Optimal LQG control over continuous fading channels

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Abstract: This paper studies the optimal control of linear systems over continuous valued fading channels. The sensor measurements are sent over a fading wireless channel to a remote controller using the analog amplify and forward technique. The controller then computes a control signal, which is transmitted over another fading channel to the actuator. Under the assumption of full channel state information (CSI) for both wireless links, we derive the optimal LQG control law. In the case where full channel state information is not available but channel statistics are available, we present the optimal linear static estimator and controller. Numerical comparisons are made between the full CSI and statistical CSI solutions.

Keywords: Fading channels, LQG control, sensor networks, stability

1. INTRODUCTION

In recent years there has been increasing growth in the use of wireless technologies in diverse applications such as home automation, telecommunications, and industrial monitoring and control. The challenges posed by wireless channels due to its time-varying nature are considerable, and much effort has been devoted to modelling and overcoming these effects.

One way to model the wireless channel is to regard it as a channel where packets can be received if the channel is of sufficiently good quality, and dropped if the channel is poor quality. The Kalman filtering problem with Bernoulli packet losses has been studied in Sinopoli et al. (2004), who showed the existence of a threshold, such that if the packet arrival rate is below this threshold then the expected error covariance becomes unbounded. This work has been extended in various directions such as e.g. Huang and Dey (2007); Epstein et al. (2008); Xu and Hespanha (2005); Schenato (2006). The problem of control over such packet dropping links has been studied in e.g. Sinopoli et al. (2005, 2006); Imer et al. (2006); Gupta et al. (2007), and conditions on the packet arrival rates for stability of the closed loop system has been derived. See also Schenato et al. (2007) and the references therein for a review of related work.

Another way in which one can view the wireless channel is to regard it as a continuous valued channel with time-varying channel gains, with commonly used channel models such as Rayleigh or Nakagami. Kalman filtering with continuous faded measurements has been studied in e.g. Mostofi and Murray (2005); Dey et al. (2009), which showed that under certain conditions on the fading distribution the expected error covariance will always remain bounded.

In this paper we extend the work of Dey et al. (2009) to LQG control over channels with continuous faded measurements. We assume fading channels between both the sensor and controller, and between the controller and actuator/plant. Under the assumption of full channel state information (CSI) we derive the optimal control law and show that the separation principle holds. We consider both the finite horizon and infinite horizon problems. In the case where channel state information is not available but channel statistics are available, the optimal linear estimator/controller will be presented, using results from De Koning (1992).

2. SYSTEM MODEL

A block diagram of the model we study in this paper can be found in Fig. 1. We consider a plant

$$x_{k+1} = Ax_k + Bu_k + w_k$$

(1)

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$. We have a sensor with measurements

$$y_k = Cx_k + v_k$$

(2)

where $y_k \in \mathbb{R}^l$. The noise processes $\{w_k\}$ and $\{v_k\}$ are i.i.d. zero mean Gaussian with covariances $\Sigma_w > 0$ and $\Sigma_v > 0$ respectively. ¹

The sensor transmits this measurement over a fading channel to a remote controller using the analog amplify and forwarding technique (Gastpar and Vetterli (2003)), i.e. the sensor simply amplifies and forwards its measurement to the controller. We assume that all the measurement components are sent separately via orthogonal channels.

¹ For a matrix $X$, we say that $X > 0$ if $X$ is positive definite, and $X \geq 0$ if $X$ is positive semi-definite.
Fig. 1. System model

(Cui et al. (2007)) within the measurement time interval. The controller thus receives
\[ z_k = H_k \alpha_k y_k + n_k \]
where \( H_k = \text{diag}(h^1_k, \ldots, h^l_k) \), with \( h^i_k \geq 0, i = 1, \ldots, l \), are the channel gains, \( \alpha_k = \text{diag}(\alpha^1_k, \ldots, \alpha^l_k) \) are the amplification factors in the analog forwarding technique, and \( n_k \) is additive noise that represents the channel noise in the communication channel between the sensor and controller. The controller computes a control signal \( u^*_k \in \mathbb{R}^m \), which is then sent over another fading channel to the actuator/plant, again using the analog forwarding technique. The control input to the plant is thus
\[ u_k = G_k \beta_k u^*_k + m_k \]
where \( G_k = \text{diag}(g^1_k, \ldots, g^m_k) \), with \( g^j_k \geq 0, j = 1, \ldots, m \), are the channel gains, \( \beta_k = \text{diag}(\beta^1_k, \ldots, \beta^m_k) \) are the amplification factors and \( m_k \) is the channel noise associated with the controller and plant. The noise processes \( \{n_k\} \) and \( \{m_k\} \) are i.i.d. zero mean Gaussian with covariances \( \Sigma_n \) and \( \Sigma_m \) respectively. In this paper we assume the block fading model (see e.g. Caire et al. (1999)), such that the channels stay constant within each fading block represented by the time index \( k \), but are independent from block to block. We also allow the fading processes \( G_k \) and \( H_k \) to have continuous distributions in general. The noise processes \( w_k, v_k, u_k, m_k \) and fading processes \( G_k, H_k \) are assumed to be mutually independent.

The system (1)-(4) above can be rewritten as
\[ x_{k+1} = Ax_k + B_k u^*_k + \bar{w}_k \]
\[ z_k = \check{C}_k x_k + \bar{v}_k \]
if we define \( B_k = BG_k \beta_k \), \( \check{C}_k = H_k \alpha_k C \), \( \bar{w}_k = Bm_k + w_k \), \( \bar{v}_k = H_k \alpha_k v_k + n_k \). The noise processes \( \{\bar{w}_k\} \) and \( \{\bar{v}_k\} \) have covariances \( \Sigma_{\bar{w}} = B \Sigma_m B^T + \Sigma_w \) and \( \Sigma_{\bar{v}} = H_k \alpha_k \Sigma_v H_k + \Sigma_n \) respectively.

3. OPTIMAL LQG CONTROL UNDER FULL CSI

We first consider the case where we have full CSI (so full knowledge of \( G_k \) and \( H_k \) is available to the controller at time \( k \)). The amplification factors \( \alpha_k \) and \( \beta_k \) are usually chosen to satisfy power constraints at the sensor transmitter and in the transmission of the control signals. Here \( \alpha_k \) and \( \beta_k \) are taken to be either constant or known functions of time.

3.1 Finite horizon

In the finite horizon case, we have a cost
\[ J_N = \mathbb{E} \left[ x_N^T Q_N x_N + \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u^*_k^T R_k u^*_k) \right] \]
where \( Q_k \geq 0, \forall k \) and \( R_k > 0, \forall k \). With full CSI, the information set available to the controller at time \( k \) is
\[ I_k = \{ z_0, \ldots, z_k, u^*_0, \ldots, u^*_k, \bar{w}_0, \ldots, \bar{w}_k, \bar{v}_0, \ldots, \bar{v}_k \} \]
The controller thus receives \( \bar{w}_k \), \( \bar{v}_k \) and \( \bar{w}_k \), \( \bar{v}_k \) being a function of the information set \( I_k \) at each time \( k \).

**Lemma 1.** The optimal control \( u^*_k \) that minimizes \( J_N \) in (6), subject to \( u^*_k \) being a function of the information set \( I_k \) in (7), is
\[ u^*_k = - (B_k^T K_{k+1} B_k + R_k)^{-1} B_k^T K_{k+1} A \bar{x}_k \]
where \( \bar{x}_k = \mathbb{E}[x_k|I_k] \), \( B_k = BG_k \beta_k \), and \( \{K_k\} \) are given recursively by
\[ K_N = Q_N, \]
\[ K_k = \mathbb{E}[(A^T (K_{k+1} - K_k B_k) + (B_k + B_k^T K_{k+1} B_k)^{-1} B_k^T K_{k+1} A) + Q_k]. \]
The expectation in (9) is with respect to \( G_k \) (since \( B_k = BG_k \beta_k \)).

**Proof.** The proof uses dynamic programming and is along similar lines to e.g. Bertsekas (2000), see also Imel et al. (2006). Define
\[ V_N(I_k) = \mathbb{E}[x_N^T Q_N x_N|I_N], \]
\[ V_k(I_k) = \min_{u^*_k} \mathbb{E}[x_k^T Q_k x_k + u^*_k^T R_k u^*_k + V_{k+1}(I_{k+1})|I_k] \]
We have
\[ V_{N-1}(I_{N-1}) = \min_{u^*_{N-1}} \mathbb{E}[x_{N-1}^T Q_{N-1} x_{N-1} + u_{N-1}^T R_{N-1} u_{N-1} + (A x_{N-1} + B_{N-1} \bar{x}_{N-1} + \bar{w}_{N-1})^T Q_N (A x_{N-1} + B_{N-1} \bar{x}_{N-1} + \bar{w}_{N-1}) + \mathbb{E}[\bar{w}_{N-1}^T Q_N \bar{w}_{N-1}] + \mathbb{E}[\bar{w}_{N-1}^T (B_{N-1}^T Q_N B_{N-1} + R_{N-1}) \bar{w}_{N-1} + 2 \mathbb{E}[x_{N-1}^T I_{N-1}] A^T Q_N B_{N-1} \bar{w}_{N-1} \bar{w}_{N-1}^T] ] \]
which gives
\[ u_{N-1}^* = - (B_{N-1}^T Q_N B_{N-1} + R_{N-1})^{-1} B_{N-1}^T Q_N A \bar{x}_{N-1} \]
Substituting back into the expression for \( V_{N-1}(I_{N-1}) \), we obtain
\[ V_{N-1}(I_{N-1}) = \mathbb{E}[\bar{w}_{N-1}^T Q_N \bar{w}_{N-1}] + \mathbb{E}[(x_{N-1} - \bar{x}_{N-1})^T P_{N-1} (x_{N-1} - \bar{x}_{N-1})|I_{N-1}] + \mathbb{E}[x_{N-1}^T K_{N-1} x_{N-1}|I_{N-1}] \]
where
\[ P_{N-1} = A^T Q_N B_{N-1} (R_{N-1} + B_{N-1}^T Q_N B_{N-1})^{-1} B_{N-1}^T Q_N A, \]
\[ K_{N-1} = A^T Q_N A + Q_{N-1} - P_{N-1} \]
For period \( N - 2 \) we have
\[ V_{N-2}(I_{N-2}) = \min_{u_{N-2}} \mathbb{E}[x_{N-2}^T Q N^{-2} x_{N-2} + V_{N-1}(I_{N-1})| I_{N-2}] \]
\[ = \mathbb{E}[x_{N-2}^T Q N^{-2} x_{N-2} + u_{N-2} R_{N-2} u_{N-2} + V_{N-1}(I_{N-1})| I_{N-2}] \]
\[ = \mathbb{E}[x_{N-2}^T Q N^{-2} x_{N-2} + \min_{u_{N-2}} v_{N-2} u_{N-2} + \mathbb{E}[x_{N-1}^T \tilde{K}_{N-1} x_{N-1}| I_{N-2}] + \mathbb{E}[x_{N-1}^T w_{N-1} | I_{N-2}] ] \]
\[ = \min_{u_{N-2}} \mathbb{E}[x_{N-2}^T R_{N-2} u_{N-2} + \mathbb{E}[x_{N-1}^T \tilde{K}_{N-1} x_{N-1}| I_{N-2}] + \mathbb{E}[x_{N-1}^T w_{N-1} | I_{N-2}] ] \]
where \( \tilde{K}_{N-1} = \mathbb{E}[\tilde{x}_{N-1}^T | I_{N-2}] \) and \( \tilde{K}_{N-1} = \mathbb{E}[\tilde{x}_{N-1}^T | I_{N-2}] \).

**Remark:** If the fading process \( \{ G_k \} \) is discrete (i.e. components of \( G_k \) take on discrete values), then the optimal controller can also be derived by using results on optimal control of jump linear systems, see Chizeck and Ji (1988).

**Remark:** System (5) is a time-varying linear system. If we attempt to apply the standard solution of the time-varying LQG problem to (5) directly, we obtain
\[ u_k^* = -(R_k + B_k^T K_{k+1} B_k)^{-1} B_k^T K_{k+1} \hat{x}_{k+1} \]
where \( K_k \) are given by
\[ K_k = Q_k, \]
\[ K_k = A_k^T (K_{k+1} - K_{k+1} B_k) \]
\[ \times (R_k + B_k^T K_{k+1} B_k)^{-1} B_k^T K_{k+1} A_k + Q_k. \]

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\[ K_k = A_k^T (K_{k+1} - K_{k+1} B_k) \]
\[ \times (R_k + B_k^T K_{k+1} B_k)^{-1} B_k^T K_{k+1} A_k + Q_k. \]

The optimal control \( u_k^* \) in (8) is a linear function of \( \hat{x}_k = \mathbb{E}[x_k|I_k] \), which can be computed with the standard time-varying Kalman filter. Hence a separation principle holds for this problem. Further define
\[ \hat{x}_{k+1|k} = \mathbb{E}[x_{k+1} | I_k], \]
\[ P_{k+1|k} = \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T | I_k]. \]

Below, we summarise the Kalman filtering equations:
\[ \hat{x}_{k+1|k} = \hat{x}_{k+1|k} = A \hat{x}_{k+1|k} + B_1 u_{k+1} \]
\[ S_k = P_{k+1|k} C_k^T (C_k P_{k+1|k} C_k^T + \Sigma_{\phi_k})^{-1} \]
\[ P_{k|k} = P_{k+1|k} - S_k C_k P_{k+1|k} \]
\[ P_{k|k} = A P_{k-1|k-1} A^T + \Sigma_{\phi_k} \]
where \( B_k = B G_k \beta_k, C_k = H_k \alpha_k C, \Sigma_{\phi} = B \Sigma_{m} B^T + \Sigma_{w} \)
and \( \Sigma_{\phi} = H_k \alpha_k \Sigma \alpha_k H_k + \Sigma_{m} \).

**Remark:** The expectation in (9) is in general difficult to compute analytically, due to the difficulty in explicitly evaluating the expectation of the nonlinear term. However in cases such as scalar control signals, closed form expressions can be obtained for specific fading distributions. See Section 5 for an example with Rayleigh fading.
where the last line holds since \( E \).

Using similar arguments to Imer et al. (2006) (see also Bertsekas (2000)), we can then show that \( E[\xi_k | k=k_0] \rightarrow 0 \) as \( k \rightarrow \infty \).

Thus for any fading processes \( G_k \) and \( H_k \) satisfying the conditions of Lemma 2, the problem of minimizing (12) is well defined, and the minimum \( J_\infty \) will be finite.

4. OPTIMAL LINEAR CONTROL WITH STATISTICAL CSI

In this section we consider the case where we don't have knowledge of the values \( G_k \) and \( H_k \) (e.g. either because it is too difficult or requires too many resources to obtain), but know their channel statistics. By similar arguments to Schenato et al. (2007), the optimal controller can be shown to be generally nonlinear and difficult to derive. One alternative is to derive the optimal linear controller and estimator. Here we will use a static linear estimator and controller of the form

\[
\hat{x}_{k+1} = F\hat{x}_k + Kz_k \\
u_k = -L\hat{x}_k
\]

(15)

We again consider the infinite horizon case where we take \( \alpha_k = \alpha, \beta_k = \beta, Q_k = Q > 0, R_k = R > 0, \forall k \), and assume that the pairs \((A, B)\) and \((A, \Sigma_k^{1/2})\) are stabilizable, and the pairs \((A, C)\) and \((A, Q)\) are detectable. The cost function that we wish to minimize is the infinite horizon cost

\[
J_\infty = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right)
\]

(16)

where the minimization is over \((F, K, L)\). The situation above falls within the framework of systems with white parameters. In De Koning (1992) necessary and sufficient conditions for minimizing (16) subject to the estimator and controller being of the form (15) is derived, using techniques such as the matrix minimum principle. A method for computing the optimal \( F, K \) and \( L \) is then also given. Below we will present the method of De Koning (1992) adapted to our situation.\(^3\)

Define the following recursions:

\[
X_{1,k+1} = A^T X_{1,k} A - L_k^T E[B_k^T X_{1,k} B_k] + R_k + E[B_k^T X_{2,k} B_k] - E[B_k^T X_{2,k} E[B_k]] L_k + Q_k + E[C_k^T K_k^T X_{2,k} K_k E[C_k]]
\]

where the last line holds since \( \xi_k \) is time-varying, however it can be verified that the role of \( W \) in De Koning (1992) can be replaced by \( E[\Sigma_{v_k}] \) here.

\[
X_{3,k+1} = A X_{3,k} A^T - K_k E[C_k X_{3,k} C_k^T] + E[\Sigma_{v_k}] + E[C_k X_{4,k} C_k^T] + E[C_k X_{4,k} E[C_k]] K_k^T + \Sigma_{w_0} + E[B_k L_k X_{4,k} L_k^T B_k^T] - E[B_k L_k X_{4,k} L_k^T E[B_k^T]] + K_k E[C_k X_{4,k} C_k^T] + E[\Sigma_{v_k}] + E[C_k X_{4,k} C_k^T] - E[C_k X_{4,k} E[C_k]] K_k^T
\]

(17)
\[ F_{k+1} = A - \mathbb{E}[B_k]L_k - K_k \mathbb{E}[C_k] \]
\[ K_{k+1} = AX_{3,k} \mathbb{E}[C_k^T](E[C_kX_{3,k}C_k^T]) + \mathbb{E}[^{\dagger}\Sigma_{\nu_k}] + \mathbb{E}[C_kX_{4,k}C_k^T] - \mathbb{E}[C_kX_{4,k}\mathbb{E}[C_k^T]) \tag{18} \]
\[ L_{k+1} = \mathbb{E}[B_k^T]X_{4,k}B_k + R + \mathbb{E}[B_k^T]X_{2,k}B_k \]
\[ - \mathbb{E}[B_k^T]X_{2,k}B_k] \]

where \( ^{\dagger} \) represents the Moore-Penrose inverse. We also have the concept of mean square compensatability introduced in De Koning (1992).

**Definition:** We say that \((A, B_k, C_k)\) is mean square compensatable if there exist \(F, K, L\) such that the system \(x_{k+1} = \Phi_k x_k\)

\[ \text{has } \mathbb{E}[|x_k|^2] \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ where } \]
\[ \Phi_k = \begin{bmatrix} A - B_k L \\ K & C_k & F \end{bmatrix}. \]

We then have the following:

**Lemma 3.** (i) Assume that \((A, B_k, C_k)\) is mean square compensatable. Then starting from \(X_{1,0} = 0, X_{2,0} = 0, X_{3,0} = 0, X_{4,0} = 0\) (where 0 here represents the zero matrix), the recursions (17)-(18) converge to limiting values as \(k \rightarrow \infty\). The optimal \(F^*, K^*, L^*\) for (15) are given by the limiting values of the \(F_k, K_k, L_k\) recursions respectively.

(ii) \((A, B_k, C_k)\) is mean square compensatable if and only if the recursions (17)-(18) converge to limiting values as \(k \rightarrow \infty\).

**Proof** (i) This is essentially Theorem 3 of De Koning (1992).


5. **NUMERICAL EXAMPLE**

We consider a scalar system, with \(g_k\) and \(h_k\) both Rayleigh distributed, so that \(g_k^2\) and \(h_k^2\) are exponentially distributed with means \(1/\lambda_g\) and \(1/\lambda_h\) respectively.

The optimal control in the case of full CSI is then

\[ u^*_k = -g_k \sigma K_{k+1} + h_k \hat{x}_k \]

with

\[ K_N = Q_N \]
\[ K_k = \mathbb{E}\left[ \frac{a^2 K_{k+1} R_k}{g_k^2 b^2 K_{k+1} + R_k} \right] + Q_k \]
\[ = \frac{\lambda_0 R_k}{\beta^2 b^2} \exp \left( \frac{\lambda_0 R_k}{\beta^2 b^2} \right) E_1 \left( \frac{\lambda_0 R_k}{\beta^2 b^2} \right) + Q_k \]

where \(E_1(x)\) is the exponential integral.

In the computation of the optimal linear controller in the case with statistical CSI, the terms in the recursions simplify to \(E[B_k] = \beta \sqrt{\frac{\pi}{4\alpha_0}} C_k, \mathbb{E}[C_k] = \alpha c \sqrt{\frac{\pi}{4\alpha_0}} B_k = \frac{b^2 \sigma_m^2 + \sigma_n^2}{\lambda_m}, \mathbb{E}[\Sigma_{\nu_k}] = \frac{\alpha^2 \sigma_m^2 + \sigma_n^2}{\lambda_m}, \mathbb{E}[B_k^T X B_k] = \mathbb{E}[B_k X B_k^T] = \frac{\beta^2 \sigma_n^2}{\lambda_n}, X, \mathbb{E}[C_k^T X C_k] = \mathbb{E}[C_k^T C_k] = \frac{\alpha^2 \sigma_n^2}{\lambda_n} X.\)

We will consider a case with \(\beta = c = 1, \alpha = \sigma_n = \sigma_m = \sigma_n^2 = \lambda_m = 1, \alpha = \beta = 1, Q = R = 1, \lambda_0 = 2, \lambda_h = 5\).

In Figure 2 we plot the finite horizon expected cost \(J_N\) for different values of \(a\). From numerical computation, we find that for values of \(a\) greater than around 1.35, the stability criteria is no longer satisfied in the case with statistical CSI and the infinite horizon cost diverges.

In Figure 3 we plot the infinite horizon cost \(J_\infty\) for the infinite horizon solution with full CSI (13)-(14), and infinite horizon solution with statistical CSI of Section 4, for different values of \(\alpha\). From numerical computation, we find that for values of \(\alpha\) greater than around 1.35, the stability criteria is no longer satisfied in the case with statistical CSI and the infinite horizon cost diverges.

![Fig. 2. Scalar system, finite horizon. Comparison between causal and non-causal control.](image-url)
6. CONCLUSION

In this paper we have considered the optimal control of a system where there are continuous valued fading channels between the sensor and controller, and between the controller and actuator. We have derived the optimal LQG controller under full CSI and statistical CSI assumptions. Future work will include jointly optimizing the powers used in transmitting the sensor measurements and control signals over the fading channels, by optimizing the choices of the amplification factors $\alpha_k$ and $\beta_k$.

REFERENCES


