IV. CONCLUSION

In this note, we have investigated a distributed $M$-ary hypothesis testing problem with correlated observations. We have developed the necessary conditions for the optimal team decision rule. A Gauss–Seidel algorithm is proposed for determining the person-by-person optimal team decision strategy. An approximation algorithm is also proposed to greatly reduce computations. It has been observed that if the approximation algorithm converges, it provides a solution very near to the optimal one. The results from our illustrative examples have shown that: i) a distributed team can achieve nearly centralized performance with very short communication messages; ii) the workload should be balanced among team members having roughly equal expertise level; iii) it is better to have a large number of short and independent messages than a smaller number of relatively long messages, when the total capacity of the communication channels is fixed and the information quality of each subordinate decision maker is identical; and iv) there exists an almost linear degradation in team performance with positive correlation of data among team members.

The basic assumption of this note is that every DM in the team is rational in that the DM strives to maximize team performance. However, the research by Mallubhatla et al. [5], [6] suggests that human decision making is far from optimal, and that they exhibit cognitive biases such as devaluation of subordinates' opinions by the primary decision maker and recency effects. A sequential Kalman filter-based model that incorporated these biases provided excellent model-data match in a distributed threat identification problem [5], [6].

REFERENCES


On the Characterization of Fixed Modes in Decentralized Control

Z. Gong and M. Aldeen

Abstract—Characterizations of decentralized fixed modes in terms of remnant zeros are presented in this technical note. A new criterion for testing the fixed modes is obtained. The new result can be used to derive an existing algebraic criterion, through a simple rank evaluation of certain matrices.

I. INTRODUCTION

The notion of "fixed modes," introduced in [1], plays a fundamental role in the stabilization of decentralized control systems. Consequently, a number of criteria for the characterization of the fixed modes has been proposed in the literature [2]–[10] over the past 20 years. In this technical note it is shown that the fixed modes can be characterized in terms of remnant zeros [11].

Consider a decentralized control system described by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t)$$ (1.1a)

$$y_i(t) = C_i x(t); \quad i = 1, 2, \ldots, N$$ (1.1b)

where $N$ is the number of its local control stations, $x(t) \in \mathbb{R}^m$ is the state of the system, $u_i(t) \in \mathbb{R}^m$ and $y_i(t) \in \mathbb{R}^l$ are the input and
output of the \( i \)th local control station, and \( A, B, \) and \( C \) are real constant matrices of appropriate sizes.

Let \( N \) denote the set \( \{1, 2, \ldots, N\} \) and \( S \) denote a subset of \( N \), i.e., \( S \subseteq N \). Let \( N - S \) denote the complementary subset of \( S \) in \( N \). Suppose \( \bar{S} = \{i_1, i_2, \ldots, i_s\} \) and \( N - S = \{i_{s+1}, i_{s+2}, \ldots, i_N\} \), then the matrices \( B_S \) and \( C_{N-S} \) are defined as

\[
C_{N-S} = \begin{bmatrix}
C_{i_1} \\
C_{i_2} \\
\vdots \\
C_{i_N}
\end{bmatrix}, \quad B_S = \begin{bmatrix}
B_{i_1} B_{i_2} \cdots B_{i_s}
\end{bmatrix}.
\]

**Definition 1:** Given the system (1.1) and a subset \( S \) of \( N \). A system described by the triple \( (C_{N-S}, A, B_S) \) is called a complementary subsystem of the system (1.1).

The definition of the complementary subsystems as stated above differs from that given in [12] only in that \( S \) is not restricted to being a proper subset of \( N \).

**II. FIXED MODES OF STRONGLY CONNECTED SYSTEMS**

In this section, the characterization of the fixed modes of strongly connected systems [12] is investigated.

The fixed modes of the system (1.1) are closely related to the remnant zeros of the complementary subsystems of this system. If the system (1.1) is strongly connected and centralized controllable and observable, then the following expressions, deduced from [12, theorem 4], hold:

\[
\Lambda' \subseteq \Lambda; \quad S \in \bar{S}^* \tag{2.1}
\]

\[
\Lambda \subseteq \bigcup_{S \in \bar{S}^*} \Lambda'_S \tag{2.2}
\]

where \( \Lambda \) denotes the set of the fixed modes of the system (1.1), \( \Lambda'_S \) denotes the set of the remnant zeros of the complementary subsystem \( (C_{N-S}, A, B_S) \) and \( \bar{S}^* \) denotes the set of all the proper subsets of \( N \).

It is clear that (2.1) implies

\[
\bigcup_{S \in \bar{S}^*} \Lambda'_S \subseteq \Lambda \tag{2.3}
\]

and that (2.2) and (2.3) imply

\[
\Lambda = \bigcup_{S \in \bar{S}^*} \Lambda'_S \tag{2.4}
\]

Hence, the following lemma.

**Lemma 1:** If the system (1.1) is strongly connected and centralized controllable and observable, then (2.4) holds.

The following lemma, deduced from [11, lemma 3] and from the definition of the remnant zeros, establishes the condition for the existence of the remnant zeros.

**Lemma 2:** Given a system \( (C, A, B) \). A number \( \lambda \), possibly a complex, is a remnant zero of the system if and only if

\[
\text{Rank} \begin{bmatrix}
\lambda I - A & B \\
C & 0
\end{bmatrix} < n. \tag{2.5}
\]

where \( I \) is the identity matrix.

Now, we introduce the following criterion, which is a direct consequence of Lemma 1 and Lemma 2, for testing the fixed modes.

**Criterion 1:** If the system (1.1) is strongly connected, centralized controllable and observable, then a number \( \lambda \) is a fixed mode of the system (1.1) if and only if there exists a proper subset \( S \) of \( N \) so that

\[
\text{Rank} \begin{bmatrix}
\lambda I - A & B_S \\
C_{N-S} & 0
\end{bmatrix} < n. \tag{2.6}
\]

It should be noted that when \( S = N \), \( \Lambda'_S \) is the set of the centralized uncontrollable modes of the system (1.1), and when \( \Lambda = \emptyset \), \( \Lambda'_S \) is the set of the centralized unobservable modes of the system (1.1). Let \( \bar{S}^* \) denote the set of all the subsets of \( N \), which includes the empty set \( \emptyset \) and the set \( N \). Then, under the assumption given in Lemma 1, (2.4) still holds if \( \bar{S}^* \) is replaced by \( \bar{S}^* \). By the same reason, (2.6) still holds if the words "a proper subset \( S \) of \( N' \)" are replaced by the words "a subset \( S \) of \( N' \)" in Criterion 1.

In the following a new theorem is introduced for the characterization of the fixed modes of strongly connected systems. This theorem extending the result of Lemma 1 to the case where the system (1.1) is not necessarily centralized controllable and observable.

**Theorem 1:** If the system (1.1) is strongly connected, then

\[
\Lambda = \bigcup_{S \in \bar{S}^*} \Lambda'_S. \tag{2.7}
\]

**Proof:** Write \( C = [C_1 \ C_2 \cdots C_s] \) and \( B = [B_1 \ B_2 \cdots B_N] \). Then, according to [13], there exists a nonsingular transformation matrix \( T \) for the system (1.1) so that

\[
TAT^{-1} = \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\
0 & A_{22} & A_{23} & A_{24} \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{bmatrix} \quad TB = \begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2 \\
\bar{B}_3 \\
\bar{B}_4
\end{bmatrix}
\]

\[
CT^{-1} = \begin{bmatrix}
0 & C_2 & 0 & C_4
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} \\
0 & A_{22}
\end{bmatrix} ; \quad \begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} ; \quad \begin{bmatrix}
C_2 \\
C_4
\end{bmatrix} ; \quad \begin{bmatrix}
\bar{A}_{22} & \bar{A}_{24} \\
0 & \bar{A}_{44}
\end{bmatrix}
\]

is a controllable and an observable pair, respectively.

Let \( \Lambda' \) denote the set of the union of the centralized uncontrollable and unobservable modes, i.e.,

\[
\Lambda' = \sigma(\bar{A}_{11}) \cup \sigma(\bar{A}_{33}) \cup \sigma(\bar{A}_{44}). \tag{2.9}
\]

Let \( \Lambda^d \) denote the set of the fixed modes of the system \( (\bar{C}_2, \bar{A}_{22}, \bar{B}_2) \) and \( \overline{\Lambda^c} \) denote the set of the remnant zeros of the complementary subsystem \( (C_{N-S}, \bar{A}_{22}, \bar{B}_2) \) of the system \( (C_2, \bar{A}_{22}, \bar{B}_2) \), where

\[
\bar{B}_2 \text{ is the matrix formed from the columns of } \bar{B}_2 \text{ in the same way that the matrix } B_S \text{ is formed from } B, \text{ and } C_{N-S} \text{ is the matrix formed from the rows of } C_2.
\]

Based on the above notations, the following holds:

\[
\Lambda'_S \cup \Lambda = \overline{\Lambda^c} \cup \Lambda^d. \tag{2.10}
\]

For the reason of space limitation, a proof of (2.10), which is simple, has been omitted from this technical note.

It is easy to see that the system \( (\bar{C}_2, \bar{A}_{22}, \bar{B}_2) \) satisfies the conditions in Lemma 1. Hence

\[
\Lambda^d = \bigcup_{S \in \bar{S}^*} \overline{\Lambda^c}_S. \tag{2.11}
\]

It is also easy to prove that

\[
\Lambda = \Lambda^d \cup \Lambda'. \tag{2.12}
\]

As pointed out above, the set of the centralized fixed modes \( \Lambda' \) is the union of the remnant zeros of the complementary subsystems \( (C_{N-S}, A, B_S) \) when \( S = N \) and \( S = \emptyset \), i.e.,

\[
\Lambda' = \Lambda'_N \cup \Lambda'_\emptyset. \tag{2.13}
\]
Since (2.10) implies
\[ \left\{ \bigcup_{s \in S^*} X_s \right\} \cup \mathcal{N} = \left\{ \bigcup_{s \in S^*} X_s \right\} \cup \mathcal{N} \] \hfill (2.14)

it follows from (2.11), (2.12), and (2.13) that
\[ \Lambda = \left\{ \bigcup_{s \in S^*} X_s \right\} \cup \mathcal{N}_p \cup \mathcal{N}_w = \bigcup_{s \in S^*} X_s. \] \hfill (2.15)

This completes the proof of Theorem 1.

Theorem 1 and Lemma 2 may now be used to obtain the following criterion for testing the fixed modes of the strongly connected systems.

**Criterion 2:** If the system (1.1) is strongly connected, then a number \( \lambda \) is a fixed mode of the system (1.1) if and only if there exists a subset \( S \) of \( \mathcal{N} \) so that
\[
\text{Rank} \left[ \begin{array}{cc}
\lambda I - A & B_S \\
C_{N-S} & 0
\end{array} \right] < n. \] \hfill (2.16)

III. CHARACTERIZATION OF FIXED MODES

In the case that the system (1.1) is not strongly connected, it can be decomposed into a number of strongly connected components as shown in [12]. Suppose that the \( p \)-th strongly connected component has \( N_p \) local control stations, and that their indices form a set \( N_p = \{ i_1^p, i_2^p, \ldots, i_{N_p}^p \} \). Then (1.1) can be rewritten as
\[
\dot{x}(t) = A_N x(t) + \sum_{p=1}^{N^*} B_N u_N(t) \] \hfill (3.1a)
\[
y_N(t) = C_N x(t); \quad p = 1, 2, \ldots, N^* \] \hfill (3.1b)

where \( N^* \) denotes the number of strongly connected components, \( u_N(t) \) and \( y_N(t) \) denote, respectively, the inputs and the output vectors consisting of all of the inputs and the outputs of the local control stations in the \( p \)-th strongly connected component. Furthermore, with the strongly connected components ordered appropriately and the state vector of the system (1.1) chosen appropriately, (3.1) can be written so that their matrices have the following block triangular structure:
\[
A = \begin{bmatrix}
\tilde{A}_1 & \times & \cdots & \times \\
0 & \tilde{A}_2 & \cdots & \times \\
\cdots & \cdots & \cdots & \times \\
0 & 0 & \cdots & \tilde{A}_{N^*}
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
\tilde{B}_N \times & \cdots & \times \\
0 & \tilde{B}_N \times & \cdots \\
\cdots & \cdots & \cdots \\
0 & 0 & \cdots \tilde{B}_{N^*}
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
\tilde{C}_N \times & \cdots & \times \\
C_n \times & \cdots & \times \\
\cdots & \cdots & \cdots \\
0 & 0 & \cdots \tilde{C}_{N^*}
\end{bmatrix}
\] \hfill (3.2)

where the \( \times \)-marks denote submatrices of no importance to the present analysis, and \( \tilde{A}_p, \tilde{B}_N, \) and \( \tilde{C}_N \) \((p = 1, 2, \ldots, N^* )\) are nonzero matrices of appropriate sizes.

It is obvious that the subsystems \( (\tilde{C}_N, \tilde{A}_p, \tilde{B}_N) \) are strongly connected. Hence, we introduce the following definition.

**Definition 2:** Given the system (1.1) and its alternative description as given by (3.1) and (3.2), the subsystems \( (\tilde{C}_N, \tilde{A}_p, \tilde{B}_N) \) \((p = 1, 2, \ldots, N^* )\) are called strongly connected subsystems of the system (1.1).

Note that the notion of strongly connected subsystems defined here is different from that defined in [12]. If \( n_p \) is defined as the order of the \( p \)-th strongly connected subsystem, i.e., the dimension of \( \tilde{A}_p \), then \( n_1 + n_2 + \cdots + n_{N^*} = n \), which is not the case in [12].

Let \( \Lambda_p \) denote the set of the fixed modes of the strongly connected subsystem \( (\tilde{C}_N, \tilde{A}_p, \tilde{B}_N) \). It is not difficult to prove that
\[
\Lambda = \bigcup_{p=1}^{N^*} \Lambda_p. \] \hfill (3.3)

The following theorem summarizes this result.

**Theorem 2:** The set of the fixed modes of the system (1.1) is the union of the fixed modes of all of its strongly connected subsystems.

We now propose a new criterion, which is a direct consequence of Theorem 2 and Criterion 2, for testing the fixed modes of the system.

**Criterion 3:** Given the system (1.1) and its alternative description as given by (3.1) and (3.2). A number \( \lambda \) is a fixed mode of the system (1.1) if and only if there exists an integer \( q \in \{1, 2, \ldots, N^* \} \) and a subset \( S_q \) of \( N_q \), so that
\[
\text{Rank} \left[ \begin{array}{cc}
\lambda I - A & B_S \\
\tilde{C}_{N-S} & 0
\end{array} \right] < n_q. \] \hfill (3.4)

Note that Criterion 3 states the condition for the existence of the fixed modes in the strongly connected subsystems, while the corresponding criteria in [2] states the condition for the overall system, as given below.

**Criterion 4:** Given the system (1.1). A number \( \lambda \) is a fixed mode of the system (1.1) if and only if there exists a subset \( S \) of \( \mathcal{N} \) so that
\[
\text{Rank} \left[ \begin{array}{cc}
\lambda I - A & B_S \\
C_{N-S} & 0
\end{array} \right] < n. \] \hfill (3.5)

Although the above two criteria are different, both convey the same information about the existence of the fixed modes. It is obvious that by using Theorem 2, Criterion 3 can be deduced from Criterion 4. On the other hand, Criterion 4 can be easily derived from Criterion 3, as shown in the Appendix.

We now propose the following theorem, which extends the results of Lemma 1 and Theorem 1, for the characterization of the fixed modes.

**Theorem 3:** Given the system (1.1), then
\[
\Lambda = \bigcup_{s \in S^*} \Lambda_s. \] \hfill (3.6)

Theorem 3 can be proven from Criterion 4 and Lemma 2, but the proof is trivial and therefore omitted in this technical note.

IV. CONCLUSION

In this technical note the relationship between the fixed modes and the remnant zeros of the complementary subsystems for the strongly connected, centralized controllable, and observable systems (Lemma 1), which is implicit in the result of [12], is extended to the general case where the decentralized control systems are not necessarily centralized controllable and observable, and are not necessarily strongly connected (Theorem 3).

When a decentralized control system is decomposed into a number of strongly connected subsystems, it is shown that the set of the fixed modes of the decentralized control system is the union of the fixed modes of all of its strongly connected subsystems. Based on
this, a new criterion for testing the fixed modes (Criterion 3) is
obtained.

This technical note also presents a proof to the criterion proposed
in [2], through the newly obtained criterion (Criterion 3). As the
 derivation of the new criterion is based on systems theory rather
than algebra, which is the case in [2], the proof in this note is
believed to offer a better understanding of the occurrence of the
fixed modes.

APPENDIX

The following lemma is needed in the proof of Criterion 4.

Lemma A1: If a matrix $H \in \mathbb{R}^{n \times m}$ has a block triangular
structure of the form:

$$H = \begin{bmatrix}
H_1 & \cdots & \times & \cdots & \times \\
0 & H_2 & \cdots & \times \cdots & H_n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & H_v
\end{bmatrix}$$  (A1)

where $H_i \in \mathbb{R}^{n_i \times m_i}$, $i = 1, 2, \cdots, v$, are possibly nonzero matrices, then
the following expressions hold:

Rank $H \geq \text{Rank } H_1 + \text{Rank } H_2 + \cdots + \text{Rank } H_v$  (A2)

Rank $H \leq n_1 + n_2 + \cdots + n_{k-1} + \text{Rank } H_k$

$+ m_{k+1} + m_{k+2} + \cdots + m_v$,  $k = 1, 2, \cdots, v$  (A3)

where the terms $n_i$, $i = 1, 2, \cdots, k$, $k = 1$ are absent when
$k = 1$ and the terms $m_j$, $j = k+1, k+2, \cdots, v$, are absent
when $k = v$.

Proof: We shall prove the lemma in the case when $v = 3$ and
$k = 2$. The proof for the general case is only a trivial extension.

Write $r_i$ for the rank of $H_i$ and $I_i$ for the identity matrix with
dimension $r_i$, $i = 1, 2, 3$. Due to the block triangular structure of
$H$, by elementary operations, matrix $H$ can be transformed into

$$H^* = \begin{bmatrix}
I_1 & 0 & 0 & 0 & 0 \\
0 & 0 & D_1 & 0 & D_3 \\
0 & 0 & I_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$  (A4)

where $D_1$, $D_2$, and $D_3$ are $(n_1 - r_1) \times (m_2 - r_2)$, $(n_2 - r_2) \times
(m_3 - r_3)$, and $(n_1 - r_1) \times (m_3 - r_3)$ matrices, respectively.

It is now easy to see that

Rank $H = \text{Rank } H^* = r_1 + r_2 + r_3 + \text{Rank } D$  (A5)

where

$$D = \begin{bmatrix}
D_1 \\
0 \\
D_2
\end{bmatrix}$$  (A6)

It is also easy to show that

Rank $D \leq (n_1 - r_1) + (m_3 - r_3)$.  (A7)

From (A5) and (A7), the following results:

Rank $H \geq r_1 + r_2 + r_3$  (A8)

Rank $H \leq n_1 + r_2 + m_3$.  (A9)

This completes the proof of Lemma A1.

Proof of Criterion 4: In the case that the system (1.1) is
strongly connected, then $N^* = 1$ and the result of Criterion 4 is
implicit in Criterion 3. Here we prove Criterion 4 when $N^* > 1$.

Let

$$M_S(\lambda) = \begin{bmatrix}
\lambda I - A & B_S \\
C_{N-S} & 0
\end{bmatrix}.$$  (A10)

Suppose that $S_p$ is a subset of $N_p$, $p = 1, 2, \cdots, N^*$, so that
$S = S_1 \cup S_2 \cup \cdots \cup S_{N^*}$. Because of the block triangular
structure of the matrices, as shown in (3.2), by some row and column
permutations, $M_S(\lambda)$ can be transformed into

$$M_S(\lambda) \times \begin{bmatrix}
M_{S_1}(\lambda) & \cdots & \times \\
0 & M_{S_2}(\lambda) & \cdots & \times \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & M_{S_{N^*}}(\lambda)
\end{bmatrix}$$  (A11)

where

$$M_S(\lambda) = \begin{bmatrix}
\lambda I - \tilde{A}_p & \tilde{B}_{S_p} \\
\tilde{C}_{N_p - S_p} & 0
\end{bmatrix},$$  (A12)

where $\tilde{B}_{S_p}$ is the matrix formed from the columns of $B_{S_p}$
in the same way that the matrix $B_{S_p}$ is formed from $B_{S_p}$
and $\tilde{C}_{N_p - S_p}$ is the matrix formed by the rows of $\tilde{C}_{N_p}$.

To prove the sufficiency of the condition of Criterion 4, we
assume that there exists a subset $S$ of $N$ so that

Rank $M_S(\lambda) < n$.  (A13)

Then there must exist a matrix $M_{S_p}(\lambda)$, $\forall p \in \{1, 2, \cdots, N^*\}$
so that

Rank $M_{S_p}(\lambda) < n_p$.  (A14)

otherwise, according to the rank inequality (A2) in Lemma A1, the
rank of $M_S(\lambda)$ will be equal to or greater than $n$. According to
Criterion 3, (A14) implies that $\lambda$ is a fixed mode of the system (1.1).
This completes the proof of the sufficiency of the condition.

To prove the necessity of the condition of Criterion 4, we assume
that the number $\lambda$ is a fixed mode of the system (1.1). Then,
according to Criterion 3, there exists a subset $S_q$ of $N_q$, $q \in
\{1, 2, \cdots, N^*\}$, so that (A14) holds.

Consider the subset $S = N_1 \cup N_2 \cup \cdots \cup N_{q-1} \cup S_q$ of $N(S
= S_q$ when $q = 1$). Then the matrices in (A12) have the forms

$$M_{S_q}(\lambda) = \begin{bmatrix}
\lambda I - \tilde{A}_p & \tilde{B}_{S_q} \\
\tilde{C}_{N_q} & 0
\end{bmatrix},$$  (A15)

$$M_{S_q}(\lambda) = \begin{bmatrix}
\lambda I - \tilde{A}_p \\
\tilde{C}_{N_q}
\end{bmatrix},$$  (A16)

It is clear that the row dimension of $M_{S_q}(\lambda)$ is $n_p$ when $p
= 1, 2, \cdots, q - 1; q > 1$ and the column dimension of $M_{S_q}(\lambda)$ is also
$n_p$ when $p = q + 1, q + 2, \cdots, N^*; q < N^*$. Using the rank
inequalities (A3) in Lemma A1, we conclude that

Rank $M_{S_q}(\lambda) \leq n_1 + n_2 + \cdots + n_{q-1} + \text{Rank } M_{S_q}(\lambda)$

$+ n_{q+1} + n_{q+2} + \cdots + n_{N_q}$.  (A17)

From (A14) and (A17), it is easy to see that (A13) holds. This
completes the proof of the necessity of the condition and also the
proof of Criterion 4.

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Orthogonal Canonical Forms for Second-Order Systems

Trevor Williams and Alan J. Laub

Abstract—It is shown in this note that a linear second-order system with arbitrary damping cannot be reduced to Hessenberg-triangular form by means of orthogonal transformations. However, it is also shown that such an orthogonal reduction is always possible for the modal damping commonly assumed for models of flexible structures. In fact, it is shown that modally damped models can be orthogonally reduced to a new triangular second-order Schur form.

I. INTRODUCTION

Second-order models arise naturally in the study of many types of physical systems, with common examples being electrical and mechanical networks. An application area of great practical interest for dynamics and control is that of flexible space structures (FSS) [2], which are commonly represented by second-order finite element models of very high dimension. Now, continuum models of structures are, to be sure, much more elegant (see, for example [1], [11]) but it is generally still the case that setting up the governing partial differential equations and solving the resulting boundary value problems can only be done for relatively simple structures. In analyzing a realistic structure (spacecraft, airplane, etc.), a continuous structure model is seldom feasible and common engineering practice has been to use some method (usually finite elements) to get an approximate "M and K model" for it. The purpose of this note is to address some numerical linear algebra aspects of the latter and we shall not discuss the modeling question further.

The large size of the matrices describing these "M and K models" makes computational efficiency especially important in applications, so it is often desirable to perform an initial reduction to some canonical form which simplifies subsequent computations. Furthermore, numerical accuracy considerations are also extremely important when dealing with large matrices, so orthogonal transformations [12], [13] should be used as much as possible. An open question that then arises is just what canonical forms are obtainable by applying orthogonal row and column operations to the symmetric matrix triple \( \{ M, C, K \} \) that describes the dynamics of a second-order system \( (M \) is the mass matrix, \( C \) is the damping matrix, and \( K \) is the stiffness matrix; see (1) below).

This note addresses that question, and shows that the answer is closely related to the type of damping which acts on the system. If \( C \) is an arbitrary symmetric nonnegative definite matrix, it is proved that the reduction to the Hessenberg-triangular form desired for a second-order generalization of the first-order efficient frequency response technique of [7], [8] is impossible. On the other hand, this reduction is trivial if damping is zero or of the special form known as Rayleigh damping [3], [4]. The type of damping of most practical interest though is modal damping [3], [4], and for this case the desired reduction is shown to be both feasible and nontrivial. In fact, a modally damped system can be reduced further to a new entirely triangular second-order Schur form. Note finally that, while the motivating class of problem considered here is that of flexible structures, the results obtained apply equally well to any second-order system.

II. PROBLEM FORMULATION

Consider an \( n \)-mode model for the structural dynamics of a nongyroscopic, noncirculatory flexible structure with \( m \) actuators and \( p \) rate and/or displacement sensors, not necessarily collocated. This can be written as [3], [4]

\[
M\ddot{q} + C\dot{q} + Kq = Vu,
\]

\[
y = W_r\ddot{q} + W_d\dot{q},
\]

where \( q(t) \in \mathbb{R}^n \) is the vector of generalized coordinates describing deflection throughout the structure. The input vector \( u(t) \in \mathbb{R}^m \) with \( V \in \mathbb{R}^{p \times m} \) while the output vector \( y(t) \in \mathbb{R}^p \) with \( W_r, W_d \in \mathbb{R}^{p \times n} \). The \( n \times n \) mass, stiffness, and damping matrices of the structure are assumed to satisfy \( M = M^T > 0 \), \( K = K^T \geq 0 \), and \( C = C^T \geq 0 \), respectively, and it is for this reason that it is generally inadvisable to transform (1) to first-order form. This symmetric structure would be lost.

Computing the \( p \times m \) frequency response matrix

\[
G(s) = (sW_r + W_d)(s^2M + sC + K)^{-1}V
\]

of (1) at many different frequencies \( s \), involving as it does the implicit inversion of \((s^2M + sC + K)\) for each \( s \), will be computationally very expensive for the large models typical of FSS [2]. We