \\ C_1^*C_2 & -A_1^* \end{bmatrix} \tag{14}$$

and let $X_{(A-BD^{-1}C)}(T)$ denote the corresponding Monodromy matrix, which can be computed by using the approach described in [11], for example. Then $A - BD^{-1}C$ defines an exponential dichotomy if, and only if, none of the eigenvalues of $X_{(A-BD^{-1}C)}(T)$ lie on the unit circle [12, p430], [7]. Furthermore, the rank of $P_{A-BD^{-1}C}$ is precisely the number of these eigenvalues that lie inside the unit disk. If this number is equal to n_2 then $\mathrm{ind}(\mathbf{T}_\tau(\mathbf{G}_2^*\mathbf{G}_1)) = 0 \ \forall\ \tau \in \mathbb{R}$. In this case, $\delta_\nu(M_1,M_2)$ can be computed as $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1)$. Otherwise, $\delta_\nu(M_1,M_2) = 1$.

An approach to computing $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1)$ is to use a bounded real lemma result and bisection algorithm as described below. Consider the state-space model

$$A_\gamma = \begin{bmatrix} A_1 + B_1F_1 & 0 \\ B_2F_1 - L_2C_1 & A_2 + L_2C_2 \end{bmatrix}; \tag{15}$$

$$B_\gamma = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \tag{16}$$

$$C_\gamma = \begin{bmatrix} -C_1 & C_2 \end{bmatrix}; \tag{17}$$

$$D_\gamma = 0, \tag{18}$$

which is a state-space realization for $\tilde{\mathbf{G}}_2\mathbf{G}_1$. Let $\delta > 0$ be a given positive scalar. Then the following statements with A_γ, B_γ and C_γ as defined in (15)-(17) above are equivalent [13, Lemma 2.6]:

(i) A_γ is stable and $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1) < \delta$.

(ii) There exists a T-periodic stabilizing positive semidefinite solution to

$$-\dot{\Pi} = A_\gamma^T\Pi + \Pi A_\gamma + C_\gamma^TC_\gamma + \delta^{-2}\Pi B_\gamma B_\gamma^T\Pi, \tag{19}$$

i.e., such that $A_\gamma + \delta^{-2}B_\gamma B_\gamma^T\Pi$ is exponentially stable.

Hence, a bisection algorithm can be constructed to compute $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1)$, see for example [11]. This approach can be problematic for long period times. For an alternative approach to computing $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1)$ see [14], which proposes an approach based on the technique of time-domain lifting.

V. Illustrative Example

Consider the (unstable) state-space models

$$M_1 = \left[\begin{array}{c|c} 0.8 + 0.1\cos t & 1 \\ \hline 1 & 0 \end{array}\right],$$

$$M_2 = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right], \quad \text{and} \quad M_3 = \left[\begin{array}{c|c} 0.8 & 1 \\ \hline 1 & 0 \end{array}\right],$$

where M_1 is a periodically perturbed version of M_3. Using LTI methods (see for example [15]):

$$\delta_\nu(M_2, M_3) = 0.1104.$$

To demonstrate the approach given in Section IV, the ν-gap distance between M_1 and M_2 is now calculated.

First, state space realizations G_1 and $\tilde{G}_2$ are obtained for the normalized graph representations $\mathbf{G}_1$ and $\tilde{\mathbf{G}}_2$. Since M_2 is LTI and one dimensional, the PRE (12) reduces to the quadratic equation

$$0 = -Q_1^2 + 2Q_1 + 1,$$

with a positive solution $Q_2 = 1 + \sqrt{2} = 2.4142$. Hence $L_2 = -2.4142$, and using (6),

$$\tilde{G}_2 = \left[\begin{array}{c|cc} -1.414 & 2.4142 & 1 \\ \hline 1 & -1 & 0 \end{array}\right].$$

To obtain the representation G_1, it is first necessary to obtain the periodic stabilizing solution P_1 for the PRE (11) corresponding to M_1:

$$-\dot{P} = 2(0.8 + 0.1\cos t)P - P^2 + 1.$$

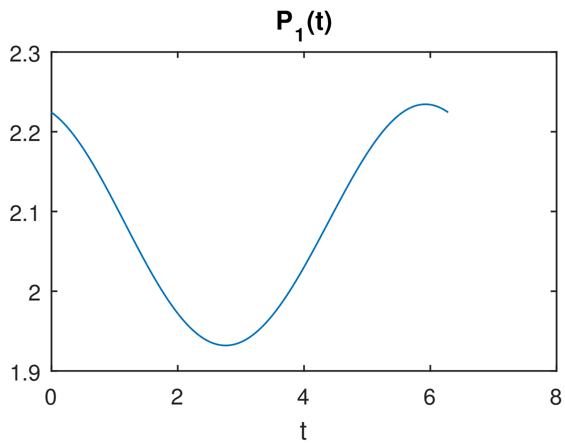

Fig. 1. Solution $P_1(t)$ of (11) for M_1

Figure 1 depicts the required solution, obtained using the periodic generator method of [4], with numerical integration implemented using a first order, symplectic Euler method [16] with 2^{16} samples over one period. Hence, $F_1(t) = -P_1(t)$, and

$$G_1 = \left[\begin{array}{c|c} 0.8 + 0.1\cos t - P_1(t) & 1 \\ \hline 1 & 0 \\ -P_1(t) & 1 \end{array}\right].$$

To verify the index condition, the Monodromy matrix associated with (14) is obtained via symplectic integration, and found to have eigenvalues of 1.1356×10^{-4} and 2.506×10^3. Hence, $\mathrm{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) = 0 \;\forall\, \tau \in \mathbb{R}$ as required.

Finally, apply (15)-(18) to obtain a state-space realization for $\tilde{\mathbf{G}}_2\mathbf{G}_1$:

$$\left[\begin{array}{cc|c} 0.8 + 0.1\cos t - P_1(t) & 0 & 1 \\ -P_1(t) + 2.4142 & -1.4142 & 1 \\ \hline -1 & 1 & 0 \end{array}\right].$$

Then,

$$\delta_\nu(M_1, M_2) = \gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1) = 0.1312,$$

where $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1)$ is computed using the bisection algorithm described in Section IV. In each iteration of the algorithm, the existence of a T-periodic stabilizing positive semidefinite solution to (19) is established by checking the following conditions:

1) The Monodromy matrix corresponding to the Hamiltonian associated with (19) has no eigenvalues on the unit circle.
2) The periodic generator $\Pi(0)$, for the solution of (19), is positive semi-definite.
3) The solution $\Pi(t)$ is positive semi-definite for all t, and stabilizing for all t.

If any of these conditions are not satisfied, the remaining conditions do not need to be checked, and $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1) \geq \delta$. If all the conditions are satisfied, then $\gamma(\tilde{\mathbf{G}}_2\mathbf{G}_1) < \delta$.

Similarly, it can verified that $\delta_\nu(M_1, M_3) = 0.0375$.

VI. Conclusion

A tractable method is provided for obtaining normalized, coprime representations for a class of periodic state-space models. Then, several important properties of these representations are verified. This leads to a tractable method for computing a generalized ν-gap metric for the class of systems studied.

References

[1] G. Vinnicombe, "Frequency domain uncertainty and the graph topology," *IEEE Transactions on Automatic Control*, vol. 38, no. 9, pp. 1371–1383, 1993.

[2] M. Cantoni, U. T. Jonsson, and S. Z. Khong, "Robust stability analysis for feedback interconnections of time-varying linear systems," *SIAM Journal on Control and Optimization*, vol. 51, no. 1, pp. 353–379, 2013.

[3] M. Mueller and M. Cantoni, "Normalized coprime representations for time-varying linear systems," in *Decision and Control (CDC), 2010 49th IEEE Conference on*. IEEE, 2010, pp. 7718–7723.

[4] A. Varga, "On solving periodic Riccati equations," *Numerical Linear Algebra with Applications*, vol. 15, no. 9, pp. 809–835, 2008.

[5] I. Gohberg, S. Goldberg, and M. Kaashoek, *Basic classes of linear operators*. Birkhäuser, 2012.

[6] M. Saeed Akram and M. Cantoni, "A note on robust stability analysis for feedback interconnections of time-varying linear systems," submitted for consideration to MTNS 2016.

[7] W. Coppel, *Stability and asymptotic behavior of differential equations*. Heath and Boston, 1965.

[8] I. Gohberg, M. Kaashoek, and F. Van Schagen, "Non-compact integral operators with semi-separable kernels and their discrete analogues: inversion and Fredholm properties," *Integral Equations and Operator Theory*, vol. 7, no. 5, pp. 642–703, 1984.

[9] A. Ben-Artzi and I. Gohberg, "Dichotomy of systems and invertibility of linear ordinary differential operators," in *Time-variant systems and interpolation*. Springer, 1992, pp. 90–119.

[10] S. Bittanti, A. J. Laub, and J. C. Willems, *The Riccati Equation*. Springer Science & Business Media, 2012.

[11] A. Varga, "On solving periodic differential matrix equations with applications to periodic system norms computation," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*. IEEE, 2005, pp. 6545–6550.

[12] R. J. Sacker and G. R. Sell, "Existence of dichotomies and invariant splittings for linear differential systems I," *Journal of Differential Equations*, vol. 15, no. 3, pp. 429–458, 1974.

[13] P. Colaneri, "Continuous-time periodic systems in H_2 and H_∞. Part I: Theoretical aspects," *Kybernetika*, vol. 36, no. 2, pp. 211–242, 2000.

[14] M. Cantoni and H. Sandberg, "Computing the L_2 gain for linear periodic continuous-time systems," *Automatica*, vol. 45, no. 3, pp. 783–789, 2009.

[15] G. Vinnicombe, *Uncertainty and Feedback: H^∞ loop-shaping and the ν-gap metric*. World Scientific, 2000.

[16] E. Hairer, C. Lubich, and G. Wanner, *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*. Springer Science & Business Media, 2006, vol. 31.